COMPOSITIONS WITH A FIXED NUMBER OF INVERSIONS

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Abstract. A composition of the positive integer n is a representation of n as an ordered sum of positive integers $n = a_1 + a_2 + \cdots + a_m$. There are 2^{n-1} unrestricted compositions of n, which can be sorted according to the number of inversions they contain. (An inversion in a composition is a pair of summands $\{a_i, a_j\}$ for which $i < j$ and $a_i > a_j$.) The number of inversions of a composition is an indication of how far the composition is from a partition of n, which by convention uses a sequence of nondecreasing summands and thus has no inversions. We count compositions of n with exactly r inversions in several ways to derive generating function identities, and also consider asymptotic results.

1. Introduction

A composition of the positive integer n is a representation of n as an ordered sum of positive integers $n = a_1 + a_2 + \cdots + a_m$. An inversion in a composition is a pair of summands $\{a_i, a_j\}$ for which $i < j$ and $a_i > a_j$.

In [4] the mean and variance for the number of inversions over all compositions of n was determined. In the current paper we study the statistics

$$ic_r(n) = \text{number of compositions of } n \text{ with } r \text{ inversions.}$$

For example, the 5-composition of 9 given as $9 = 1+3+2+1+2$ has four inversions, comparing the second summand to the third, fourth, and fifth, and comparing the third summand to the fourth. A partition of n written in standard nondecreasing order has no inversions.

A table of values of $ic_r(n)$ is provided below. Rows are indexed by n, columns are indexed by r and give values for $ic_r(n)$.

Not surprisingly, $ic_0(n) = p(n)$, the number of partitions of n. $ic_1(n)$ is [6, A058884]: partial sums of the partition function $p(n)$ with the last term subtracted.

2. Identities

Classical results about Mahonian statistics on permutations apply for compositions as well. Recall that the major index of the permutation τ, $MAJ(\tau)$, is the sum of the indices at which descents occur. We can make calculations more efficient by working with MAJ, rather than explicitly enumerating the number of inversions, by virtue of MacMahon's result [5] that the distribution of the major index on all

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$n\backslash r$	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	2	0	0	0	0	0	0	0	0	0	0	0	0
3	3	1	0	0	0	0	0	0	0	0	0	0	0
4	5	2	1	0	0	0	0	0	0	0	0	0	0
5	7	5	3	1	0	0	0	0	0	0	0	0	0
6	11	8	7	4	2	0	0	0	0	0	0	0	0
7	15	15	14	10	6	3	1	0	0	0	0	0	0
8	22	23	26	21	17	10	6	2	1	0	0	0	0
9	30	37	44	42	36	27	19	11	6	3	1	0	0
10	42	55	73	74	73	60	50	34	24	13	8	4	2
11	56	83	115	128	133	123	109	87	68	48	32	20	12
12	77	118	177	209	235	230	223	192	166	129	100	70	51

TABLE 1. Compositions with r inversions, $ic_r(n), 1 \leq n \leq 12, 0 \leq r \leq 12$

permutations (compositions) of a fixed length is the same as the distribution of inversions.

The approach we take is to build generating functions by studying MAJ. First we focus on the smallest part.

The generating function for partitions with first (smallest) part k is

$$f(k,z) = z^k \prod_{i=0}^{\infty} \frac{1}{1-z^{i+k}}.$$

This representation can be extended to the generating function for partitions with first part $< k$:

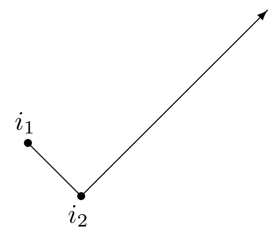
$$P_1(k,z) = \sum_{i=1}^{k-1} f(i,z).$$

The complementary sum provides the generating function for partitions with first part $\geq k$:

$$P_2(k,z) = \sum_{i=k}^{\infty} f(i,z).$$

Note $P_1(n,k) + P_2(n,k) = P(z) = \prod_{n=1}^{\infty} \frac{1}{1-z^n}$ for any choice of k. This is the generating function $F_0(z)$ for column $r = 0$.

Now we exhibit the generating function for column $r = 1$, the second column in Table 1, $F_1(z) = \sum_{i=0}^{\infty} ic_1(n)z^n$. We observe that by the MacMahon result, compositions of n with one inversion are equinumerous with compositions of n with one descent. Such compositions must have exactly one descent at the first summand i_1, with the remaining parts viewed as a partition with smallest part $i_2 < i_1$. This situation is diagrammed below.



