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THE LATTICE OF IDEALS OF $M_R(R^2)$, R A COMMUTATIVE PIR

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Abstract

In this paper we characterize the ideals of the centralizer near-ring $N = M_R(R^2)$, where R is a commutative principle ideal ring. The characterization is used to determine the radicals $J_\nu(N)$ and the quotient structures $N/J_\nu(N)$, $\nu = 0, 1, 2$.

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1. Introduction

Let R be a ring with identity and let G be a unitary (right) R-module. Then $M_R(G) = \{f\colon G \to G \mid f(ar) = f(a)\cdot r,\ a \in G, r \in R\}$ is a near-ring under function addition and composition, called the *centralizer near-ring determined by the pair* (R, G). When G is the free R-module on a finite number of (say n) generators, then $M_R(R^n)$ contains the ring $\mathscr{M}_n(R)$ of $n \times n$ matrices over R, and in this case the known structure of $\mathscr{M}_n(R)$ can be used to obtain structural results for $M_R(R^n)$. An investigation of these relationships was initiated in [5]. (As in [5] we restrict our attention to the case $n = 2$, which shows all the salient features, for ease of exposition.)

When R is an integral domain, it was shown in [5] that $M_R(R^2)$ is a simple near-ring. Moreover, when R is a principal ideal domain, there is a lattice isomorphism between the ideals of R and the lattice of two-sided invariant subgroups of $M_R(R^2)$. In this work we turn to the case in which R is a commutative principal ideal ring and investigate the lattice of ideals of $M_R(R^2)$. Here the situation is quite different from that of the principal ideal domain.

Let R be a commutative principal ideal ring with identity. It is well-known ([1], [8]) that R is the direct sum of principal ideal domains (PID) and special principal ideal rings (PIR). A special PIR is a principal ideal ring which has a unique prime ideal and this ideal is nilpotent. Thus a special PIR is a local ring with nilpotent radical $J = \langle\theta\rangle$ (the principal ideal generated by θ). If m is the index of nilpotency of $\langle\theta\rangle$, then every non-zero element in a special PIR, R, can be written in the form $a\theta^l$ where a is a unit in R, $0 \leq l < m$, l is unique and a is unique modulo θ^{m-l}. Furthermore every ideal of R is of the form $\langle\theta^j\rangle$, $0 \leq j \leq m$. We mention that special PIR's are chain rings. (See [3] and the references there for information and examples of finite chain rings.)

Our work also has geometric connections. Specifically, let R be a principal ideal ring and let $\mathscr{C}$ be a cover (see [2]) of R^2 by cyclic submodules. Then for each $f \in M_R(R^2)$ and each $\mathscr{C}_\alpha \in \mathscr{C}$, there exists $\mathscr{C}_\beta \in \mathscr{C}$ such that $f(\mathscr{C}_\alpha) \subseteq \mathscr{C}_\beta$. Hence $M_R(R^2)$ is a set of operators for the geometry $\langle R^2, \mathscr{C}\rangle$ and we obtain a *generalized translation space with operators* as investigated in [4].

Throughout the remainder of this paper all rings R will be commutative principal ideal rings, unless specified to the contrary, with identity and all R-modules will be unitary. We let $N = M_R(R^2)$ denote the centralizer near-ring and all near-rings will be right near-rings. For details about near-rings we refer the reader to the books by Meldrum [6] or Pilz [7]. Also, for any set S, let $S^* = S\backslash\{0\}$.

The objective of this investigation is to determine the ideals of $N = M_R(R^2)$. After developing some general results in the next section we establish the characterization of the ideals of N in Section 3. As mentioned above, the situation here differs from the PID situation. In fact, we find for a special PIR, R, a very nice bijection between the ideals of R and the ideals of $M_R(R^2)$. In the final section we use our results to determine the radicals $J_\nu(N)$, $\nu = 0, 1, 2$, and we find the quotient structure $N/J_\nu(N)$.

