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Lie properties in associative algebras

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A B S T R A C T

For a field K containing $\frac{1}{2}$, we exhibit two matrices in the full $n \times n$ matrix algebra $\mathrm{M}_n(K)$ which generate $\mathrm{M}_n(K)$ as a Lie K-algebra with the commutator Lie product. We also study Lie centralizers of a not necessarily commutative unitary algebra and obtain results which we hope will eventually be a step in the direction of, firstly, proving that, for any field K, a Lie-nilpotent K-subspace (or a Lie K-subalgebra) of a finite-dimensional associative algebra over K of index k (say), generates a Lie-nilpotent associative subalgebra of much higher nilpotency index, and secondly, in consideration of the sharp upper bound for the maximum (K-)dimension of a Lie-nilpotent K-subalgebra of $\mathrm{M}_n(K)$ of index k obtained in [13], finding an upper bound for the maximum dimension of a Lie-nilpotent (of index k) Lie K-subalgebra of $\mathrm{M}_n(K)$. Finally, the constructive elementary proof of the Skolem-Noether theorem for the matrix algebra $\mathrm{M}_n(K)$ in [14], in conjunction with the well-known characterization of Lie automorphisms of $\mathrm{M}_n(K)$ (if the characteristic of K is different from 2 and 3) in terms of, amongst others, automorphisms and anti-automorhisms

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of $\mathrm{M}_n(K)$, leads us to a unifying approach to constructively describe automorphisms and anti-automorphisms of $\mathrm{M}_n(K)$.

1. Introduction and motivation

Throughout the paper an algebra R means a not necessarily commutative unitary algebra over a field K. In many of the results K can be replaced by a commutative unitary ring satisfying certain mild extra conditions. The centralizer of an element $a \in R$ is denoted by $\mathrm{Cen}(a) = \{r \in R : ra = ar\}$, and the center of R is denoted by $\mathrm{Z}(R) = \{r \in R : rs = sr \text{ for all } s \in R\}$. Clearly, $\mathrm{Z}(R) \subseteq \mathrm{Cen}(a)$ are K-subalgebras of R.

We start with the following simple observation.

1.1. Proposition. *If the elements $a_1, a_2, \ldots, a_t \in R$ generate R as an associative algebra, then the intersection of their centralizers is trivial, i.e.,*

$$\mathrm{Cen}(a_1) \cap \mathrm{Cen}(a_2) \cap \cdots \cap \mathrm{Cen}(a_t) = \mathrm{Z}(R).$$

The full $n \times n$ matrix algebra over K is denoted by $\mathrm{M}_n(K)$. The standard matrix unit in $\mathrm{M}_n(K)$ with 1 in the (i, j) position and zeros in all other positions is denoted by $E_{i,j}$, and I_n denotes the $n \times n$ identity matrix.

The fact that $\mathrm{M}_n(K)$ can be generated as a K-algebra by the two matrices $E_{n,1}$ and

$$S := E_{1,2} + E_{2,3} + \cdots + E_{n-1,n},$$

i.e.,

$$\mathrm{M}_n(K) = \langle E_{1,1}, S \rangle_K, \tag{1.1}$$

played a prominent role in [14], in which a constructive elementary proof of the Skolem-Noether theorem (see, e.g., [3], [9] and [12]) for the matrix algebra $\mathrm{M}_n(K)$, with K any field, was given. To be more precise, given a K-automorphism φ of $\mathrm{M}_n(K)$, an invertible matrix $A \in \mathrm{M}_n(K)$ yielding the conjugation,

$$\varphi(X) = AXA^{-1}$$

for all $X \in \mathrm{M}_n(K)$, was constructed from only the two φ-images $\varphi(E_{n,1})$ and $\varphi(S)$ of the matrices $E_{n,1}$ and S, respectively, and a nonzero vector $\mathbf{a}$ in the kernel of the matrix $I_n - \big(\varphi(S)\big)^{n-1}\varphi(E_{n,1}) \in \mathrm{M}_n(K)$, as follows:

$$A = \Big[\big(\varphi(S)\big)^{n-1}\varphi(E_{n,1})\mathbf{a} \big| \big(\varphi(S)\big)^{n-2}\varphi(E_{n,1})\mathbf{a} \big| \cdots \big| \varphi(S)\varphi(E_{n,1})\mathbf{a} \big| \varphi(E_{n,1})\mathbf{a}\Big]. \tag{1.2}$$

A Lie automorphism ψ of a K-algebra R is a one-to-one K-linear map from R onto itself which preserves the commutator Lie product (also called the Lie bracket in the literature), i.e.,

$$\psi([x,y]) = [\psi(x), \psi(y)],$$

equivalently,

$$\psi(xy - yx) = \psi(x)\psi(y) - \psi(y)\psi(x),$$

for all $x, y \in R$. We note that $\mathrm{M}_n(K)$ with the commutator Lie product plays an exceptional role in the theory of finite-dimensional Lie algebras. The fundamental Ado-Iwasava theorem (see [5]) asserts that every finite-dimensional Lie K-algebra can be embedded into $\mathrm{M}_n(K)$ for some $n \geq 1$.

If K is any field of characteristic different from 2 and 3, then (see, e.g., [4], [6], [7] and [8]) every Lie automorphism ψ of $\mathrm{M}_n(K)$ can be presented as a sum

$$\psi = \sigma + \tau, \tag{1.3}$$

where σ is either an automorphism of $\mathrm{M}_n(K)$ (as a K-algebra) or the negative of an anti-automorphism of $\mathrm{M}_n(K)$, and τ is an additive mapping from $\mathrm{M}_n(K)$ to K which maps commutators into zero. In the light of this significant result we apply (1.2) in Section 4, where we present a unifying approach to constructively describe automorphisms and anti-automorphisms of the ring $\mathrm{M}_n(K)$.

First we show in Section 2, in a vein similar to [14], that if $\frac{1}{2} \in K$, then the matrix $E_{1,1}$ and the permutation matrix $P = S + E_{n,1}$ generate $\mathrm{M}_n(K)$ as a Lie K-algebra.