Hence

$$F_1(z) = \sum_{i_1=2}^{\infty} z^{i_1} P_1(i_1, z) = \sum_{i_1=2}^{\infty} z^{i_1} \sum_{i_2=1}^{i_1-1} z^{i_2} \prod_{j=0}^{\infty} \frac{1}{1-z^{j+i_2}}$$

Thus we have

$$F_1(z) = \sum_{i_1=2}^{\infty} z^{i_1} \sum_{i=1}^{i_1-1} \frac{z^i}{(z^i;z)_\infty}. \tag{1}$$

Alternatively, using the equivalence with the sum of positions of descents, for $ic_1(n)$ we need to count compositions whose only descent occurs in position 1. Thus we have a partition into one part followed by a non-empty partition. The generating function for this is

$$\frac{z}{(1-z)}(P(z)-1).$$

However, we must subtract off the case where the combined combination is itself a partition with two or more parts (i.e. when there is no descent at position 1). The generating function for partitions into at least two parts is

$$P(z) - \frac{1}{1-z}.$$

Therefore,

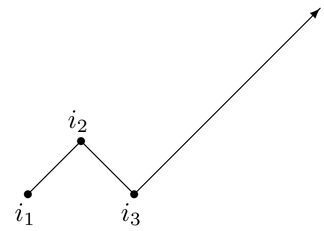
$$F_1(z) = \frac{z}{1-z}(P(z)-1) - \left(P(z) - \frac{1}{1-z}\right) = \frac{2z-1}{1-z}P(z) + 1. \tag{2}$$

Our first identity equates these two representations of the first column entries. Since (1) and (2) both represent the generating function for compositions with exactly one inversion, we have

Theorem 2.1.

$$\sum_{i_1=2}^{\infty} z^{i_1} \sum_{i=1}^{i_1-1} \frac{z^i}{(z^i;z)_\infty} = \frac{2z-1}{1-z}P(z) + 1.$$

For compositions counted by $ic_2(n)$ we need exactly one descent, occurring at the second summand. We will sum on i_2, constrain i_1 to be $\leq i_2$, and constrain summands after i_2 to form a partition with smallest part $i_3 < i_2$. The pattern is indicated below. Note there is no inequality implied in the picture between values of i_1 and i_3: i_3 could be greater than, equal to, or less than i_1, as long as $i_2 > i_1$ and $i_2 > i_3$.



$$F_2(z) = \sum_{i_1=2}^{\infty} z^{i_2} P_1(i_2, z) \frac{z(1-z^{i_2})}{1-z}.$$

This can be written as

$$F_2(z) = \frac{z}{1-z} \sum_{i_2=2}^{\infty} z^{i_2} \left(1 - z^{i_2}\right) \sum_{i=1}^{i_2-1} \frac{z^i}{(z^i; z)_\infty}. \tag{3}$$

An alternate formula involving only $P(z)$ is possible in this case as well. By the equivalence with the sum of positions of descents, for $ic_2(n)$ we see that we need to count compositions whose only descent occurs in position 2. Here and below a partition will denote a weakly increasing composition, unless specified as decreasing. Thus we have a partition into two parts followed by a non-empty partition. The generating function for this is

$$\frac{z^2}{(1-z)(1-z^2)}(P(z) - 1).$$

However we must subtract off the case where the combined composition is in fact itself a partition with 3 or more parts. (This is the case where the combined composition does not have a descent at position 2). The generating function for partitions into at least 3 parts is

$$P(z) - \frac{1}{(1-z)(1-z^2)}.$$

Therefore the generating function for $ic_2(n)$ is

$$F_2(z) = \left(\frac{z^2}{(1-z)(1-z^2)} - 1\right) P(z) + \frac{1}{1-z}. \tag{4}$$

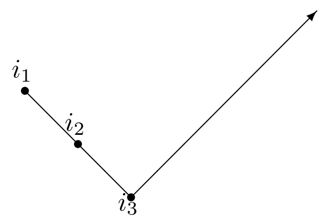
Since (3) and (4) both account for the $r = 2$ column in Table 1, we have

Theorem 2.2.

$$\frac{z}{1-z} \sum_{i_2=2}^{\infty} z^{i_2} \left(1 - z^{i_2}\right) \sum_{i=1}^{i_2-1} \frac{z^i}{(z^i; z)_\infty} = \left(\frac{z^2}{(1-z)(1-z^2)} - 1\right) P(z) + \frac{1}{1-z}.$$

For F_3 we need go no further than the third summand i_3. There are two cases: descents in positions 1 and 2 or a descent in position 3. To analyze the first case we need P_2.

