On the commutativity of finite products and coequalisers

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Limit/Colimit commutation conditions

finite products	finite coproducts
equalisers	coequalisers

limit/colimit commutation conditions

finite products | finite coproducts

finitely complete categories

equalisers | coequalisers

limit/colimit commutation conditions

finite products

finitely complete categories

equalisers

finite coproducts

finitely cocomplete categories

coequalisers

Commutativity binary products and binary coproducts

Let $\mathbb{C}$ have finite products and coproducts and suppose that if $X_1 \xrightarrow{x_1} X \xleftarrow{x_2} X_2$ and $Y_1 \xrightarrow{y_1} Y \xleftarrow{y_2} Y_2$ are coproduct diagrams in $\mathbb{C}$ then the diagram $\mathbb{C}$.

$$X_1 \times Y_1 \xrightarrow{x_1 \times y_1} X \times Y \xleftarrow{x_2 \times y_2} X_2 \times Y_2$$

is a coproduct diagram in $\mathbb{C}$.

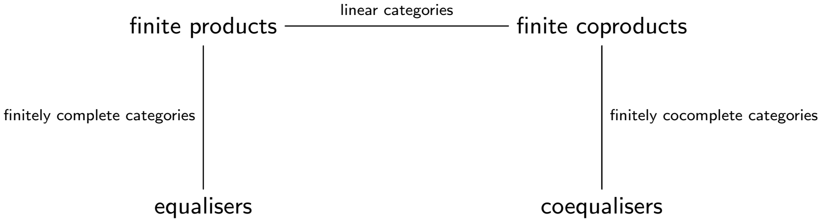
Linear categories

Then $\mathbb{C}$ is pointed (the initial and terminal objects coincide) and since $X \to X \leftarrow 0$ and $0 \to Y \leftarrow Y$ are coproducts we have

$$X \xrightarrow{(1,0)} X \times Y \xleftarrow{(0,1)} Y$$

is a coproduct. Thus $\mathbb{C}$ admits biproducts, and hence $\mathbb{C}$ is a linear (or half additive) category.

limit/colimit commutation conditions



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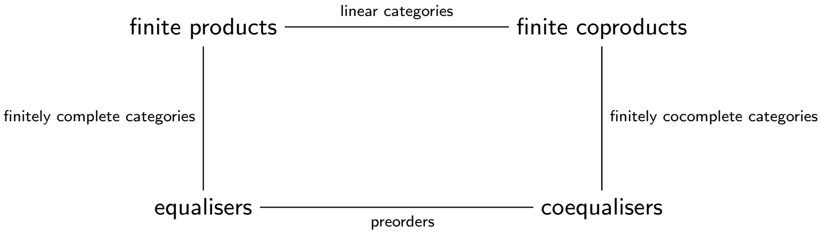

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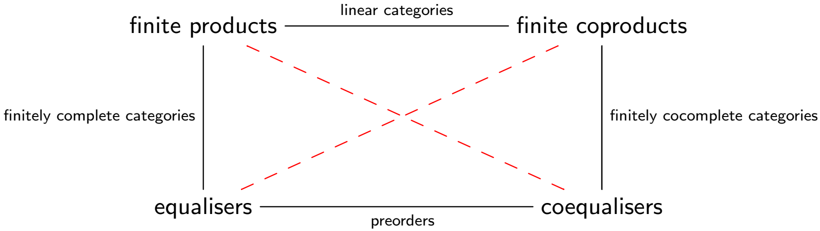

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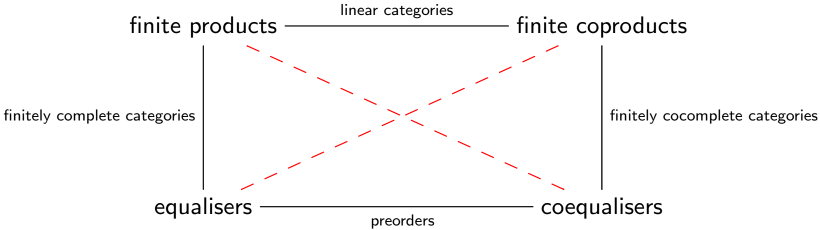

A natural question

What, if anything, can be said of categories in which finite products commute with coequalisers?