In Section 3 we study Lie centralizers in a (not necessarily commutative) unitary algebra R. We obtain results which we hope will eventually pave the way towards, firstly, proving that a Lie-nilpotent K-subspace (or a Lie K-subalgebra) of a finite-dimensional associative algebra over K of index k (say), for any field K, generates a Lie-nilpotent associative subalgebra (of much higher nilpotency index), and secondly, finding an upper bound (perhaps even a sharp upper bound) for the maximum dimension of a Lie-nilpotent (of index k) Lie K-subalgebra of $\mathrm{M}_n(K)$ (see Conjecture 3.9). In this context the sharp upper bound for the maximum dimension of a Lie-nilpotent K-subalgebra of $\mathrm{M}_n(K)$ of index $k \geq 1$ is important (see [13]).

2. Two matrices generating $\mathrm{M}_n(K)$ as a Lie K-algebra if $\frac{1}{2} \in K$

We shall make use of the well known multiplication rule for standard matrix units:

$$E_{i,j}E_{k,l} = \begin{cases} E_{i,l}, & \text{if } j = k; \\ 0, & \text{if } j \neq k. \end{cases}$$

The permutation matrix $P \in \mathrm{M}_n(K)$ is defined as follows:

$$P = E_{1,2} + E_{2,3} + \cdots + E_{n-1,n} + E_{n,1}.$$

We now show that if $\frac{1}{2} \in K$, then $\mathrm{M}_n(K)$ can be generated as a Lie K-algebra by two matrices.

2.1. Theorem. *If K contains $\frac{1}{2}$, then the matrices P and $E_{1,1}$ generate $\mathrm{M}_n(K)$ as a Lie K-algebra with the commutator Lie product.*

Proof. Let $\mathcal{G} = \langle P, E_{1,1} \rangle_{\mathrm{Lie}}$ denote the Lie K-subalgebra of $\mathrm{M}_n(K)$ generated by the matrices P and $E_{1,1}$. Clearly,

$$E_{1,2} - E_{n,1} = E_{1,1}P - PE_{1,1} = [E_{1,1}, P] \in \mathcal{G}$$

and

$$E_{1,2} + E_{n,1} = E_{1,1}(E_{1,2} - E_{n,1}) - (E_{1,2} - E_{n,1})E_{1,1} = [E_{1,1}, E_{1,2} - E_{n,1}] \in \mathcal{G}$$

ensure that

$$E_{1,2} = \frac{1}{2}\Big((E_{1,2} + E_{n,1}) + (E_{1,2} - E_{n,1})\Big) \in \mathcal{G}$$

and

$$E_{n,1} = \frac{1}{2}\Big((E_{1,2} + E_{n,1}) - (E_{1,2} - E_{n,1})\Big) \in \mathcal{G}.$$

Starting from $E_{1,2} \in \mathcal{G}$, assume that $E_{1,j} \in \mathcal{G}$ for some $2 \leq j \leq n-1$. Using

$$S = E_{1,2} + E_{2,3} + \cdots + E_{n-1,n} = P - E_{n,1} \in \mathcal{G},$$

we obtain that $E_{1,j+1} = [E_{1,j}, S] \in \mathcal{G}$. Therefore, it follows that

$$E_{1,1}, E_{1,2}, E_{1,3}, \ldots, E_{1,n} \in \mathcal{G}.$$

Next, starting from $E_{n,1} \in \mathcal{G}$, assume that $E_{i,1} \in \mathcal{G}$ for some $3 \leq i \leq n$. Now

$$E_{i-1,1} - E_{i,2} = SE_{i,1} - E_{i,1}S = [S, E_{i,1}] \in \mathcal{G}$$

and

$$E_{i,2} = E_{i,1}E_{1,2} - E_{1,2}E_{i,1} = [E_{i,1}, E_{1,2}] \in \mathcal{G}$$

give that

$$E_{i-1,1} = (E_{i-1,1} - E_{i,2}) + E_{i,2} \in \mathcal{G}.$$

Consequently, we have that

$$E_{n,1}, E_{n-1,1}, \dots, E_{2,1}, E_{1,1} \in \mathcal{G}.$$

Finally, if $i \neq j$, then

$$E_{i,j} = E_{i,1}E_{1,j} - E_{1,j}E_{i,1} = [E_{i,1}, E_{1,j}] \in \mathcal{G},$$

and if $i = j$, then

$$E_{i,i} = E_{1,1} + (E_{i,1}E_{1,i} - E_{1,i}E_{i,1}) = E_{1,1} + [E_{i,1}, E_{1,i}] \in \mathcal{G}.$$

Thus we have that $E_{i,j} \in \mathcal{G}$ for all $1 \leq i, j \leq n$, and so $\mathcal{G} = \mathrm{M}_n(K)$. □

2.2. Remark. We note that, in the above theorem, K can be any commutative unitary ring such that $\frac{1}{2} \in K$. Another observation is that the Lie generation of $\mathrm{M}_n(K)$ is much stronger than the associative generation. Indeed, $E_{1,1} = SE_{2,1}$ implies that S and $E_{2,1}$ also generate $\mathrm{M}_n(K)$ as an associative K-algebra. Since S and $E_{2,1}$ have zero traces, it follows that all matrices in $\langle S, E_{2,1}\rangle_{\mathrm{Lie}}$ have zero traces and $\langle S, E_{2,1}\rangle_{\mathrm{Lie}} \neq \mathrm{M}_n(K)$.

3. The Lie centralizer

For a sequence $x_1, x_2, \dots, x_m$ of elements in a not necessarily commutative unitary algebra R over (a field) K we use the notation $[x_1, x_2, \dots, x_m]_m$ for the left normed commutator (or Lie) product:

$$[x_1]_1 = x_1 \text{ and } [x_1, x_2, \dots, x_m]_m = [\dots[[x_1, x_2], x_3], \dots, x_m].$$

The k-th Lie centralizer of a subset $H \subseteq R$ is

$$\mathrm{L}_k(H) = \big\{ r \in R : [r, x_1, \dots, x_k]_{k+1} = 0 \text{ for all } x_i \in H,\ 1 \leq i \leq k \big\},$$

a K-subspace (submodule) of R.