2. General results

We start out with an arbitrary (not necessarily commutative principal ideal) ring S with identity and suppose $S = S_1 \oplus \cdots \oplus S_t$ is the direct sum of the ideals $S_1, S_2, \ldots, S_t$. Then $1 = e_1 + e_2 + \cdots + e_t$ where $\{e_i\}$ is a set of orthogonal idempotents, e_i the identity of S_i. Note further that $S^2 = S_1^2 \oplus \cdots \oplus S_t^2$, and let $\binom{x}{y} \in S^2$, $\binom{x}{y} = \binom{x_1}{y_1} + \cdots + \binom{x_t}{y_t}$, $\binom{x_i}{y_i} \in S_i^2$. For $f \in M_S(S^2)$, $f\binom{x}{y} = f\left(\binom{x_1}{y_1} + \cdots + \binom{x_t}{y_t}\right) = \binom{a_1}{b_1} + \cdots + \binom{a_t}{b_t}$, $\binom{a_i}{b_i} \in S_i^2$. But $f\binom{x}{y}e_i = f\left(\binom{x}{y}e_i\right)$ implies $f\binom{x_i}{y_i} = \binom{a_i}{b_i}$, so we obtain $f\binom{x}{y} = f\binom{x_1}{y_1} + \cdots + f\binom{x_t}{y_t}$ and $f(S_i^2) \subseteq S_i^2$.

If $M_i = M_S(S_i^2)$, then $\varphi\colon M \to M_1 \oplus \cdots \oplus M_t$ defined by $\varphi(f) = (f_1, \ldots, f_t)$, where $f_i = f|S_i^2$, is a near-ring homomorphism. Moreover, φ is onto. For, if $(g_1, \ldots, g_t) \in M_1 \oplus \cdots \oplus M_t$, define $g\colon S^2 \to S^2$ by $g\binom{x}{y} = g_1\binom{x_1}{y_1} + \cdots + g_t\binom{x_t}{y_t}$, where $\binom{x}{y} = \binom{x_1}{y_1} + \cdots + \binom{x_t}{y_t}$. Then $g \in M$ and $\varphi(g) = (g_1, \ldots, g_t)$. Next, suppose $f \in M$ and $\varphi(f) = 0$. This means that $f|S_i^2 = 0$, $i = 1, 2, \ldots, t$, so $f \equiv 0$, and hence φ is an isomorphism.

Since $S_i \subseteq S$, we have $M_S(S_i^2) \subseteq M_{S_i}(S_i^2)$. On the other hand, for $s \in S$, $s = s_1 + \cdots + s_t$, $s_i \in S_i$, and for $\binom{a_i}{b_i} \in S_i^2$, $\binom{a_i}{b_i}s = \binom{a_i}{b_i}(e_1s_1 + \cdots + e_ts_t) = \binom{a_i}{b_i}s_i$. Thus if $f \in M_{S_i}(S_i^2)$, then $f\left(\binom{a_i}{b_i}s\right) = f\left(\binom{a_i}{b_i}s_i\right) = f\binom{a_i}{b_i}s_i = f\binom{a_i}{b_i}s$, i.e., $f \in M_S(S_i^2)$. We have established the following result.

Theorem 2.1. *Let $S = S_1 \oplus \cdots \oplus S_t$ be a direct sum of ideals $S_1, \ldots, S_t$. Then $M_S(S^2) \cong M_{S_1}(S_1^2) \oplus \cdots \oplus M_{S_t}(S_t^2)$.*

Let $K = K_1 \oplus \cdots \oplus K_t$ be a direct sum of near-rings with identities e_i, and let B denote an ideal of K. Note that $B \cap K_i$ is an ideal of K_i, and for $b \in B$, $b = (b_1, \ldots, b_t)$, we have $be_i = b_ie_i = b_i$, which implies $b_i \in B \cap K_i$. Thus $B = (B \cap K_1) \oplus \cdots \oplus (B \cap K_t)$, and so, from the previous theorem, to determine the ideals of $M_S(S^2)$ it suffices to determine the ideals of the individual components.

If R is a commutative PIR, then, as stated above, R is the direct sum of principal ideal domains (PID) and special PIR's, say $R = R_1 \oplus \cdots \oplus R_t$. From Theorem 2.1, $N = M_R(R^2) \cong M_{R_1}(R_1^2) \oplus \cdots \oplus M_{R_t}(R_t^2)$, so we are going to determine the ideals of $M_{R_i}(R_i^2)$. We know, however, if R_i is a PID then $M_{R_i}(R_i^2)$ is simple, so the only ideals are $M_{R_i}(R_i^2)$ and $\{0\}$. (See [5, Theorem II.12].) It remains to determine the ideals of $M_{R_i}(R_i^2)$ when R_i is a special PIR.

To this end, let R be a special PIR with unique maximal ideal $J = \langle\theta\rangle$, and let m be the index of nilpotency of J, i.e., $\theta^m = 0$ and $\theta^{m-1} \neq 0$.