Case 1: $i_1 > i_2 > i_3 \leq i_4 \leq i_5 \ldots.$



We insure that $i_1 > i_2 > i_3$ in the generating function via the factor

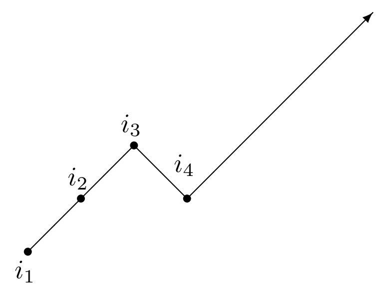
$$\sum_{i_2=i_3+1}^{\infty} z^{i_2} \sum_{i_1=i_2+1}^{\infty} z^{i_1} = \frac{z(z^2)^{1+i_3}}{(1-z)(1-z^2)}.$$

The first term in the trailing partition must be at least i_3 to guarantee no more inversions.

Thus

$$F_{3,1}(z) = \sum_{i_3=1}^{\infty} z^{i_3} \frac{z^{2i_3+3}}{(1-z)(1-z^2)} P_2(i_3, z).$$

Case 2: $i_1 \le i_2 \le i_3 > i_4 \le i_5 \le \dots$.



In this case the trailing partition is like that enumerated for ic_2. Let us get the contribution from $i_1 \le i_2 \le i_3$ by evaluating a sum.

$$\sum_{i_2=1}^{i_3} z^{i_2} \sum_{i_1=1}^{i_2} z^{i_1} = \frac{z^2(1-z^{i_3})(1-z^{1+i_3})}{(1-z)(1-z^2)}.$$

Hence

$$F_{3,2}(z) = \sum_{i_3=1}^{\infty} z^{i_3} \frac{z^2(1-z^{i_3})(1-z^{1+i_3})}{(1-z)(1-z^2)} P_1(i_3, z)$$

and

$$F_3(z) = F_{3,1}(z) + F_{3,2}(z)$$

is equivalent to

$$F_3(z) = \frac{1}{(1-z)(1-z^2)} \sum_{i_3=1}^{\infty} z^{i_3} \Big(z^{2i_3+3} \left(\sum_{i=i_3}^{\infty} \frac{z^i}{(z^i;z)_\infty} + 1 \right)$$

$$+ z^2(1-z^{i_3})(1-z^{i_3+1}) \left(\sum_{i=1}^{i_3-1} \frac{z^i}{(z^i;z)_\infty} \right) \Big) \tag{5}$$

In an alternate formulation there are also two case to consider. Firstly, the case where the only descent is in position 3. We then have a partition into 3 parts followed by a nonempty partition, with generating function

$$\frac{z^3}{(1-z)(1-z^2)(1-z^3)}(P(z)-1).$$

From this we subtract the case where the combined composition forms a partition into 4 or more parts, that is

$$P(z) - \frac{1}{(1-z)(1-z^2)(1-z^3)}.$$

Combining and simplifying gives the generating function

$$\left(\frac{z^3}{(1-z)(1-z^2)(1-z^3)} - 1 \right) P(z) + \frac{1}{(1-z)(1-z^2)}. \tag{6}$$

In the second case we have descents at both positions 1 and 2. Thus we have a strictly decreasing partition of 3 parts, followed by a possibly empty partition, with generating function

$$\frac{z^6}{(1-z)(1-z^2)(1-z^3)} P(z).$$

We must subtract the case of a strictly decreasing partition into 4 parts followed by a partition

$$\frac{z^{10}}{(1-z)(1-z^2)(1-z^3)(1-z^4)} P(z).$$

However we must add back the case of a strictly decreasing partition into 5 parts, and continue to consider the cases of even longer strictly decreasing partitions by an inclusion-exclusion argument, which leads to the generating function

$$P(z) \sum_{k=3}^{\infty} (-1)^{k-1} \frac{z^{k(k+1)/2}}{\prod_{j=1}^{k}(1-z^j)}.$$

Applying Euler's partition identity

$$\sum_{k=0}^{\infty} (-1)^k \frac{z^{k(k+1)/2}}{\prod_{j=1}^{k}(1-z^j)} = P(z)^{-1}$$

this becomes

$$\left(1 - \frac{z}{1-z} + \frac{z^3}{(1-z)(1-z^2)} \right) P(z) - 1. \tag{7}$$

Combining the two cases gives that the generating function for $ic_3(n)$ is

$$\left(-\frac{z}{1-z}+\frac{z^3}{(1-z)(1-z^2)}+\frac{z^3}{(1-z)(1-z^2)(1-z^3)}\right)P(z)$$
$$+\frac{1}{(1-z)(1-z^2)}-1. \tag{8}$$

We can note the equality of the representations of $F_3(z)$ provided in (5) and (8) to obtain another identity.