As a consequence of $[rs, x_1] = [r, sx_1] + [s, x_1 r]$, we can see that the containment

$$\{shr : s, r \in R \text{ and } h \in H\} \subseteq H$$

implies that $\mathrm{L}_k(H)$ is a (unitary) K-subalgebra of R. Clearly,

$$\bigcap_{h \in H} \mathrm{Cen}(h) = \mathrm{L}_1(H) \subseteq \mathrm{L}_2(H) \subseteq \dots \subseteq \mathrm{L}_k(H) \subseteq \mathrm{L}_{k+1}(H) \subseteq \cdots$$

follows from

$$[r, x_1, \ldots, x_k, x_{k+1}]_{k+2} = [[r, x_1, \ldots, x_k]_{k+1}, x_{k+1}].$$

The ω-Lie centralizer of $H \subseteq R$ is defined as

$$\mathrm{L}_\omega(H) = \bigcup_{k=1}^{\infty} \mathrm{L}_k(H).$$

A subset $H \subseteq R$ is called Lie-nilpotent of index $k \geq 1$ if $H \subseteq \mathrm{L}_k(H)$. A natural further step is the following: H is called ω-Lie-nilpotent (or almost Lie-nilpotent) if $H \subseteq \mathrm{L}_\omega(H)$.

3.1. Proposition. *If $r \in \mathrm{L}_k(H)$ and $1 \leq j \leq k$, then*

$$[x_1, \ldots, x_j, r, x_{j+1}, \ldots, x_k]_{k+1} = 0$$

for all $x_i \in H$, $1 \leq i \leq k$.

Proof. It is a well known consequence of the Jacobian identity that, in any Lie ring, $[x_1, \ldots, x_j, r]_{j+1}$ can be written as a sum of 2^{j-1} terms of the form

$$\pm[r, x_{\pi(1)}, \ldots, x_{\pi(j)}]_{j+1},$$

where π is some permutation of $\{1, 2, \ldots, j\}$. We note that an easy induction on j works. It follows that $[x_1, \ldots, x_j, r, x_{j+1}, \ldots, x_k]_{k+1}$ can be written as a sum of 2^{j-1} terms of the form

$$\pm[r, x_{\pi(1)}, \ldots, x_{\pi(j)}, x_{j+1}, \ldots, x_k]_{k+1},$$

whence $[x_1, \ldots, x_j, r, x_{j+1}, \ldots, x_k]_{k+1} = 0$ follows. □

3.2. Proposition. *If $\mathrm{L}_k(H) = \mathrm{L}_{k+1}(H)$, then $\mathrm{L}_{k+1}(H) = \mathrm{L}_{k+2}(H)$.*

Proof. For the elements $x_1 \in H$ and $r \in \mathrm{L}_{k+2}(H)$ we have

$$[[r, x_1], x_2, \ldots, x_{k+2}]_{k+3} = [r, x_1, \ldots, x_{k+2}]_{k+3} = 0$$

for all $x_i \in H$, $2 \leq i \leq k+2$. Thus we obtain that $[r, x_1] \in \mathrm{L}_{k+1}(H)$ for all $x_1 \in H$, whence $[r, x_1] \in L_k(H)$ and

$$[r, x_1, \ldots, x_{k+1}]_{k+2} = [[r, x_1], x_2, \ldots, x_k, x_{k+1}]_{k+1} = 0$$

follow for all $x_i \in H$, $1 \leq i \leq k+1$. In view of the above argument, $r \in \mathrm{L}_{k+1}(H)$ and $\mathrm{L}_{k+2}(H) = \mathrm{L}_{k+1}(H)$ can be derived. □

3.3. Proposition. *Let R be a finite-dimensional algebra over a field K with $\dim_K(R) = d$. Then for any subset $H \subseteq R$ we have $\mathrm{L}_\omega(H) = \mathrm{L}_d(H)$.*

Proof. The finite-dimensionality of R implies that

$$\{0\} \subseteq \mathrm{L}_1(H) \subseteq \mathrm{L}_2(H) \subseteq \cdots \subseteq \mathrm{L}_k(H) \subseteq \mathrm{L}_{k+1}(H) \subseteq \cdots$$

cannot be a strictly ascending infinite chain of K-subspaces. In view of Proposition 3.2, the shape of the above chain is

$$\{0\} \subset \mathrm{L}_1(H) \subset \mathrm{L}_2(H) \subset \cdots \subset \mathrm{L}_t(H) = \mathrm{L}_{t+1}(H) = \mathrm{L}_{t+2}(H) = \cdots$$

for some $t \geq 1$ (notice that $1_R \in \mathrm{L}_1(H)$). Now

$$t \leq \dim_K\bigl(\mathrm{L}_t(H)\bigr) \leq \dim_K(R) = d$$

and $\mathrm{L}_\omega(H) = \mathrm{L}_d(H)$ follows. $\square$

3.4. Corollary. *Let R be a finite-dimensional algebra over a field K such that* $\dim_K(R) = d$. *If $H \subseteq R$ is ω-Lie-nilpotent, then H is Lie-nilpotent of index d.*

3.5. Theorem. *For any subset $H \subseteq R$, we have* $\mathrm{L}_p(H)\mathrm{L}_q(H) \subseteq \mathrm{L}_{p+q-1}(H)$ *for all $p, q \geq 1$, and* $\mathrm{L}_\omega(H)$ *is a K-subalgebra of R.*

Proof. Using an induction on $k \geq 1$, we prove that for all $r, s, x_1, \dots, x_k \in R$,

$$[rs, x_1, \dots, x_k]_{k+1} = \sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_t \leq k \\ j_1 < j_2 < \cdots < j_{k-t}}} [r, x_{i_1}, \dots, x_{i_t}]_{t+1} \cdot [s, x_{j_1}, \dots, x_{j_{k-t}}]_{k-t+1}, \qquad *(k)$$

where the sum is taken over all strictly increasing sequences

$$1 \leq i_1 < i_2 < \cdots < i_t \leq k \quad \text{and} \quad 1 \leq j_1 < j_2 < \cdots < j_{k-t} \leq k,$$

with $0 \leq t \leq k$ and

$$\{j_1, j_2, \dots, j_{k-t}\} = \{1, 2, \dots, k\} \setminus \{i_1, i_2, \dots, i_t\}.$$

In the above the empty and the full sequences are allowed with $[r, \varnothing]_{0+1} = r$ and $[s, \varnothing]_{0+1} = s$.