We know that the ideals of R are of the form $\langle\theta^k\rangle$, $k = 0, 1, 2, \ldots, m$. We denote $\langle\theta^k\rangle$ by A_k and remark that $A_k^2 = \{\binom{a_1}{a_2} \mid a_1, a_2 \in A_k\}$ is an R-submodule of R^2 with the property $f(A_k^2) \subseteq A_k^2$ for each $f \in N = M_R(R^2)$, because $f\binom{r\theta^2}{s\theta^2} = f\binom{r}{s}\theta^2$ for all $r, s \in R$. But then $(\{\binom{0}{0}\}: A_k^2)$ is an ideal of N. For $r, s \in R$ and $f \in (\{\binom{0}{0}\}: A_k^2)$, we have $\binom{0}{0} = f\binom{r\theta^k}{s\theta^k} = f\binom{r}{s}\theta^k$, so $f\binom{r}{s} \in \langle\theta^{m-k}\rangle^2 = A_{m-k}^2$. Therefore $(\{\binom{0}{0}\}: A_k^2) \subseteq (A_{m-k}^2 : R^2)$. Since the reverse inclusion is straightforward, we have the next result.

Proposition 2.2. *If R is a special PIR with $J = \langle\theta\rangle$ and index of nilpotency m, and if $A_k = \langle\theta^k\rangle$, then $(\{\binom{0}{0}\}: A_k^2) = (A_{m-k}^2 : R^2)$, $k = 0, 1, 2, \ldots, m$.*

We know that if I is an ideal of N, then there exists a unique ideal A_k of R with $I \cap \mathscr{M}_2(R) = \mathscr{M}_2(A_k)$. In particular from [5], if $f \in I$, say $f\binom{x}{y} = \binom{a}{b}$, then $f \circ \left[\begin{smallmatrix} x & 0 \\ y & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right]$. This in turn implies $f(R^2) \subseteq A_k^2$, so we have $I \subseteq (A_k^2 : R^2)$.

Proposition 2.3. *If R is a special PIR with $J = \langle\theta\rangle$ and index of nilpotency m, then for each non-trivial ideal I of $N = M_R(R^2)$ there is a unique integer k, $0 < k < m$, such that $I \subseteq (A_l^2 : R^2)$ for $l \leq k$, and $I \not\subseteq (A_l^2 : R^2)$ for $l > k$.*

In the next section we develop the machinery to show that $I = (A_k^2 : R^2)$. (Of course, if $I = \{0\}$, then $I = (\{(\binom{0}{0})\} : R^2) = (A_m^2 : R^2)$, and if $I = M_R(R^2)$, then $I = (R^2 : R^2) = (A_0^2 : R^2)$.) This will complete a proof of our major result.

Theorem 2.4. *Let R be a commutative principal ideal ring with $R = R_1 \oplus \cdots \oplus R_t$, where R_i is a PID or a special PIR. Then $N = M_R(R^2) = M_{R_1}(R_1^2) \oplus \cdots \oplus M_{R_t}(R_t^2)$, and if I is an ideal of N, then $I = I_1 \oplus \cdots \oplus I_t$, where I_i is an ideal of $M_{R_i}(R_i^2)$. If R_i is a PID, then $I_i = \{0\}$ or $I_i = M_{R_i}(R_i^2)$. If R_i is a special PIR with $J = \langle\theta\rangle$ and index of nilpotency m, then $I_i = (A_k^2 : R_i^2) = (\{(\binom{0}{0})\} : A_{m-k}^2)$ for some k, $0 \leq k \leq m$, where $A_k = \langle\theta^k\rangle$.*

3. Ideals in $M_R(R^2)$, R a special PIR

Unless otherwise stated, in this section R will denote a special PIR with unique maximal ideal $J = \langle\theta\rangle$ and index of nilpotency m. Let I be an ideal of $N = M_R(R^2)$ with $I \subseteq (A_k^2 : R^2)$ as given in Proposition 2.3. From the fact that $\mathscr{M}_2(A_k) \subseteq I$ our plan is to show that an arbitrary function in $(A_k^2 : R^2)$ can be constructed from functions in I. This will then give the desired equality. To aid in the construction of functions in N we recall from [5] that $x, y \in (R^2)^*$ are *connected* if there exist $x = a_0, a_1, \dots, a_s = y$ in $(R^2)^*$ such that $a_i R \cap a_{i+1} R \neq \{\binom{0}{0}\}$, $i = 0, 1, 2, \dots, s-1$. This defines an equivalence relation on $(R^2)^*$ and the equivalence classes are called *connected components*. We first determine the connected components of $(R^2)^*$.