The analysis of $F_3(z)$ provides a general template for constructing an arbitrary generating function $F_r(z)$. The technique is to consider cases indexed by the partitions of r into distinct parts, and in each case set up the primary sum over the last summand a_k before the trailing partition. All the terms before i_k consolidate to contribute a rational function factor to the generating function, and the trailing partition contributes a factor of form either P_1 or P_2. Summing over cases gives the generating function for $F_r(z)$. If we cannot find a nice simplification to write the general case in a compact form, we can at least claim that in principle the g.f. is constructible, see also Theorem 2.3 later.

We use this method to find F_4 and F_5.

$$F_4(z)=\frac{1}{(1-z)(1-z^2)(1-z^3)}\sum_{i_4=1}^{\infty} z^{i_4}(z^{4+i_4}(z^3-1-(z-1)(z^{2+2i_4}))P_2(i_4,z)+$$
$$z^3(1-z^{i_4})(1-z^{1+i_4})(1-z^{2+i_4})P_1(i_4,z)),$$

which we can write as

$$F_4(z)=\frac{1}{(z;z)_3}\sum_{i_4=1}^{\infty} z^{i_4}\Bigg((-z^{2i_4+2}+z^2+z+1)z^{i_4+4}\left(\sum_{i=i_4}^{\infty}\frac{z^i}{(z^i;z)_\infty}+1\right)$$
$$+z^3(1-z^{i_4})(1-z^{i_4+1})(1-z^{i_4+2})\left(\sum_{i=1}^{i_4-1}\frac{z^i}{(z^i;z)_\infty}\right)\Bigg) \tag{9}$$

Alternatively, again there are two case to consider. Firstly, the case where the only descent is in position 4. We then have a partition into 4 parts followed by a nonempty partition, with generating function

$$\frac{z^4}{(1-z)(1-z^2)(1-z^3)(1-z^4)}(P(z)-1).$$

From this we subtract the case where the combined composition forms a partition into 5 or more parts, that is

$$P(z)-\frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^4)}.$$

Combining and simplifying gives the generating function

$$\left(\frac{z^4}{(1-z)(1-z^2)(1-z^3)(1-z^4)}-1\right)P(z)+\frac{1}{(1-z)(1-z^2)(1-z^3)}. \tag{10}$$

In the second case we have descents at both positions 1 and 3. Firstly we attach the generating function for a single part to that found for $ic_2(n)$ in order to make a descent at position 3, getting

$$\frac{z}{1-z}\left(\left(\frac{z^2}{(1-z)(1-z^2)}-1\right)P(z)+\frac{1}{1-z}\right). \tag{11}$$

We must subtract the case where we do not have a descent also at position 1, which from (6) is

$$\left(\frac{z^3}{(1-z)(1-z^2)(1-z^3)}-1\right)P(z)+\frac{1}{(1-z)(1-z^2)}. \tag{12}$$

Combining (10), (11) and (12) gives

$$\begin{aligned}
&\Big(-\frac{z}{1-z}+\frac{z^3}{(1-z)^2\,(1-z^2)}-\frac{z^3}{(1-z)\,(1-z^2)\,(1-z^3)}\\
&\quad+\frac{z^4}{(1-z)\,(1-z^2)\,(1-z^3)\,(1-z^4)}\Big)P(z)\\
&\quad+\frac{z}{(1-z)^2}-\frac{1}{(1-z)\,(1-z^2)}+\frac{1}{(1-z)\,(1-z^2)\,(1-z^3)}\\
=&\ \frac{z\left(z^9-z^8-2z^7-z^6-z^5+2z^4+3z^3+z^2-1\right)}{(z-1)^4(z+1)^2\,(z^2+1)\,(z^2+z+1)}P(z)\\
&\quad+\frac{z\left(z^4+z^3-z^2-z-1\right)}{(z-1)^3(z+1)\,(z^2+z+1)}.
\end{aligned} \tag{13}$$

We can build a complicated identity from (9) and (13).