If $k = 1$, then

$$[rs, x_1]_2 = [rs, x_1] = r[s, x_1] + [r, x_1]s = [r, \varnothing]_1 \cdot [s, x_1]_2 + [r, x_1]_2 \cdot [s, \varnothing]_1$$

is well known.

Assume that $*(k)$ holds for some $k \geq 1$. We use

$$[ab, x_{k+1}] = a[b, x_{k+1}] + [a, x_{k+1}]b$$

repeatedly in the following calculations:

$$\begin{aligned}
&[rs, x_1, \ldots, x_k, x_{k+1}]_{k+2} = [[rs, x_1, \ldots, x_k]_{k+1}, x_{k+1}] \\
&= \left[\left(\sum_{\substack{1 \le i_1 < i_2 < \cdots < i_t \le k \\ j_1 < j_2 < \cdots < j_{k-t}}} [r, x_{i_1}, \ldots, x_{i_t}]_{t+1} \cdot [s, x_{j_1}, \ldots, x_{j_{k-t}}]_{k-t+1} \right), x_{k+1} \right] \\
&= \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_t \le k \\ j_1 < j_2 < \cdots < j_{k-t}}} \big[[r, x_{i_1}, \ldots, x_{i_t}]_{t+1} \cdot [s, x_{j_1}, \ldots, x_{j_{k-t}}]_{k-t+1}, x_{k+1} \big] \\
&= \left(\sum_{\substack{1 \le i_1 < i_2 < \cdots < i_t \le k \\ j_1 < j_2 < \cdots < j_{k-t}}} [r, x_{i_1}, \ldots, x_{i_t}]_{t+1} \cdot [[s, x_{j_1}, \ldots, x_{j_{k-t}}]_{k-t+1}, x_{k+1}] \right) \\
&\quad + \left(\sum_{\substack{1 \le i_1 < i_2 < \cdots < i_t \le k \\ j_1 < j_2 < \cdots < j_{k-t}}} [[r, x_{i_1}, \ldots, x_{i_t}]_{t+1}, x_{k+1}] \cdot [s, x_{j_1}, \ldots, x_{j_{k-t}}]_{k-t+1} \right) \\
&= \left(\sum_{\substack{1 \le i_1 < i_2 < \cdots < i_t \le k \\ j_1 < j_2 < \cdots < j_{k-t}}} [r, x_{i_1}, \ldots, x_{i_t}]_{t+1} \cdot \big[s, x_{j_1}, \ldots, x_{j_{k-t}}, x_{k+1}\big]_{k-t+2} \right) \\
&\quad + \left(\sum_{\substack{1 \le i_1 < i_2 < \cdots < i_t \le k \\ j_1 < j_2 < \cdots < j_{k-t}}} [r, x_{i_1}, \ldots, x_{i_t}, x_{k+1}]_{t+2} \cdot [s, x_{j_1}, \ldots, x_{j_{k-t}}]_{k-t+1} \right) \\
&= \sum_{\substack{1 \le i'_1 < i'_2 < \cdots < i'_m \le k+1 \\ j'_1 < j'_2 < \cdots < j'_{(k+1)-m}}} [r, x_{i'_1}, \ldots, x_{i'_m}]_{m+1} \cdot [s, x_{j'_1}, \ldots, x_{j'_{(k+1)-m}}]_{k-m+2}. \qquad *(k+1)
\end{aligned}$$

The last equality is a consequence of the fact that a strictly increasing sequence $1 \le i'_1 < i'_2 < \cdots < i'_m \le k+1$ can appear either as

$$1 \le i'_1 = i_1 < i'_2 = i_2 < \cdots < i'_m = i_t \le k \text{ (with } m = t)$$

or as

$$1 \le i'_1 = i_1 < i'_2 = i_2 < \cdots < i'_{m-1} = i_t < i'_m = k+1 \text{ (with } m = t+1).$$

If $r \in \mathrm{L}_p(H)$, $s \in \mathrm{L}_q(H)$, $x_1, \ldots, x_{p+q-1} \in H$ and $0 \le t \le p+q-1$, then either $p \le t$ or $q \le (p+q-1)-t$, and each summand in

$$[rs, x_1, \dots, x_{p+q-1}]_{p+q} = \sum_{\substack{1 \le i_1 < i_2 < \dots < i_t \le p+q-1 \\ j_1 < j_2 < \dots < j_{k-t}}} [r, x_{i_1}, \dots, x_{i_t}]_{t+1} \cdot [s, x_{j_1}, \dots, x_{j_{(p+q-1)-t}}]_{p+q-t}$$

is zero. Indeed, if $p \le t$, then $r \in \mathrm{L}_p(H)$ implies that $[r, x_{i_1}, \dots, x_{i_t}]_{t+1} = 0$, and if $q \le (p+q-1) - t$, then $s \in \mathrm{L}_q(H)$ implies that

$$[s, x_{j_1}, \dots, x_{j_{(p+q-1)-t}}]_{p+q-t} = 0.$$

It follows that $rs \in \mathrm{L}_{p+q-1}(H)$.

Since $\mathrm{L}_{p+q-1}(H) \subseteq \mathrm{L}_\omega(H)$, we derive that $\mathrm{L}_\omega(H)$ is a K-subalgebra of R. □

3.6. Remark. A property $\mathcal{P}$ which is defined for any finite sequence $x_1, \dots, x_m$ of elements in R is called hereditary if $\mathcal{P}$ holds for any subsequence $x_{i_1}, \dots, x_{i_t}$ with $1 \le i_1 < i_2 < \dots < i_t \le m$. Two typical examples are $\mathcal{D}$ and $\mathcal{L}$. For a sequence $x_1, \dots, x_m \in R$ the meaning of $\mathcal{D}$ is that the elements $x_1, \dots, x_m$ are distinct and the meaning of $\mathcal{L}$ is that the elements $x_1, \dots, x_m$ are linearly independent over the base field K.

