The matrix taxonomy of finitely complete categories

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What is a matrix property?

Various properties in *categorical algebra* may be presented as a finite matrix of positive integers such as the matrix below:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

This particular matrix captures the essential categorical property of a category to be what is know as a *Mal'tsev category* [1] in categorical algebra. Other properties in categorical algebra can similarly be represented by a matrix, such as for instance the property of a category to be a *majority category* [2], which is given by the 3×3 identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The general theory of matrix properties were first introduced in the paper [5].

The problem of determining implications of matrix properties

If M is a matrix, then the class of all (finitely complete) categories which have the matrix property corresponding to M is denoted by $\mathrm{mclex}\{M\}$. This is called the matrix class of M. Given two such matrices M and N, we are interested in determining when does M "imply" N, i.e., when is $\mathrm{mclex}\{M\}$is contained in $\mathrm{mclex}\{N\}$. A valuable intuition about matrix properties is given by their interpretation in algebra. Roughly speaking, a matrix property corresponds to a system of equations. For example the Mal'tsev matrix above determines a system of equations as follows:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$p(x_0, x_1, x_1) = x_0$$
$$p(x_1, x_0, x_1) = x_0$$

In the algebraic case, an implication of matrix properties reduces to determining when does one such system of equations determine another.

The algorithm

In a recent paper together with my collaborators [7], we have determined a simple algorithm for determining such implications/inclusions. This algorithm can be implemented on a computer, allowing us to calculate implications of matrix properties. Such a calculation (done by the computer) looks like:

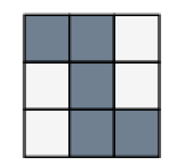
	0 0 1	0 0 1
0 1 1	0 1 0	0 1 1
1 0 1	1 0 0	1 0 1
	0 0 1	0
	1 0 1	1
	0 1 1	1
0 0 0		0
0 1 1		0
1 0 1		0

Computational results

For visual purposes we represent matrices with cells of different colors. For example the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

is represented by



Then using the algorithm we may generate images such as the image below, which represents all matrix properties with at most 3 rows and two distinct variables. The arrows represent entailment of matrix properties.

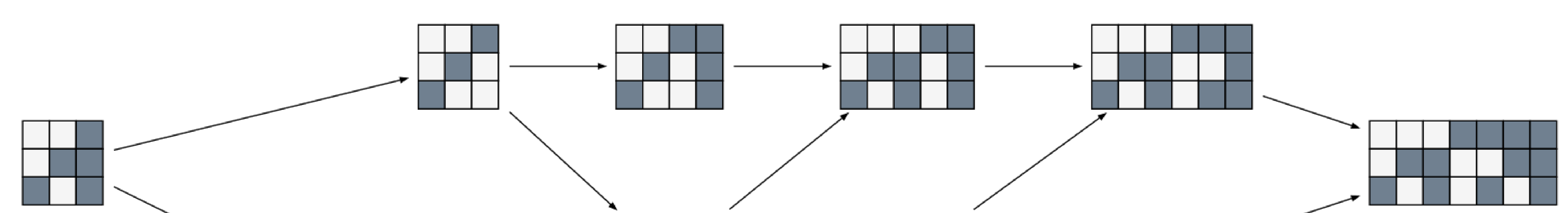


Fig. 1: All matrix properties with at most 3 rows and two distinct variables.

The image below was the first to be calculated, but since then we have produced many more such images. For example, the image below represents all matrix properties (and their relations) which may be represented by a matrix with at most 4 rows, 4 columns and two variables.