Let F be a set of representatives for the classes R/J, where we choose 0 for the class J. Thus for $\alpha \in F^*$, α is a unit in R. We know for each $r \in R$ there is a unique $\alpha_0 \in F$ such that $r = \alpha_0 + r_0\theta$, $r_0 \in R$. But $r_0 = \alpha_1 + r_1\theta$, with $\alpha_1 \in F$, $r_1 \in R$, implies $r = \alpha_0 + \alpha_1\theta + r_1\theta^2$. Continuing, we find that every element $r \in R$ has a unique "base θ" representation, $r = \alpha_0 + \alpha_1\theta + \cdots + \alpha_{m-1}\theta^{m-1}$, $\alpha_i \in F$, $i = 0, 1, 2, \dots, m-1$.

In the sequel, for ease of exposition we let # denote a symbol not in F, and we let $\widehat{F} = F \cup \{\#\}$.

LEMMA 3.1. *Let* $M_\# = \left\langle \binom{\theta^{m-1}}{0} \right\rangle$ *and let* $M_\alpha = \left\langle \binom{\alpha\theta^{m-1}}{\theta^{m-1}} \right\rangle$, $\alpha \in F$. *The submodules* M_β, $\beta \in \widehat{F}$, *are the minimal submodules of* R^2.

PROOF. Let H be an R-submodule of R^2, $\{\binom{0}{0}\} \subsetneq H \subseteq M_\beta$, where $\beta \in F$, and let $\binom{0}{0} \neq x \in H$. Then $x = \binom{\beta\theta^{m-1}}{\theta^{m-1}} s$ for some $s \in R$, and since $x \neq 0$, we have $s \notin J$, so s is a unit in R. But then $xs^{-1} \in H$, hence $M_\beta \subseteq H$. In the same manner if $\beta = \#$, then $H = M_\#$.

To show that the M_β, $\beta \in \widehat{F}$, are the only minimal submodules, we show that every non-zero submodule L of R^2 must contain some M_β, $\beta \in \widehat{F}$. Let $y = \binom{u_1\theta^{l_1}}{u_2\theta^{l_2}}$ be a non-zero element in L, where u_1, u_2 are units in R. Suppose $l_1 \geq l_2$. Then $yu_2^{-1}\theta^{m-l_2-1} = \binom{u_1u_2^{-1}\theta^{l_1-l_2+m-1}}{1\theta^{m-1}}$. If $l_1 > l_2$, then $yu_2^{-1}\theta^{m-l_2-1} = \binom{0}{\theta^{m-1}}$, so $M_0 \subseteq L$. We have $u_1u_2^{-1} = \alpha + r\theta$ for some $\alpha \in F$, $r \in R$, and $u_1u_2^{-1}\theta^{m-1} = \alpha\theta^{m-1}$, and so if $l_1 = l_2$, then $yu_2^{-1}\theta^{m-l_2-1} = \binom{\alpha\theta^{m-1}}{\theta^{m-1}}$, i.e., $M_\alpha \subseteq L$. A similar argument for $l_1 < l_2$ gives $M_\# \subseteq L$ and the proof is complete.

LEMMA 3.2. *For* $x, y \in (R^2)^*$, *the following are equivalent*:

(i) x *and* y *are connected*;

(ii) xR *and* yR *contain the same minimal submodule* M;

(iii) *there exist positive integers* l_1, l_2 *such that* $x\theta^{l_1} \in M^*$ *and* $y\theta^{l_2} \in M^*$ *for some minimal submodule* M.