The k-th Lie centralizer of a subset $H \subseteq R$ with respect to the property $\mathcal{P}$ is

$$\mathrm{L}_k^{\mathcal{P}}(H) = \{r \in R \colon [r, x_1, \dots, x_k]_{k+1} = 0 \text{ for all } x_1, \dots, x_k \in H \text{ having property } \mathcal{P}\}.$$

Using the same calculations as in the above proof, the following interesting (and probably far reaching) generalization of Theorem 3.5 can be obtained: *If $\mathcal{P}$ is a hereditary property, then for any subset $H \subseteq R$, we have $\mathrm{L}_p^{\mathcal{P}}(H)\mathrm{L}_q^{\mathcal{P}}(H) \subseteq \mathrm{L}_{p+q-1}^{\mathcal{P}}(H)$ (and the union $\mathrm{L}_\omega^{\mathcal{P}}(H) = \cup_{k=1}^{\infty} \mathrm{L}_k^{\mathcal{P}}(H)$ is a K-subalgebra of R).*

3.7. Remark. Unfortunately we were not able to prove the following:

Let R be a finite-dimensional algebra over a field K with $\dim_K(R) = d$. If $V \subseteq R$ is a Lie-nilpotent K-subspace (or a Lie K-subalgebra) of index $k \ge 1$, then the associative K-subalgebra $\langle V \rangle_K$ of R generated by V is Lie-nilpotent of index $f(k, d)$.

The main result in [13] states that if K is any field and R is any Lie-nilpotent K-subalgebra of $\mathrm{M}_n(K)$ of index $k \ge 1$, then

$$\dim_K(R) \le g(k+1, n),$$

where $g(k+1, n)$ is the maximum of

$$\frac{1}{2}\left(n^2 - \sum_{i=1}^{k+1} n_i^2\right) + 1,$$

subject to the constraint $\sum_{i=1}^{k+1} n_i = n$, with $n_1, n_2, \dots, n_{k+1}$ non-negative integers. To be precise:

3.8. Theorem *(see [13]). If R is a Lie-nilpotent K-subalgebra of $\mathrm{M}_n(K)$ of index $k \geq 1$, with (according to the Division Algorithm)*

$$n = (k+1)\left\lfloor \frac{n}{k+1} \right\rfloor + r, \quad 0 \leq r < k+1,$$

then

$$\frac{1}{2}\left(n^2 - (k+1-r)\left\lfloor \frac{n}{k+1} \right\rfloor^2 - r\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)^2\right) + 1$$

is a sharp upper bound for $\dim_K(R)$.

Using Theorem 3.8 and the statement formulated in Remark 3.7 we had hoped that we would be able to obtain an upper bound for the maximum dimension of a Lie nilpotent (of index k) Lie K-subalgebra of the full matrix algebra $\mathrm{M}_n(K)$. To be more precise, we had been hopeful that the foregoing results would lead to a proof of the following conjecture, but unfortunately we fell short:

3.9. Conjecture. *If $\mathcal{L} \subseteq \mathrm{M}_n(K)$ is an ω-Lie-nilpotent Lie K-subalgebra, then $\dim_K(\mathcal{L}) \leq 1 + \frac{1}{2}(n^2 - n)$.*

4. A unifying approach to constructively describe automorphisms and anti-automorphisms of matrix algebras

Evidence in the literature is abundant for the importance of automorphisms and anti-automorphisms of a matrix ring $\mathrm{M}_n(K)$ over a field K. We apply (1.2) in this section by presenting a unifying approach to constructively describe automorphisms and anti-automorphisms of $\mathrm{M}_n(K)$.

In particular, first consider the following setting: for an automorphism f of a field K, i.e., for $f \in \mathrm{Aut}(K)$, and for any $X \in \mathrm{M}_n(K)$, let X_f denote the matrix obtained from X by applying f entrywise, i.e., $X_f = [x_{i,j}]_f = [f(x_{i,j})]$, and let B be any invertible matrix in $\mathrm{M}_n(K)$. Then the function $\beta : \mathrm{M}_n(K) \to \mathrm{M}_n(K)$, defined by

$$\beta(X) = BX_fB^{-1}$$

for all $X \in \mathrm{M}_n(K)$, is a ring automorphism of $\mathrm{M}_n(K)$, but it needs not be a K-automorphism of $\mathrm{M}_n(K)$. In fact, it is easily verified that β is a K-automorphism of $\mathrm{M}_n(K)$ if and only if f is the identity automorphism of K. Nevertheless, we obtain the following constructive description in the above vein (see also [11, Corollary 1.2]):

4.1. Proposition. *Let $f \in \mathrm{Aut}(K)$ (K any field), let B be any invertible matrix in $\mathrm{M}_n(K)$, and let $\beta : \mathrm{M}_n(K) \to \mathrm{M}_n(K)$ be the function defined by*

$$\beta(X) = BX_f B^{-1}$$

for all $X \in \mathrm{M}_n(K)$*. Then*

$$\beta(X) = \overline{B} X_f \overline{B}^{-1}$$

for all $X \in \mathrm{M}_n(K)$*, where* $\overline{B} \in \mathrm{M}_n(K)$ *is the invertible matrix*

$$\overline{B} = \left[\big(\beta(S)\big)^{n-1} \beta(E_{n,1})\mathbf{b} \big| \big(\beta(S)\big)^{n-2} \beta(E_{n,1})\mathbf{b} \big| \cdots \big| \beta(S)\beta(E_{n,1})\mathbf{b} \big| \beta(E_{n,1})\mathbf{b} \right],$$

and $\mathbf{b}$ *is a nonzero vector in the kernel of* $I_n - \big(\beta(S)\big)^{n-1}\beta(E_{n,1}) \in \mathrm{M}_n(K)$*.*

Proof. Since

$$\beta(X_{f^{-1}}) = BXB^{-1}$$

for all $X \in \mathrm{M}_n(K)$, it follows that $\alpha : \mathrm{M}_n(K) \to \mathrm{M}_n(K)$, defined by

$$\alpha(X) = \beta(X_{f^{-1}})$$

for all $X \in \mathrm{M}_n(K)$, is indeed a K-automorphism of $\mathrm{M}_n(K)$. Hence, by (1.2), we can constructively find an invertible matrix $\overline{B}$ (say) in $\mathrm{M}_n(K)$ such that

$$\alpha(X) = \overline{B} X \overline{B}^{-1}$$

for all $X \in \mathrm{M}_n(K)$, where $\overline{B} \in \mathrm{M}_n(K)$ is the invertible matrix