Applying the first technique to five inversions, we find

$$\begin{aligned}
F_5(z)=&\ \frac{1}{(z;z)_4}\sum_{i_5=1}^{\infty}z^{i_5+4}(1-z^{i_5})(1-z^{i_5+1})(1-z^{i_5+2})(1-z^{i_5+3})\sum_{i=1}^{i_5-1}\frac{z^i}{(z^i;z)_\infty}\\
&+\frac{1}{(z;z)_3}\sum_{i_4=1}^{\infty}z^{i_4}\Big((-z^{i_4+2}+z^2+z+1)z^{2i_4+4}(\sum_{i=i_4}^{\infty}\frac{z^i}{(z^i;z)_\infty}+1)\\
&+z^4(1-z^{i_4})(1-z^{i_4+1})(z^{i_4+1}+z+1)\left(\sum_{i=1}^{i_4-1}\frac{z^i}{(z^i;z)_\infty}\right)\Big).
\end{aligned}$$

For the second technique there are three case to consider. Firstly, the case where the only descent is in position 5. We then have a partition into 5 parts followed by a nonempty partition, which leads in the same way as previously to the generating function

$$\begin{aligned}
&\left(\frac{z^5}{(1-z)\,(1-z^2)\,(1-z^3)\,(1-z^4)\,(1-z^5)}-1\right)P(z)\\
&\quad+\frac{1}{(1-z)\,(1-z^2)\,(1-z^3)\,(1-z^4)}.
\end{aligned} \tag{14}$$

In the second case we have descents at both positions 1 and 4. Firstly we attach the generating function for a single part to that found in (6) in order to make a descent at position 4, getting

$$\frac{z}{1-z}\left(\left(\frac{z^3}{(1-z)(1-z^2)(1-z^3)}-1\right)P(z)+\frac{1}{(1-z)(1-z^2)}\right). \tag{15}$$

We must subtract the case where we do not have a descent also at position 1, that is, where the only descent is in position 4, found already in (10).

In the third case we have descents at both positions 2 and 3. Firstly, attach a single part in front of the compositions counted in (7) leading to

$$\frac{z}{1-z}\left(\left(1-\frac{z}{1-z}+\frac{z^3}{(1-z)(1-z^2)}\right)P(z)-1\right) \tag{16}$$

and subtract the case in which the extra initial part would lead to descents at positions 1, 2 and 3. Using inclusion-exclusion as in (7) leads to

$$\begin{aligned} &P(z)\sum_{k=4}^{\infty}(-1)^{k-1}\frac{z^{k(k+1)/2}}{\prod_{j=1}^{k}(1-z^j)} \\ &= \left(1-\frac{z}{1-z}+\frac{z^3}{(1-z)(1-z^2)}-\frac{z^6}{(1-z)(1-z^2)(1-z^3)}\right)P(z)-1. \end{aligned} \tag{17}$$

Now we combine (14), (15), (10), (16) and (17) to obtain our result for $ic_5(n)$

$$\begin{aligned} &\left(1-\frac{z}{1-z}+\frac{z^3}{(z;z)_2}-\frac{z^6}{(z;z)_3}-\frac{z^4}{(z;z)_4}+\frac{z^5}{(z;z)_5}+\frac{z}{1-z}\left(-\frac{z}{1-z}+\frac{z^3}{(z;z)_2}+\frac{z^3}{(z;z)_3}\right)\right)P(z) \\ &\quad+\frac{1}{(z;z)_4}+\frac{z}{(1-z)(z;z)_2}-\frac{1}{(z;z)_3}-\frac{z}{1-z}-1. \end{aligned}$$

Alternatively this is

$$\begin{aligned} &\frac{2z^{15}-3z^{14}-3z^{13}+z^{11}+4z^{10}+5z^9+4z^8-2z^7-3z^6-6z^5-2z^4-z^3+2z^2+2z-1}{(z-1)^5(z+1)^2\left(z^2+1\right)\left(z^2+z+1\right)\left(z^4+z^3+z^2+z+1\right)}P(z) \\ &\quad+\frac{z^9-2z^7-2z^6-2z^5+2z^4+2z^3+2z^2+z-1}{(z-1)^4(z+1)^2\left(z^2+1\right)\left(z^2+z+1\right)}. \end{aligned}$$

With these special cases as a template, we now formulate a general theorem on the structure of the generating function of the sequence $ic_r(n)$.

Theorem 2.3. *For every fixed positive integer r, we have*

$$F_r(z)=\sum_{n=0}^{\infty}ic_r(n)z^n=f_r(z)P(z)+g_r(z)$$

for certain rational functions f_r and g_r. Neither f_r nor g_r has poles of modulus less than 1, and we have, as $z\to 1$,

$$f_r(z)\sim\frac{1}{r!}(1-z)^{-r}.$$

Proof. We use the approach via the major index. For a fixed set $A=\{a_1,a_2,\ldots,a_\ell\}$ of positive integers, we consider the set of compositions with ascents at all positions that do not lie in A (in other words, descents can only occur at positions that are elements of A). Following the approach shown in the examples above, we see that any such composition can be decomposed as follows: if its length is at least $\max A=a_\ell$, then it consists of

- a partition of length a_1,
- a partition of length a_2-a_1,
- $\cdots$
- a partition of length $a_\ell-a_{\ell-1}$,
- a partition of arbitrary length.