Proof. (i) $\Rightarrow$ (ii). Suppose x and y are connected. As we showed in the previous proof, xR and yR contain minimal submodules, say $xR \supseteq M' = cR$ and $yR \supseteq M'' = dR$. Thus there exist $r, s \in R^*$ such that $c = xr$ and $d = ys$. Since x and y are connected, so are c and d, say $cr_1 = b_1 s_1 \neq 0$, $b_1 r_2 = b_2 s_2 \neq 0, \ldots, b_{t-1} r_t = ds_t \neq 0$. Since $cr_1 \in (M')^*$, it follows that $cr_1 R = cR$, so there exists $r' \in R$ such that $cr_1 r' = c$, hence $c = cr_1 r' = b_1 s_1 r'$. Now c has the form $\binom{a}{b}\theta^{m-1}$, so if $b_1 = \binom{u_1\theta^{l_1}}{u_2\theta^{l_2}}$ and $s_1 r' = v_1\theta^{l_3}$, then $b_1\theta^{l_3} = cv_1^{-1} \in (cR)^*$. If $r_2 = v_2\theta^{l_4}$, then $0 \neq b_1 r_2 = b_1 v_2 \theta^{l_3 + (l_4 - l_3)}$, and since $b_1\theta^{l_3} \in cR$, a minimal submodule, it follows from Lemma 3.1 that $l_4 \leq l_3$, otherwise $b_1 r_2 = 0$. Therefore $r_2\theta^{l_3 - l_4} = v_2\theta^{l_3}$, which in turn implies $b_1 r_2 \theta^{l_3 - l_4} = b_1 v_2 \theta^{l_3} \in (cR)^*$. Hence $b_2 s_2 \theta^{l_3 - l_4} \in (cR)^*$, so there exists $r'' \in R$ such that $b_2 r'' = c$. Continuing in this manner we get $\hat{r}$ such that $d\hat{r} = c$ for some $\hat{r} \in R$. But this means $M' = M''$.
(ii) $\Rightarrow$ (iii). If $xR \supseteq M$ and $yR \supseteq M$, then there exist $r, s \in R$ such that $xr, ys \in M^*$, say $r = u\theta^{l_1}$, $s = v\theta^{l_2}$, u, v units. But then $x\theta^{l_1}$ and $y\theta^{l_2}$ are non-zero in M.
(iii) $\Rightarrow$ (i). From $x\theta^{l_1} \in M^*$ we have $\{\binom{0}{0}\} \subsetneq M \cap xR = M$. Hence $M \subseteq xR$, and similarly, $M \subseteq yR$. Therefore, for some $r, s \in R^*$, $xr = ys \neq 0$, i.e., x and y are connected.

From this lemma we have that every minimal submodule M determines a connected component $\mathscr{C}$, where $\mathscr{C} = (\bigcup\{xR \mid xR \supseteq M\}) \backslash \{\binom{0}{0}\}$.

Consider the minimal submodule M_α, for some $\alpha \in F$. We consider the submodules $H(\alpha, \alpha_1, \ldots, \alpha_{m-1}) = \langle \binom{\alpha + \alpha_1\theta + \cdots + \alpha_{m-1}\theta^{m-1}}{1} \rangle$, where $\alpha_1, \ldots, \alpha_{m-1}$ range over F. We note that $H(\alpha, \alpha_1, \ldots, \alpha_{m-1}) \cap H(\beta, \beta_1, \ldots, \beta_{m-1}) = \{\binom{0}{0}\}$ if and only if $\alpha \neq \beta$. For if $\alpha = \beta$, then $\binom{\alpha + \alpha_1\theta + \cdots + \alpha_{m-1}\theta^{m-1}}{1}\theta^{m-1} = \binom{\alpha\theta^{m-1}}{\theta^{m-1}} = \binom{\beta + \beta_1\theta + \cdots + \beta_{m-1}\theta^{m-1}}{1}\theta^{m-1}$, so

$$H(\alpha, \alpha_1, \ldots, \alpha_{m-1}) \cap H(\beta, \beta_1, \ldots, \beta_{m-1}) \supseteq M_\alpha.$$

Conversely, suppose $\binom{\alpha + \alpha_1\theta + \cdots + \alpha_{m-1}\theta^{m-1}}{1} r = \binom{\beta + \beta_1\theta + \cdots + \beta_{m-1}\theta^{m-1}}{1} s$ for some non-zero $r, s \in R$. Then if $r = a\theta^{l_1}$, $s = b\theta^{l_2}$, we get $l_1 = l_2$ and $\binom{\alpha\theta^{m-1}}{\theta^{m-1}} = \binom{\beta\theta^{m-1}}{\theta^{m-1}}$. Hence $\alpha = \beta$, since $\alpha, \beta \in F$. In the same way we see that $H(\#, \alpha_1, \dots, \alpha_{m-1}) = \langle {}_{\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}}^{1} \rangle$ contains $M_\#$ and that $H(\#, \alpha_1, \dots, \alpha_{m-1}) \cap H(\beta, \beta_1, \dots, \beta_{m-1}) = \{\binom{0}{0}\}$ for all $\beta \in F$.