Commutativity of finite products and coequalisers

Let $\mathbb{C}$ admit finite products and coequalisers and suppose that $\mathbb{C}$ satisfies the property:

- for any two coequalizer diagrams

$$C_1 \underset{v_1}{\overset{u_1}{\rightrightarrows}} X \xrightarrow{q_1} Q_1 \qquad C_2 \underset{v_2}{\overset{u_2}{\rightrightarrows}} Y \xrightarrow{q_2} Q_2,$$

  in $\mathbb{C}$, the diagram

$$C_1 \times C_2 \underset{v_1 \times v_2}{\overset{u_1 \times u_2}{\rightrightarrows}} X \times Y \xrightarrow{q_1 \times q_2} Q_1 \times Q_2,$$

  is a coequalizer diagram in $\mathbb{C}$.

Examples

- **Grp** - the category of groups.
- **Mon** - the category of monoids.
- The category **Imp** of implication algebras, or $\mathbf{Lat}_*$ - pointed lattices.
- The category **Frm** of frames.
- Every congruence modular variety of algebras (which admits at least one constant).
- Every factor permutable category with coequalisers of H.P. Gumm (which admits at least one constant).
- Every regular unital or weakly unital category, with coequalisers.
- Every coextensive category with coequalisers.

Varieties in which finite products commute with coequalisers.

A universal algebraic question

Is it possible to give an equational description of varieties of algebras in which finite products commute with coequalisers?

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Answer

For pointed varieties of algebras, this was answered in the paper

- *M. Hoefnagel. Products and coequalizers in pointed categories. Theory and Applications of Categories 34, 1386–1400, 2019.*

The Mal'tsev condition for products commuting with coequalizers.

Theorem

Finite products commute with coequalisers in a pointed variety if and only if it admits binary terms $b_i(x, y)$ and unary terms $c_i(x)$ for each $1 \leqslant i \leqslant m$ and $(m+2)$-ary terms $p_1, p_2, ..., p_n$ satisfying the equations:

$$p_1(x, y, b_1(x, y), b_2(x, y), ..., b_m(x, y)) = x,$$
$$p_i(y, x, b_1(x, y), ..., b_m(x, y)) = p_{i+1}(x, y, b_1(x, y), ..., b_m(x, y)),$$
$$p_n(y, x, b_1(x, y), b_2(x, y), ..., b_m(x, y)) = y,$$
$$p_i(0, 0, c_1(z), ..., c_m(z)) = z.$$

What about general varieties?

Finite products commute with coequalisers in many non-pointed varieties of algebras. For instance **Ring**, or in any congruence modular variety with constants.

Common coequalisers

Given any diagram

$$C_1 \underset{v_1}{\overset{u_1}{\rightrightarrows}} X \underset{u_2}{\overset{v_2}{\leftleftarrows}} C_2 \qquad (*)$$

in any category, we say that it *admits a common coequaliser* if there exists a morphism $q : X \to Q$ which is simultaneously a coequaliser of both of the parallel pairs above.

$$\begin{array}{ccccc} C_1 & \underset{v_1}{\overset{u_1}{\rightrightarrows}} & X & \underset{u_2}{\overset{v_2}{\leftleftarrows}} & C_2 \\ & & \downarrow & & \\ & & Q & & \end{array}$$

Preservation of common coequalisers

Binary products are said to preserve common coequalisers if the following property holds: if given any two diagrams

$$C_1 \underset{v_1}{\overset{u_1}{\rightrightarrows}} X \underset{u_2}{\overset{v_2}{\leftleftarrows}} C_2 \qquad C_1' \underset{v_1'}{\overset{u_1'}{\rightrightarrows}} X' \underset{u_2'}{\overset{v_2'}{\leftleftarrows}} C_2'$$

each of which admits a common coequaliser, their pointwise product

$$C_1 \times C_1' \underset{v_1 \times v_1'}{\overset{u_1 \times u_1'}{\rightrightarrows}} X \times X' \underset{u_2 \times u_2'}{\overset{v_2 \times v_2'}{\leftleftarrows}} C_2 \times C_2'$$

admits a common coequaliser.

Varieties without constants

Suppose $\mathcal{V}$ is a variety with no constants (so that $\emptyset$ is initial). If X is a non-empty algebra in $\mathcal{V}$, then

$$X^2 \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} X \underset{\pi_1}{\overset{\pi_2}{\leftleftarrows}} X^2 \qquad X \underset{1_X}{\overset{1_X}{\rightrightarrows}} X \underset{0}{\overset{0}{\leftleftarrows}} 0$$

both admit a common coequaliser, so that the diagram

$$X \times X^2 \underset{1_X \times \pi_1}{\overset{1_X \times \pi_2}{\rightrightarrows}} X^2 \underset{0}{\overset{0}{\leftleftarrows}} 0$$

admits a common-coequaliser. Since X is non-empty the coequaliser of the left parallel pair is the product projection $\pi_1 : X \times X \to X$, which must be an isomorphism as it is the coequaliser of the right-most parallel pair.