Otherwise, it decomposes as follows, for some positive integer $k \leq \ell$:

- a partition of length a_1,
- a partition of length $a_2 - a_1$,
- $\cdots$
- a partition of length $a_{k-1} - a_{k-2}$,
- a partition of length at most $a_k - a_{k-1} - 1$.

Here, we set $a_0 = 0$ for $k = 1$. The generating function associated with such compositions is thus

$$S_A(z) = z^{a_1} \prod_{j=1}^{a_1} (1-z^j)^{-1} \cdot z^{a_2-a_1} \prod_{j=1}^{a_2-a_1} (1-z^j)^{-1} \cdots z^{a_\ell - a_{\ell-1}} \prod_{j=1}^{a_\ell - a_{\ell-1}} (1-z^j)^{-1} \cdot \prod_{j=1}^{\infty} (1-z^j)^{-1}$$
$$+ \sum_{k=1}^{\ell} z^{a_1} \prod_{j=1}^{a_1} (1-z^j)^{-1} \cdot z^{a_2-a_1} \prod_{j=1}^{a_2-a_1} (1-z^j)^{-1} \cdots \prod_{j=1}^{a_k - a_{k-1}-1} (1-z^j)^{-1}.$$

The first term rewrites as

$$z^{a_\ell} \prod_{k=1}^{\ell} \prod_{j=1}^{a_k - a_{k-1}} (1-z^j)^{-1} \cdot P(z), \tag{18}$$

while the rest is a rational function. We remark that $S_\emptyset(z) = P(z)$.

Now we obtain the generating function for compositions with descents *exactly* at all positions that are elements of A, by means of the inclusion-exclusion principle:

$$T_A(z) = \sum_{B \subseteq A} (-1)^{|A|-|B|} S_B(z). \tag{19}$$

Finally, let $\mathbb{N}$ be the set of all positive integers, and let $\Sigma(A) = a_1 + a_2 + \cdots + a_\ell$ denote the sum of elements in A. By definition, the generating function for compositions whose major index is r is given by

$$F_r(z) = \sum_{A \subseteq \mathbb{N} : \Sigma(A) = r} T_A(z). \tag{20}$$

We observe that each function $S_A(z)$ has the form $f_A(z)P(z) + g_A(z)$ for certain rational functions f_A and g_A. Since T_A and eventually F_r are linear combinations of such functions, they must have the same shape. We also note that all poles of these rational functions lie on the unit circle, hence there are no poles of modulus less than 1. It only remains to prove the last statement about the asymptotic behaviour of f_r.

To this end, note first that f_A has a pole of order

$$\sum_{k=1}^{\ell} (a_k - a_{k-1}) = a_\ell = \max A$$

at 1, since each factor $(1-z^j)^{-1}$ in (18) contains a single factor $(1-z)^{-1}$. It follows from (19) that the order of 1 as a pole of the rational factor occurring in T_A is at most $a_\ell = \max A$. Considering (20) next, we see that the dominant contribution to the sum comes from the set $A = \{r\}$:

$$T_{\{r\}}(z) = S_{\{r\}}(z) - S_\emptyset(z) = z^r \prod_{j=1}^{r} (1-z^j)^{-1} \cdot P(z) + \prod_{j=1}^{r-1} (1-z^j)^{-1} - P(z),$$

which contributes a pole of order r to the function f_r, while all other terms in (20) have poles of lower order. Since $1 - z^j \sim j(1-z)$ as $z \to 1$, we have

$$z^r \prod_{j=1}^{r} (1-z^j)^{-1} \sim \frac{1}{r!}(1-z)^{-r}$$

as $z \to 1$, which completes the proof. ∎

3. Asymptotic Estimates

In the analysis of partition statistics, one often has to study generating functions of the form $P(x)F(x)$, where

$$P(x) = \prod_{j=1}^{\infty} (1-x^j)^{-1}$$

is the generating function for the number of partitions. In the paper [3], a general asymptotic scheme was derived that allows one to derive an asymptotic formula for the n-th coefficient of $P(x)F(x)$ from the behaviour of $F(x)$ as $x \to 1$. It is well known that $p(n) = [x^n]P(x)$ essentially behaves like $\frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{2n/3}\right)$, which is made much more precise by Rademacher's celebrated formula

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \frac{d}{dn} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-1/24}},$$

a sum formula that is both exact and asymptotic (in the sense that the asymptotic order of the summands is decreasing).