Let a be an arbitrary non-zero element of R^2, say $a = \binom{a_1\theta^{l_1}}{a_2\theta^{l_2}}$. If $l_1 \geq l_2$, then $a = \binom{a_1\theta^{l_1-l_2}}{a_2}\theta^{l_2} = \binom{a_1a_2^{-1}\theta^{l_1-l_2}}{1}a_2\theta^{l_2}$ implies a is in some $H(\alpha, \alpha_1, \dots, \alpha_{m-1})$, $\alpha \in F$. If $l_1 < l_2$, then

$$a = \begin{pmatrix} a_1 \\ a_2\theta^{l_2-l_1} \end{pmatrix} \theta^{l_1} = \begin{pmatrix} 1 \\ a_2a_1^{-1}\theta^{l_2-l_1} \end{pmatrix} a_1\theta^{l_1}$$

implies a is in some $H(\#, \alpha_1, \dots, \alpha_{m-1})$. Thus we see that the collection of submodules $\{H(\beta, \alpha_1, \dots, \alpha_{m-1}) \mid \beta \in \widehat{F}, \ \alpha_1, \dots, \alpha_{m-1} \in F\}$ is a cover for R^2 (see [2]) and we call the submodules $H(\beta, \alpha_1, \dots, \alpha_{m-1})$ *covering submodules*.

Therefore, to define a function f in N it suffices to define f on the generators of the covering submodules, use the homogeneous property $f(xr) = f(x)r$ to extend f to all of R^2 and then verify that f is well-defined. That is, if x and y are generators of covering submodules and $0 \neq xr = ys$ for $r, s \in R$, then one must show that $f(x)r = f(y)s$. Suppose $r = a_1\theta^{l_1}$, $s = a_2\theta^{l_2}$ and $x = \binom{x_1}{1}$, $y = \binom{y_1}{1}$. (A similar argument works for $x = \binom{1}{x_1}$, $y = \binom{1}{y_1}$.) Thus we have $x_1a_1\theta^{l_1} = y_1a_2\theta^{l_2}$ and $a_1\theta^{l_1} = a_2\theta^{l_2}$. Thus $l_1 = l_2$, and so $a_2 = a_1 + r\theta^{m-l_1}$ for some $r \in R$. Thus $xr = ys$ implies $x\theta^{l_1} = y\theta^{l_1}$. Consequently, to show that f is well-defined, it suffices to show that $x\theta^l = y\theta^l$ implies $f(x)\theta^l = f(y)\theta^l$, where x and y are generators of covering submodules.

For convenience in manipulating functions in N we give the next result.

LEMMA 3.3. *If $f \in N$, then for any j, $1 \leq j \leq m-1$,* $f\binom{\alpha+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}}{1}$ $= f\binom{\alpha+\alpha_1\theta+\cdots+\alpha_j\theta^j}{1} + \sigma_{j+1}\theta^{j+1} + \cdots + \sigma_{m-1}\theta^{m-1}$ *and* $f\binom{1}{\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}} = f\binom{1}{\alpha_1\theta+\cdots+\alpha_j\theta^j} + \sigma'_{j+1}\theta^{j+1} + \cdots + \sigma'_{m-1}\theta^{m-1}$, *where* $\sigma_{j+1}, \dots, \sigma_{m-1}$, $\sigma'_{j+1}, \dots, \sigma'_{m-1} \in R^2$.

PROOF. We note that $f\binom{\alpha+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}}{1}\theta = f\binom{\alpha+\alpha_1\theta+\cdots+\alpha_{m-2}\theta^{m-2}}{1}\theta$ implies $f\binom{\alpha+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}}{1} = f\binom{\alpha+\alpha_1\theta+\cdots+\alpha_{m-2}\theta^{m-2}}{1} + \sigma_{m-1}\theta^{m-1}$ for some $\sigma_{m-1} \in R^2$. The result now follows by induction. The second equality follows similarly.

Some additional notation will now be introduced. Let x be a generator of a covering submodule. We denote by $m_{\theta^k} f(x)$ the *multiplier of* θ^k *in* $f(x)$. If $x = \binom{\alpha+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}}{1}$ and $j+1 \geq k$, then from the above lemma, $f(x) = f\binom{\alpha+\alpha_1\theta+\cdots+\alpha_j\theta^j}{1} + \sigma_{j+1}\theta^{j+1} + \cdots + \sigma_{m-1}\theta^{m-1}$ and so $m_{\theta^k} f(x) = m_{\theta^k} f\binom{\alpha+\alpha_1\theta+\cdots+\alpha_j\theta^j}{1} + \sigma_{j+1}\theta^{j+1-k} + \cdots + \sigma_{m-1}\theta^{m-1-k}$.