$$\overline{B} = \left[\big(\alpha(S)\big)^{n-1} \alpha(E_{n,1})\mathbf{b} \big| \big(\alpha(S)\big)^{n-2} \alpha(E_{n,1})\mathbf{b} \big| \cdots \big| \alpha(S)\alpha(E_{n,1})\mathbf{b} \big| \alpha(E_{n,1})\mathbf{b} \right],$$

with $\mathbf{b}$ a nonzero vector in the kernel of the matrix $I_n - \big(\alpha(S)\big)^{n-1}\alpha(E_{n,1})$ in $\mathrm{M}_n(K)$. Since $\alpha(S) = \beta(S_{f^{-1}})$ and $\alpha(E_{n,1}) = \beta\big((E_{n,1})_{f^{-1}}\big)$, and since every entry of S and $E_{n,1}$ is 0 or 1, with $f \in \mathrm{Aut}(K)$, we have that $\alpha(S) = \beta(S)$ and $\alpha(E_{n,1}) = \beta(E_{n,1})$. Therefore,

$$\beta(X) = \beta\big((X_f)_{f^{-1}}\big) = \alpha(X_f) = \overline{B} X_f \overline{B}^{-1}$$

for all $X \in \mathrm{M}_n(K)$, where

$$\overline{B} = \left[\big(\beta(S)\big)^{n-1} \beta(E_{n,1})\mathbf{b} \big| \big(\beta(S)\big)^{n-2} \beta(E_{n,1})\mathbf{b} \big| \cdots \big| \beta(S)\beta(E_{n,1})\mathbf{b} \big| \beta(E_{n,1})\mathbf{b} \right],$$

with $\mathbf{b}$ a nonzero vector in the kernel of $I_n - \big(\beta(S)\big)^{n-1}\beta(E_{n,1}) \in \mathrm{M}_n(K)$. $\square$

For our purposes we state explicitly a result from [11], using our notation:

4.2. Corollary *([11, Corollary 1.2]). Let K be an arbitrary field, and let $\phi : \mathrm{M}_n(K) \to \mathrm{M}_n(K)$ be a bijective additive function satisfying $\phi(XY) = \phi(X)\phi(Y)$ for all $X, Y \in \mathrm{M}_n(K)$. Then there exists an automorphism f of the field K and an invertible matrix $A \in \mathrm{M}_n(K)$ such that*

$$\phi(X) = AX_f A^{-1}$$

for all $X \in \mathrm{M}_n(K)$.

Next, combining Proposition 4.1 and Corollary 4.2, and denoting the transpose of a matrix $X \in \mathrm{M}_n(K)$ by $X^\top$, we also obtain the following constructive and explicit description of an invertible matrix yielding any anti-automorphism of $\mathrm{M}_n(K)$. In this regard it is noteworthy that, just as S and $E_{n,1}$ generate $\mathrm{M}_n(K)$ as a K-algebra, so do their transposes $S^\top$ and $E_{1,n}$, respectively.

4.3. Theorem. *If ϕ is a ring anti-automorphism of $\mathrm{M}_n(K)$, then*

$$\phi(X) = \overline{A} X_f^\top \overline{A}^{-1}$$

for all $X \in \mathrm{M}_n(K)$, where $\overline{A} \in \mathrm{M}_n(K)$ is the invertible matrix

$$\overline{A} = \Big[\big(\phi(S^\top)\big)^{n-1} \phi(E_{1,n})\mathbf{a} \big| \big(\phi(S^\top)\big)^{n-2} \phi(E_{1,n})\mathbf{a} \big| \cdots \big| \phi(S^\top)\phi(E_{1,n})\mathbf{a} \big| \phi(E_{1,n})\mathbf{a} \Big],$$

with $\mathbf{a}$ a nonzero vector in the kernel of $I_n - \big(\phi(S^\top)\big)^{n-1}\phi(E_{1,n}) \in \mathrm{M}_n(K)$.

Proof. Let $\mathcal{T}$ denote the transposition map $X \mapsto X^\top$ on $\mathrm{M}_n(K)$. Since $\mathcal{T}$ is also a ring anti-automorphism of $\mathrm{M}_n(K)$, the composition $\phi \circ \mathcal{T}$ is a ring automorphism of $\mathrm{M}_n(K)$, and so by Corollary 4.2, there is an automorphism f of K and an invertible matrix $A \in \mathrm{M}_n(K)$ such that

$$(\phi \circ \mathcal{T})(X) = AX_f A^{-1}$$

for all $X \in \mathrm{M}_n(K)$. Hence, by Proposition 4.1,

$$(\phi \circ \mathcal{T})(X) = \overline{A} X_f \overline{A}^{-1}$$

for all $X \in \mathrm{M}_n(K)$, where

$$\begin{aligned} \overline{A} &= \Big[\big((\phi \circ \mathcal{T})(S)\big)^{n-1} \big((\phi \circ \mathcal{T})(E_{n,1})\big)\mathbf{a} \big| \big((\phi \circ \mathcal{T})(S)\big)^{n-2} \big((\phi \circ \mathcal{T})(E_{n,1})\big)\mathbf{a} \big| \cdots \\ &\qquad \cdots \big| \big((\phi \circ \mathcal{T})(S)\phi(E_{n,1})\big)\mathbf{a} \big| \big((\phi \circ \mathcal{T})(E_{n,1})\big)\mathbf{a} \Big] \\ &= \Big[\big(\phi(S^\top)\big)^{n-1} \phi(E_{1,n})\mathbf{a} \big| \big(\phi(S^\top)\big)^{n-2} \phi(E_{1,n})\mathbf{a} \big| \cdots \end{aligned}$$

$$\cdots|\phi(S^\top)\phi(E_{1,n})\mathbf{a}|\phi(E_{1,n})\mathbf{a}\Big].$$

(Here $S^\top = E_{2,1} + E_{3,2} + \cdots + E_{n,n-1}$, and $\mathbf{a}$ is a nonzero vector in the kernel of $I_n - \big(\phi(S^\top)\big)^{n-1}\phi(E_{1,n})$.)

In particular,

$$(\phi \circ \mathcal{T})(X^\top) = AX_f^\top A^{-1},$$

i.e.,

$$\phi(X) = AX_f^\top A^{-1}$$

for all $X \in \mathrm{M}_n(K)$. $\square$

We illustrate the construction of $\bar{A}$ in Theorem 4.3 with the (canonical) symplectic involution as a special case of an anti-automorphism ϕ.