It is necessary that $F(z)$ does not grow too quickly as $|z| \to 1$. Specifically, we assume that

$$|F(z)| = O\left(e^{C/(1-|z|)^{\eta}}\right) \text{ as } |z| \to 1 \text{ for some } C > 0 \text{ and } \eta < 1. \tag{21}$$

These technical conditions are easily seen to hold if $F(z)$ is a rational function without poles of modulus less than 1, and this is the case in all our examples.

Theorem 3.1. *Suppose that the function $F(z)$ satisfies (21) and that*

$$\frac{F(e^{-t+iu})}{F(e^{-t})} \to 1$$

if $|u| \le At^{1+\epsilon}$ for some $A > 0$ and some $\epsilon < \frac{1-\eta}{2}$, uniformly in u as $t \to 0$. Then one has

$$\frac{1}{p(n)}[x^n]P(x)F(x) = F\left(e^{-\pi/\sqrt{6n}}\right)(1+o(1)) + O\left(\exp\left(-Bn^{1/2-\epsilon}\right)\right)$$

as $n \to \infty$ for some $B > 0$.

To obtain more precise asymptotic formulae in the case that $F(e^{-t})$ can be expanded into powers of t around $t = 0$, we also need the following result from [3]:

Theorem 3.2. *Suppose that the function $F(z)$ satisfies (21) and $F(e^{-t}) = at^b + O(f(|t|))$ as $t \to 0$ for real numbers a, b. Then one has*

$$\frac{1}{p(n)}[x^n]P(x)F(x) = a\left(\frac{2\pi}{\sqrt{24n-1}}\right)^b \cdot \frac{I_{|b+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}$$
$$+ O\left(\exp\left(-n^{1/2-\epsilon}\right) + f\left(\frac{\pi}{\sqrt{6n}} + O\left(n^{-1/2-\epsilon}\right)\right)\right)$$

as $n \to \infty$ for any $0 < \epsilon < \frac{1-\eta}{2}$, where I_ν denotes a modified Bessel function of the first kind.

For computational purposes we can make use of the following asymptotic formula

$$\frac{I_{|b+3/2|}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)}{I_{3/2}\left(\sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}\right)} = \frac{m}{m-1} \cdot \sum_{j=0}^{J} \binom{h+j}{2j} \frac{(2j)!}{j!}\left(-\frac{1}{2m}\right)^j + O(m^{-J-1})$$

for any fixed J, with $h = |b+3/2| - 1/2$ and $m = \sqrt{\frac{2\pi^2}{3}\left(n-\frac{1}{24}\right)}$.

Of course the theorem generalises to asymptotic expansions of the form

$$F(e^{-t}) = \sum_{j=1}^{J} a_j t^{b_j} + O(f(|t|)).$$

Using the above formulas, together with computer algebra to carry out the lengthy calculations, we can obtain precise asymptotic estimates for the number of compositions with a fixed number of inversions.

Recall that the generating function for $ic_1(n)$ is

$$\frac{2z-1}{1-z}P(z) + 1.$$

Here

$$F(e^{-t}) = \frac{2e^{-t}-1}{1-e^{-t}} = \frac{1}{t} - \frac{3}{2} + \frac{t}{12} + O(t^3),$$

and we obtain

$$\frac{ic_1(n)}{p(n)} = \frac{\sqrt{6}\sqrt{n}}{\pi} - \frac{3\left(\pi^2-2\right)}{2\pi^2} + \frac{216 - 3\pi^2 + 2\pi^4}{24\sqrt{6}\pi^3\sqrt{n}} + O\left(\frac{1}{n}\right).$$

The generating function for $ic_2(n)$ is

$$\left(\frac{z^2}{(1-z)(1-z^2)} - 1\right)P(z) + \frac{1}{1-z}.$$

Here

$$F(e^{-t}) = \frac{1}{2t^2} - \frac{1}{4t} - \frac{25}{24} + O(t),$$

and we obtain

$$\frac{ic_2(n)}{p(n)} = \frac{3n}{\pi^2} - \frac{\sqrt{\frac{3}{2}}\left(\pi^2-6\right)\sqrt{n}}{2\pi^3} - \frac{7}{8\pi^2} + \frac{9}{2\pi^4} - \frac{25}{24} + O\left(\frac{1}{\sqrt{n}}\right).$$