As at the beginning of this section, let $I \subseteq (A_k^2 : R^2)$. We consider two cases, F finite and F infinite.

First, suppose F is finite, and let $f \in (A_k^2 : R^2)$. Since F is finite, there are only a finite number of connected components, namely $\mathscr{C}_\beta$ where $\beta \in \widehat{F}$, $\mathscr{C}_\beta$ determined by M_β. We show how to find a function in I which agrees with f on a single component and is zero off this component. Then by adding we get $f \in I$. We work first with the component $\mathscr{C}_\#$. We know the generators of the covering submodules for this component have the form $\binom{1}{\alpha_1\theta+\alpha_2\theta^2+\cdots+\alpha_{m-1}\theta^{m-1}}$, $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in F$.

For the fixed k above (determined by $I \subseteq (A_k^2 : R^2)$) we partition these generators of the covering submodules of $\mathscr{C}_\#$ into sets determined by the $(k-1)$-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_{k-1})$, $\alpha_1, \alpha_2, \ldots, \alpha_{k-1} \in F$, where we take $k \geq 2$. (The case $k = 1$ will be handled separately.) That is, given $(\alpha_1, \ldots, \alpha_{k-1})$, in one set we have all generators $\binom{1}{\beta_1\theta+\cdots+\beta_{m-1}\theta^{m-1}}$ where $(\beta_1, \ldots, \beta_{k-1}) = (\alpha_1, \ldots, \alpha_{k-1})$. Define $p_{k-1} : R^2 \to R^2$ by $p_{k-1}\binom{1}{\beta_1\theta+\cdots+\beta_{m-1}\theta^{m-1}} = \binom{1}{\beta_k\theta^k+\cdots+\beta_{m-1}\theta^{m-1}}$ if

$$(\beta_1, \ldots, \beta_{k-1}) = (\alpha_1, \ldots, \alpha_{k-1}), \quad p_{k-1}\binom{1}{\beta_1\theta + \cdots + \beta_{m-1}\theta^{m-1}} = \binom{0}{0}$$

if $(\beta_1, \ldots, \beta_{k-1}) \neq (\alpha_1, \ldots, \alpha_{k-1})$, extend using the homogeneous property, and define $p_{k-1}(x) = \binom{0}{0}$ if $x \notin \mathscr{C}_\#$. We show that p_{k-1} is well-defined. Let $\overline{\alpha} = \alpha_1\theta + \cdots + \alpha_{m-1}\theta^{m-1}$, $\overline{\beta} = \beta_1\theta + \cdots + \beta_{m-1}\theta^{m-1}$ and suppose $\binom{1}{\overline{\alpha}}\theta^l = \binom{1}{\overline{\beta}}\theta^l$. This means $(\alpha_1, \ldots, \alpha_{m-l-1}) = (\beta_1, \ldots, \beta_{m-l-1})$. If $l \leq m-k-1$, then $m-l-1 \geq k$ and so $\binom{1}{\overline{\alpha}}$ and $\binom{1}{\overline{\beta}}$ are in the same set of the partition, thus $p_{k-1}\binom{1}{\overline{\alpha}}\theta^l = \binom{\theta^l}{\alpha_k\theta^{k+l}+\cdots+\alpha_{m-1-l}\theta^{m-1}+\cdots+\alpha_{m-1}\theta^{m-1+l}} = p_{k-1}\binom{1}{\overline{\beta}}\theta^l$. If $l > m-k-1$, then $l \geq m-k$ and so $p_{k-1}\binom{1}{\overline{\alpha}}\theta^l = \binom{0}{0} = p_{k-1}\binom{1}{\overline{\beta}}\theta^l$. Thus $p_{k-1} \in M_R(R^2)$. Also, since $\begin{bmatrix} 0 & 0 \\ \theta^k & 0 \end{bmatrix} \in I$, $\hat{f} = \begin{bmatrix} 0 & 0 \\ \theta^k & 0 \end{bmatrix} p_{k-1} \in I$.