4.4. Example. Consider the symplectic involution ϕ on $\mathrm{M}_8(K)$ (see, e.g., [1] or [10]), i.e., ϕ is the anti-automorphism of $\mathrm{M}_8(K)$ defined by

$$\phi\left(\left[\begin{array}{c|c} U & P \\ \hline Q & V \end{array}\right]\right) = \left[\begin{array}{c|c} V^\top & -P^\top \\ \hline -Q^\top & U^\top \end{array}\right]$$

for all $U, V, P, Q \in \mathrm{M}_4(K)$. In order to construct $\overline{A}$ above, we need certain powers of $\phi(S^\top)$. Since $S^\top = E_{2,1} + E_{3,2} + \cdots + E_{8,7}$, we have

$$\begin{aligned}
(S^\top)^2 &= E_{3,1} + E_{4,2} + E_{5,3} + E_{6,4} + E_{7,5} + E_{8,6},\\
(S^\top)^3 &= E_{4,1} + E_{5,2} + E_{6,3} + E_{7,4} + E_{8,5},\\
(S^\top)^4 &= E_{5,1} + E_{6,2} + E_{7,3} + E_{8,4},\\
(S^\top)^5 &= E_{6,1} + E_{7,2} + E_{8,3},\\
(S^\top)^6 &= E_{7,1} + E_{8,2},\\
(S^\top)^7 &= E_{8,1},
\end{aligned}$$

and so

$$\begin{aligned}
\phi(S^\top) &= E_{1,2} + E_{2,3} + E_{3,4} + E_{5,6} + E_{6,7} + E_{7,8} - E_{8,1},\\
\big(\phi(S^\top)\big)^2 &= E_{1,3} + E_{2,4} + E_{5,7} + E_{6,8} - E_{7,1} - E_{8,2},\\
\big(\phi(S^\top)\big)^3 &= E_{1,4} + E_{5,8} - E_{6,1} - E_{7,2} - E_{8,3},\\
\big(\phi(S^\top)\big)^4 &= -E_{5,1} - E_{6,2} - E_{7,3} - E_{8,4},
\end{aligned}$$

$$\big(\phi(S^\top)\big)^5 = -E_{5,2} - E_{6,3} - E_{7,4},$$
$$\big(\phi(S^\top)\big)^6 = -E_{5,3} - E_{6,4},$$
$$\big(\phi(S^\top)\big)^7 = -E_{5,4}.$$

Hence, $\phi(S^\top)\big)^7\phi(E_{1,8}) = (-E_{5,4})(-E_{4,5}) = E_{5,5}$. Consequently, $\mathbf{a} := e_5$ is a nonzero vector in the kernel of

$$I_8 - \phi(S^\top)\big)^7\phi(E_{1,8}) = E_{1,1} + E_{2,2} + E_{3,3} + E_{4,4} + E_{6,6} + E_{7,7} + E_{8,8}.$$

(Here e_j denotes the 8×1 column vector with 1 in position j, and with 0 elsewhere.) Therefore, since $\phi(E_{1,8})\mathbf{a} = -E_{4,5}e_5 = -e_4$, the foregoing presentations of $\big(\phi(S^\top)\big)^i$, $i = 1, 2, \dots, 7$, together with the construction of $\overline{A}$ in Theorem 4.3, yield

$$\begin{aligned}\overline{A} &= \Big[\big(\phi(S^\top)\big)^7\phi(E_{1,8})\mathbf{a}\big|\big(\phi(S^\top)\big)^6\phi(E_{1,8})\mathbf{a}\big|\cdots\big|\phi(S^\top)\phi(E_{1,8})\mathbf{a}\big|\phi(E_{1,8})\mathbf{a}\Big] \\ &= \left[\begin{array}{c|c} 0 & -I_4 \\ \hline I_4 & 0 \end{array}\right],\end{aligned}$$

the latter being the negative of the matrix y on the last line of the first page of [10]. (Of course, $\overline{A}X^\top\overline{A}^{-1} = (\lambda\overline{A})X^\top(\lambda\overline{A})^{-1}$ for every $0 \neq \lambda \in K$.) This concludes the example.

Next, consider the setting following (1.2), with the only difference that the function $\beta : \mathrm{M}_n(K) \to \mathrm{M}_n(K)$ is defined by

$$\beta(X) = BX_f^\top B^{-1}$$

for all $X \in \mathrm{M}_n(K)$ (instead of $\beta(X) = BX_fB^{-1}$). Then, as before, β is a ring anti-automorphism of $\mathrm{M}_n(K)$, but it needs not be a K-anti-automorphism of $\mathrm{M}_n(K)$. In this case we have the following result:

4.5. Corollary. *Let $f \in \mathrm{Aut}(K)$ (K any field), let B be any invertible matrix in $\mathrm{M}_n(K)$, and let $\beta : \mathrm{M}_n(K) \to \mathrm{M}_n(K)$ be the function defined by*

$$\beta(X) = BX_f^\top B^{-1}$$

for all $X \in \mathrm{M}_n(K)$. Then

$$\beta(X) = \overline{B}X_f^\top\overline{B}^{-1}$$

for all $X \in \mathrm{M}_n(K)$, where $\overline{B} \in \mathrm{M}_n(K)$ is the invertible matrix

$$\overline{B} = \Big[\big(\beta(S^\top)\big)^{n-1}\beta(E_{1,n})\mathbf{b}\big|\big(\beta(S^\top)\big)^{n-2}\beta(E_{1,n})\mathbf{b}\big|\cdots\big|\beta(S^\top)\beta(E_{1,n})\mathbf{b}\big|\beta(E_{1,n})\mathbf{b}\Big]$$

and $\mathbf{b}$ *is a nonzero vector in the kernel of* $I_n - \big(\beta(S^\top)\big)^{n-1}\beta(E_{1,n}) \in \mathrm{M}_n(K)$.