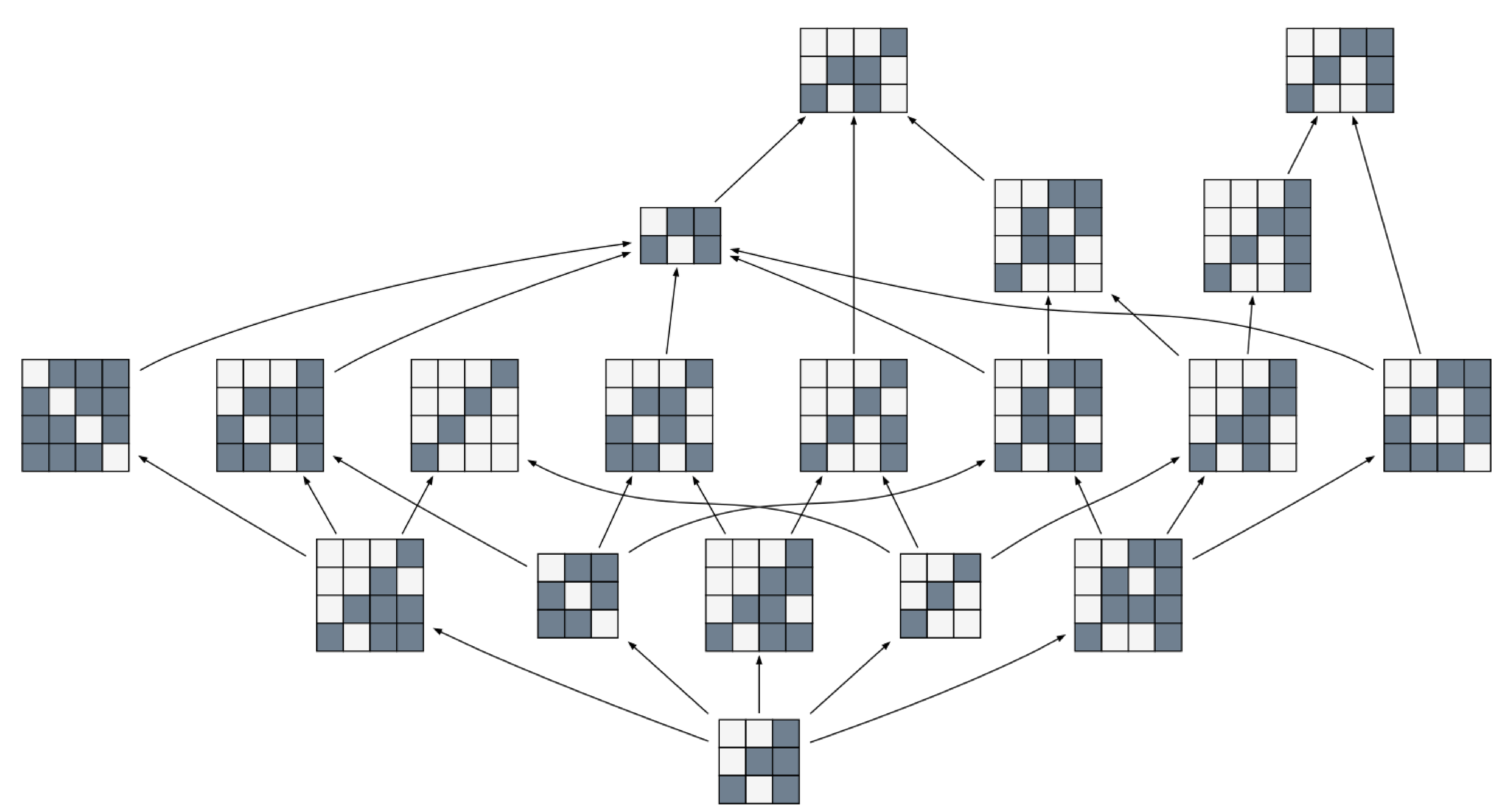


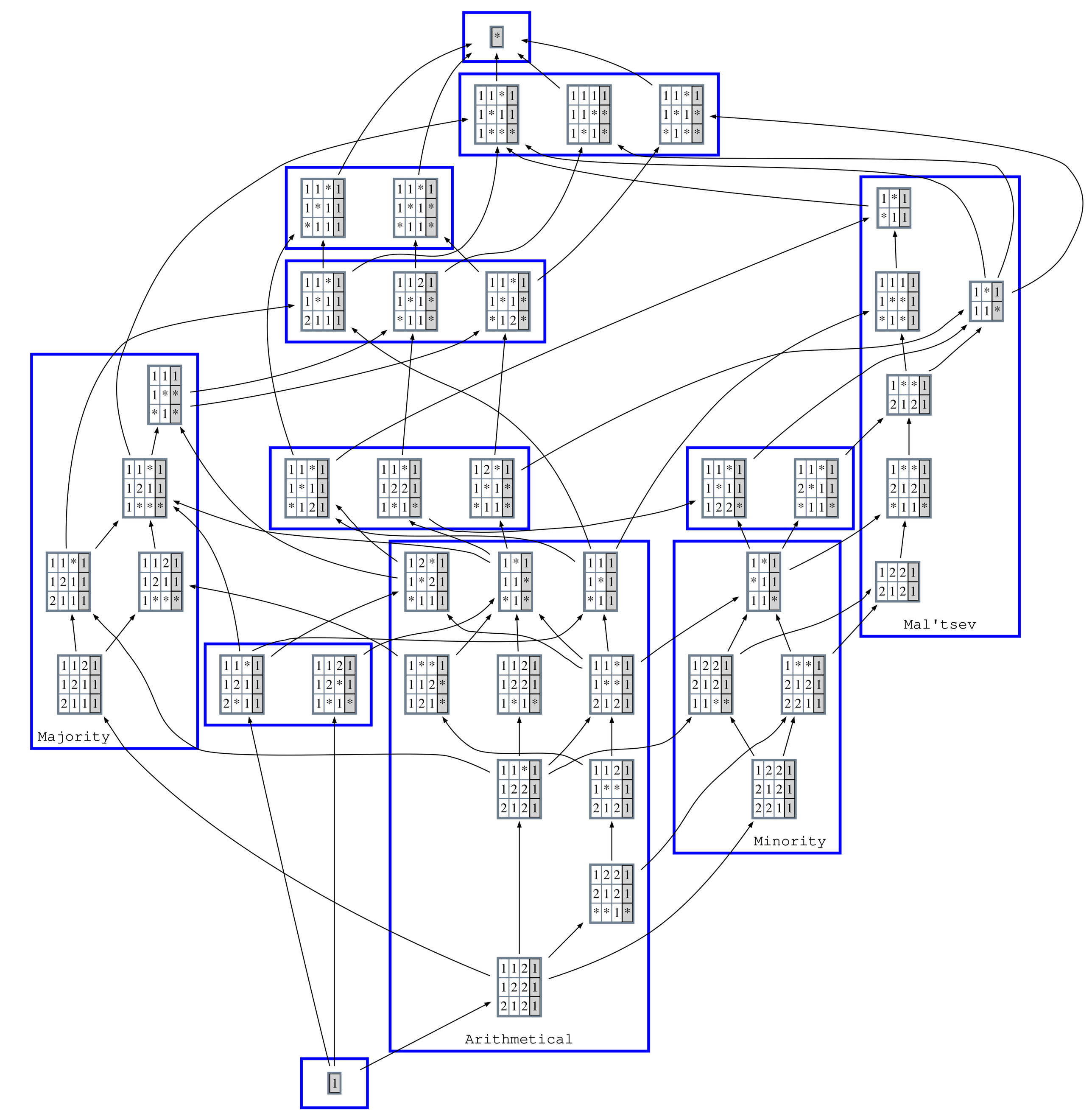
Fig. 2: All matrix properties with at most 4 rows/columns and two distinct variables.

These two pictures already illustrate a general fact about matrix properties: there is only one non trivial matrix property represented by a matrix with two rows — the Mal'tsev matrix!

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Bourn localisation

Matrix properties can have other relationships to each other. For instance a matrix can "Bourn localize" [6] another matrix. Together with P. A. Jacqmin in [3] we have extended the algorithm mentioned earlier to include this relation. The result can then be incorporated to generate pictures such as the one below. The blue groups represent matrices which have the same Bourn localisation.



Concluding remarks

Many open problems exist in this project ranging from the theoretical to the concrete, for more information see [7, 3, 4, 8].

References

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[4] M. Hoefnagel and P.-A. Jacqmin. "When a matrix property implies the Mal'tsev property". In: *Cahiers de topologie et géométrie différentielle catégoriques. (Submitted)* (2022).

[5] Z. Janelidze. "Closedness properties of internal relations I: a unified approach to Mal'tsev, unital and subtractive categories". In: *Theory and Applications of Categories* 16 (2006), pp. 236–261.

[6] Z. Janelidze. "Closedness properties of internal relations II: Bourn localization". In: *Theory and Applications of Categories* 16 (2006), pp. 262–282.

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[8] Z. Janelidze M. Hoefnagel P.-A. Jacqmin and E. Van der Walt. "On Binary matrix properties". In: *In preparation* ().