Examples

In each of the categories below, binary products preserve common coequalisers.

Example

Every category $\mathbb{C}$ in which finite products commute with coequalisers necessarily has that it preserves common coequalisers.

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Non-preservation of common coequalisers

A variety with just two distinct constant symbols.

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A variety with just two distinct constant symbols.

A significant example

Pointed sets $\mathbf{Set}_*$ preserve common coequalisers. In fact, any regular pointed category with coequalisers preserves common coequalisers.

Universal-Algebraic description

Commutativity of finite products with coequalisers is the conjunction of:

- binary products preserving common coequalisers,
- a certain Mal'tsev condition.

The Mal'tsev condition

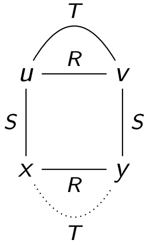
$\mathcal{V}$ admits binary terms b_i and unary terms u_i, v_i, c_i and terms $p_1, p_2, \dots, p_n$ satisfying the equations

$$\begin{aligned} p_1(u_1(x), \dots, u_r(x), v_1(y) \dots, v_r(y), b_1(x,y), \dots, b_m(x,y)) = x, \\ p_i(u_1(x), \dots, u_r(x), v_1(y) \dots, v_r(y), b_1(x,y), \dots, b_m(x,y)) = \\ p_{i+1}(u_1(y), \dots, u_r(y), v_1(x) \dots, v_r(x), b_1(x,y), \dots, b_m(x,y)), \\ p_n(u_1(y), \dots, u_r(y), v_1(x) \dots, v_r(x), b_1(x,y), \dots, b_m(x,y)) = y \end{aligned}$$

and for each $i = 1, ..., n$ we have $p_i(k_1, \dots, k_r, t_1, \dots, t_r, c_1(z), ..., c_m(z)) = z$.

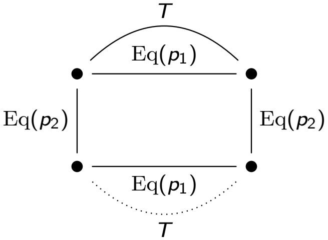
H.P. Gumm's shifting lemma

A variety $\mathcal{V}$ of universal algebras satisfies the *shifting lemma* if for any three congruences R, S, T on any algebra X in $\mathcal{V}$ such that $R \cap S \leqslant T$, the relation between elements of X indicated by the dotted arrow may be deduced from the relations indicated by the solid arrows.



The shifting lemma on pullbacks

Is the restricted case when $R = \mathrm{Eq}(p_1)$ and $S = \mathrm{Eq}(p_2)$ where p_1 and p_2 are complementary pullback projections.



Using the fibration of points $\pi : \mathrm{Pt}(\mathbb{C}) \to \mathbb{C}$ we can Bourn-localise:

Theorem

Finite products commute with coequalisers locally in a variety if and only if it satisfies the shifting lemma on pullbacks.

Some questions

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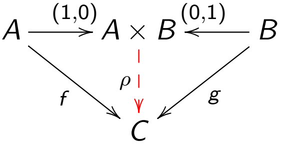
- To be more precise, are we able to establish some centrality properties (of congruences) for varieties satisfying the shifting lemma on pullbacks?

Some questions

- Given how much can be said of congruence modular varieties, what could we say of varieties satisfying the shifting lemma on pullbacks?
- To be more precise, are we able to establish some centrality properties (of congruences) for varieties satisfying the shifting lemma on pullbacks?
- What about centrality simply in pointed categories in which finite products commute with coequalisers?

Huq-commutes

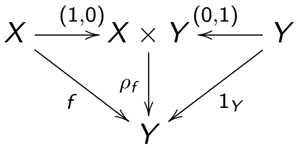
Two morphisms $f : A \to C$ and $g : B \to C$ in a pointed category $\mathbb{C}$ with finite products are said to **commute** if there exists a morphism $\rho : A \times B \to C$ making the diagram



commute.

Central morphisms

A morphism $f : X \to Y$ is called <u>central</u> if f and 1_Y commute.



Commutative objects

An object X is called *commutative* if 1_X is central. The full subcategory of $\mathbb{C}$ of commutative objects is denoted by $\mathrm{Com}(\mathbb{C})$.