The generating function for $ic_3(n)$ is

$$\left(-\frac{z}{1-z}+\frac{z^3}{(1-z)(1-z^2)}+\frac{z^3}{(1-z)(1-z^2)(1-z^3)}\right)P(z)$$
$$+\frac{1}{(1-z)(1-z^2)}-1.$$

Here

$$F(e^{-t})=\frac{1}{6t^3}+\frac{1}{2t^2}-\frac{133}{72t}+O(1),$$

and we obtain

$$\frac{ic_3(n)}{p(n)}=\frac{\sqrt{6}n^{3/2}}{\pi^3}+\frac{3n}{\pi^2}-\frac{\left(266\pi^2-207\right)\sqrt{n}}{24\sqrt{6}\pi^3}+O\left(1\right).$$

The generating function for $ic_4(n)$ is

$$\frac{z\left(z^9-z^8-2z^7-z^6-z^5+2z^4+3z^3+z^2-1\right)}{(z-1)^4(z+1)^2\left(z^2+1\right)\left(z^2+z+1\right)}P(z)$$
$$+\frac{z\left(z^4+z^3-z^2-z-1\right)}{(z-1)^3(z+1)\left(z^2+z+1\right)}.$$

Here

$$F(e^{-t})=\frac{1}{24t^4}+\frac{3}{8t^3}-\frac{17}{32t^2}+O\left(t^{-1}\right),$$

and we obtain

$$\frac{ic_4(n)}{p(n)}=\frac{3n^2}{2\pi^4}+\frac{3\sqrt{\frac{3}{2}}\left(3\pi^2-2\right)n^{3/2}}{2\pi^5}+\frac{\left(36-2\pi^2-51\pi^4\right)n}{16\pi^6}+O\left(\sqrt{n}\right).$$

The generating function for $ic_5(n)$ is $P(z)F(z)+R(z)$ where

$$F(z)=\frac{2z^{15}-3z^{14}-3z^{13}+z^{11}+4z^{10}+5z^9+4z^8-2z^7-3z^6-6z^5-2z^4-z^3+2z^2+2z-1}{(z-1)^5(z+1)^2\left(z^2+1\right)\left(z^2+z+1\right)\left(z^4+z^3+z^2+z+1\right)}.$$

Here

$$F(e^{-t})=\frac{1}{120t^5}+\frac{7}{48t^4}+\frac{31}{144t^3}+O\left(t^{-2}\right),$$

and we obtain

$$\frac{ic_5(n)}{p(n)}=\frac{3\sqrt{\frac{3}{2}}n^{5/2}}{5\pi^5}+\frac{3\left(7\pi^2-6\right)n^2}{4\pi^6}+\frac{\left(432-507\pi^2+124\pi^4\right)n^{3/2}}{16\sqrt{6}\pi^7}+O\left(n\right).$$

Having seen these examples, we would now like to prove a more general theorem in which we identify the leading term of the asymptotics for arbitrary r. This is easily achieved by means of Theorem 2.3:

Theorem 3.3. *For every fixed positive integer r, the number of compositions of n with r inversions satisfies the asymptotic formula*

$$ic_r(n)\sim\frac{1}{r!}\Big(\frac{\sqrt{6n}}{\pi}\Big)^r p(n). \tag{22}$$

Proof. As mentioned before, the technical conditions of Theorem 3.2 are satisfied, since the factor f_r is a rational function. The additional rational function g_r only makes a contribution to $ic_r(n)$ that grows polynomially, so it is irrelevant.

Theorem 2.3 tells us that

$$f_r(e^{-t}) \sim \frac{1}{r!}(1-e^{-t})^{-r} \sim \frac{1}{r!}t^{-r},$$

so the result follows immediately by means of Theorem 3.2. ∎

Remark 3.4. *A heuristic argument for Theorem 3.3 can be given as follows: for large n, most compositions with r inversions are obtained from a partition by switching a pair of two consecutive distinct summands in r different places (there are of course other ways to achieve precisely r inversions, but those provide fewer choices). It is well known that the number of distinct parts in a random partition of n satisfies a central limit theorem with an average that is asymptotically equal to $\sqrt{6n}/\pi$. Since we are choosing r pairs of consecutive distinct parts, this means that the number of ways to turn a partition into a composition with r inversions is typically about*

$$\binom{\sqrt{6n}/\pi}{r} \sim \frac{1}{r!}\Big(\frac{\sqrt{6n}}{\pi}\Big)^r,$$

which explains the asymptotic formula (22).

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