Proof. Since

$$\beta(X_{f^{-1}}) = BX^\top B^{-1}$$

for all $X \in \mathrm{M}_n(K)$, it follows that $\alpha : \mathrm{M}_n(K) \to \mathrm{M}_n(K)$, defined by

$$\alpha(X) = \beta(X_{f^{-1}})$$

for all $X \in \mathrm{M}_n(K)$, is a K-anti-automorphism of $\mathrm{M}_n(K)$. Hence, by Theorem 4.3, we can constructively find an invertible matrix $\overline{B}$ (say) in $\mathrm{M}_n(K)$ such that

$$\alpha(X) = \overline{B}X^\top\overline{B}^{-1}$$

for all $X \in \mathrm{M}_n(K)$, where $\overline{B} \in \mathrm{M}_n(K)$ is the invertible matrix

$$\overline{B}=\Big[\big(\alpha(S^\top)\big)^{n-1}\alpha(E_{1,n})\mathbf{b}\big|\big(\alpha(S^\top)\big)^{n-2}\alpha(E_{1,n})\mathbf{b}\big|\cdots\big|\alpha(S^\top)\alpha(E_{1,n})\mathbf{b}\big|\alpha(E_{1,n})\mathbf{b}\Big]$$

and $\mathbf{b}$ is a nonzero vector in the kernel of the matrix $I_n - \big(\alpha(S^\top)\big)^{n-1}\alpha(E_{1,n})$ in $\mathrm{M}_n(K)$. We have $\alpha(S^\top) = \beta(S^\top_{f^{-1}})$ and $\alpha(E_{1,n}) = \beta\big((E_{n,1})_{f^{-1}}\big)$, and so, since every entry of both $S^\top$ and $E_{1,n}$ is 0 or 1, and since $f^{-1} \in \mathrm{Aut}(K)$, we have that $\alpha(S^\top) = \beta(S^\top)$ and $\alpha(E_{1,n}) = \beta(E_{1,n})$. Therefore,

$$\beta(X) = \beta\big((X_f)_{f^{-1}}\big) = \alpha(X_f) = \overline{B}X_f^\top\overline{B}^{-1}$$

for all $X \in \mathrm{M}_n(K)$, where

$$\overline{B} = \Big[\big(\beta(S^\top)\big)^{n-1}\beta(E_{1,n})\mathbf{b}\big|\big(\beta(S^\top)\big)^{n-2}\beta(E_{1,n})\mathbf{b}\big|\cdots\big|\beta(S^\top)\beta(E_{1,n})\mathbf{b}\big|\beta(E_{1,n})\mathbf{b}\Big],$$

with $\mathbf{b}$ a nonzero vector in the kernel of $I_n - \big(\beta(S^\top)\big)^{n-1}\beta(E_{1,n}) \in \mathrm{M}_n(K)$. $\square$

Consider again (1.3). By Proposition 4.1, Corollary 4.2 and Theorem 4.3 we have an exact description of σ in (1.3) in the terms of the images of generators of $\mathrm{M}_n(K)$. Regarding τ, recall that it is an additive mapping from $\mathrm{M}_n(K)$ to K which maps commutators into zero. With $\mathrm{tr}(X)$ denoting the trace of a matrix X in $\mathrm{M}_n(K)$, we have the following:

4.6. Proposition. *Let* τ *be as in (1.3), and let* $X = \sum_{i,j=1}^n k_{ij}E_{i,j} \in \mathrm{M}_n(K)$. *Then* $\tau(X) = \tau\big(\mathrm{tr}(X)\cdot E_{1,1}\big)$.

Proof. If $i \neq j$, then $[k_{ij}E_{i,j}, E_{j,j}] = k_{ij}E_{i,j}$, and so, since τ maps commutators to zero, we have $\tau(k_{ij}E_{i,j}) = 0$. Hence, $\tau(X) = \tau\big(\sum_{i=1}^{n} k_{ii}E_{i,i}\big)$. Note also that $k_{ii}E_{i,i} = [k_{ii}E_{i,1}, E_{1,i}] + k_{ii}E_{1,1}$ for every i, and so $\tau(k_{ii}E_{i,i}) = \tau(k_{ii}E_{1,1})$. Consequently,

$$\tau(X) = \tau\left(\left(\sum_{i=1}^{n} k_{ii}\right)E_{1,1}\right) = \tau\big(\mathrm{tr}(X) \cdot E_{1,1}\big). \quad \square$$

Unfortunately, we do not seem to be able to describe $\tau\big(\mathrm{tr}(X) \cdot E_{1,1}\big)$ any better. In general, if ψ in (1.3) is not a Lie K-automorphism, then we do not necessarily have $\tau(\mathrm{tr}(X) \cdot E_{1,1}) = \mathrm{tr}(X)\tau(E_{1,1})$.

The following result by Dolinar et al. should be mentioned here:

4.7. Theorem *(see [2]). Let K be a field, and let $\psi : \mathrm{M}_n(K) \to \mathrm{M}_n(K)$ be a bijective map which preserves the commutator Lie product. Then there is an invertible matrix $T \in \mathrm{M}_n(K)$, a field authomorphism f of K, and a function $\tau : \mathrm{M}_n(K) \to K$, where $\tau(X) = 0$ for all matrices of trace zero such that:*

(i) *for $n \geq 3$ and K with a least 2^{n-1} elements, either*

$$\psi(X) = TX^fT^{-1} + \tau(X)I \text{ for all } X \in \mathrm{M}_n(K),$$

or

$$\psi(X) = -T(X^f)^\top T^{-1} + \tau(X)I \text{ for all } X \in \mathrm{M}_n(K);$$

(ii) *for $n = 2$ and $\mathrm{char}K \neq 2$,*

$$\psi(X) = TX^fT^{-1} + \tau(X)I \text{ for all } X \in \mathrm{M}_n(K).$$

Considering Theorem 4.7, we note that if we consider the functions

$$\sigma_1, \sigma_2 : \mathrm{M}_n(K) \to \mathrm{M}_n(K),$$

defined by

$$\sigma_1(X) = TX^fT^{-1} \qquad \text{and} \qquad \sigma_2(X) = -T(X^f)^\top T^{-1}$$

for all $X \in \mathrm{M}_n(K)$, then by the foregoing constructions and considerations, σ_1 is an automorphism of $\mathrm{M}_n(K)$ and σ_2 is the negative of an anti-automorphism of $M_n(K)$ (in both cases as rings), and as before, we have exact descriptions of them in the terms of generators of $\mathrm{M}_n(K)$. However, we know nothing more about τ.

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Declaration of competing interest

There is no competing interest.

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