Huq-centrality for groups/monoids

In the category **Grp** a morphism $f : G \to H$ is central if and only if $\mathrm{im}(f) \subseteq Z(H)$. This is true of **Mon** as well as any Jónsson-Tarski variety of algebras, that is, any variety admitting a binary operation $+$ satisfying

$$x + 0 = x = 0 + x.$$

In **Mon**/**Grp** if we consider two objects X and Y then the central morphisms $Z(X, Y)$ from X to Y from a commutative monoid, which acts canonically on $\hom(X, Y)$

$$Z(X, Y) \times \hom(X, Y) \to \hom(X, Y).$$

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Moreover, commutative objects is each respective category is simply the commutative objects in each respective category.

Additive core

In above structure is part of what is called the "additive core" in the paper

- D. Bourn. Intrinsic centrality and associated classifying properties Journal of Algebra, 256:126–145, 2002.

Unital categories

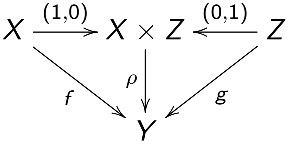
A category is said to be unital (D. Bourn) if for any two objects X and Y the products inclusions

$$X \xrightarrow{(1,0)} X \times Y \xleftarrow{(0,1)} Y$$

are jointly strongly epimorphic. It is called weakly unital (N.-M. Ferreira) if they are epimorphic.

Unital and weakly categories

In a (weakly) unital category the morphism ρ in



is unique when f is central.

Remark

This is one of the main properties that is necessary to abstractly obtain this additive core of a (weakly) unital category

$$Z(X,Y) \times \hom(X,Y) \to \hom(X,Y).$$

Centralic categories

Definition

A pointed category with finite products is called centralic if for any morphism $f : X \times Y \to Z$ we have

$$f(x, 0) = f(x', 0) \implies f(x, y) = f(x', y)$$

for any $y \in Y$.

Theorem

A pointed regular category with coequalisers is centralic if and only if finite products commute with coequalisers.

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A pointed regular category with coequalisers is centralic if and only if finite products commute with coequalisers.

Theorem

In a centralic category $\mathbb{C}$, *there exists a unique cooperator between a central morphism* $f : X \to Y$ *and any other morphism* $g : X' \to Y$ *in* $\mathbb{C}$.

Theorem

In a centralic category $\mathbb{C}$, there exists a unique cooperator between a central morphism $f : X \to Y$ and any other morphism $g : X' \to Y$ in $\mathbb{C}$.

Using this fact, together with some other properties of centralic categories, we have that every centralic category admits the additive core mentioned earlier

$$Z(X, Y) \times \hom(X, Y) \to \hom(X, Y).$$

Examples

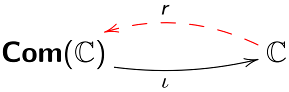
- Every unital or weakly unital category is centralic.
- Every pointed Gumm category is centralic.
- Every factor permutable category is centralic.
- Every congruence hyperextensible category in the sense of D. Bourn is centralic.

Remark

These examples serve to show how general the concept of a centralic category is, and therefore how common the property for finite products to commute with arbitrary coequalisers is.

Abelianization

If $\mathbb{C}$ is the category of groups/monoids then the inclusion functor ι admits a left adjoint given by "abelianisation"



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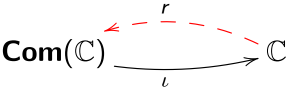

Strongly centralic categories

There is a strengthening of the concept of centralic category, namely, of <u>strongly centralic category</u>, which every unital or factor permutable category is.

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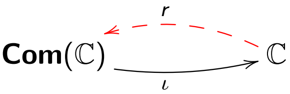

Strongly centralic categories

There is a strengthening of the concept of centralic category, namely, of <u>strongly centralic category</u>, which every unital or factor permutable category is. In the context of a regular strongly centralic category $\mathbb{C}$ with coequalisers, the inclusion $\mathbf{Com}(\mathbb{C}) \to \mathbb{C}$ admits a left adjoint, i.e., we can abelianize.

Concluding remarks

- The general question of what can be proved about centralic and strongly centralic categories is very much open.
- What about locally (strongly) centralic categories, i.e., categories for which every category of points is centralic. How much of the theory of congruence modular varieties generalises to this context?

Thank you for listening

References

- M. Hoefnagel, *Products and coequalizers in pointed categories*, Theory and Applications of Categories, 2019, accepted.
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