Define h: $R^2 \to R^2$ by $h\binom{1}{\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}} = \binom{\alpha_k\theta^k+\cdots+\alpha_{m-1}\theta^{m-1}}{0}$, extend, and define $h(x) = \binom{0}{0}$ if $x \notin \mathscr{C}_\#$. As above one shows that h is well-defined, i.e., $h \in M_R(R^2)$. Thus for each $g \in M_R(R^2)$, $\hat{q} = g(\hat{f} + h) -$

$gh \in I$. For $x \notin \mathscr{C}_\#$ we have $\hat{q}(x) = \binom{0}{0}$, because $p_{k-1}(x) = \binom{0}{0}$ if $x \notin \mathscr{C}_\#$. Further, $\hat{q}(\frac{1}{\beta}) = g(\hat{f}(\frac{1}{\beta}) + h(\frac{1}{\beta})) - gh(\frac{1}{\beta})$. If $(\beta_1, \dots, \beta_{k-1}) \neq (\alpha_1, \dots, \alpha_{k-1})$, then $\hat{f}(\frac{1}{\beta}) = \binom{0}{0}$ and in this case $\hat{q}(\frac{1}{\beta}) = \binom{0}{0}$. Thus we focus on $(\frac{1}{\beta})$ where $(\beta_1, \dots, \beta_{k-1}) = (\alpha_1, \dots, \alpha_{k-1})$. Here, $\hat{q}(\frac{1}{\beta}) = g(\binom{0}{\theta^k} + \binom{\beta_k\theta^k + \cdots + \beta_{m-1}\theta^{m-1}}{0}) - g\binom{1}{0}(\beta_k\theta^k + \cdots + \beta_{m-1}\theta^{m-1})$. We wish to define g so that $\hat{q}$ agrees with f on all generators $\binom{1}{\beta_1\theta + \cdots + \beta_{m-1}\theta^{m-1}}$ with $(\beta_1, \dots, \beta_{k-1}) = (\alpha_1, \dots, \alpha_{k-1})$. First define $g(\mathscr{C}_\#) = \{\binom{0}{0}\}$. Then define

$$
\begin{aligned}
g & \begin{pmatrix} \beta_0 + \beta_1\theta + \cdots + \beta_{m-1}\theta^{m-1} \\ 1 \end{pmatrix} \\
& = \cdots = g \begin{pmatrix} \beta_0 + \beta_1\theta + \cdots + \beta_{m-k-1}\theta^{m-k-1} \\ 1 \end{pmatrix} \\
& = m_{\theta^k} f \begin{pmatrix} 1 \\ \alpha_1\theta + \cdots + \alpha_{k-1}\theta^{k-1} + \beta_0\theta^k + \cdots + \beta_{m-k-1}\theta^{m-1} \end{pmatrix}.
\end{aligned}
$$

We show that g is well-defined. Let $\beta = \beta_0 + \beta_1\theta + \cdots + \beta_{m-k-1}\theta^{m-k-1}$ and $\gamma = \gamma_0 + \gamma_1\theta + \cdots + \gamma_{m-k-1}\theta^{m-k-1}$, and suppose $\binom{\beta}{1}\theta^l = \binom{\gamma}{1}\theta^l$. Then

$$(\beta_0, \beta_1, \dots, \beta_{m-l-1}) = (\gamma_0, \gamma_1, \dots, \gamma_{m-l-1}).$$

If $l \le k$, then $m - l - 1 \ge m - k - 1$ and

$$g\begin{pmatrix} \beta \\ 1 \end{pmatrix} = m_{\theta^k} f \begin{pmatrix} 1 \\ \alpha_1\theta + \cdots + \alpha_{k-1}\theta^{k-1} + \beta_0\theta^k + \cdots + \beta_{m-k-1}\theta^{m-1} \end{pmatrix} = g\begin{pmatrix} \gamma \\ 1 \end{pmatrix}.$$

If $l \ge k + 1$, then

$$
\begin{aligned}
g\begin{pmatrix} \beta \\ 1 \end{pmatrix} = m_{\theta^k} & \left[f \begin{pmatrix} 1 \\ \alpha_1\theta + \cdots + \alpha_{k-1}\theta^{k-1} + \beta_0\theta^k + \cdots + \beta_{m-l-1}\theta^{m+k-l-1} \end{pmatrix} \right] \\
& + \rho_l\theta^{m-l} + \cdots + \rho_{k+1}\theta^{m-k-1},
\end{aligned}
$$

where $\rho_{k+1}, \dots, \rho_l \in R^2$. A similar expression holds for $g\binom{\gamma}{1}$. But then $g\binom{\beta}{1}\theta^l = g\binom{\gamma}{1}\theta^l$ as desired.

Thus,

$$\begin{aligned}\hat{q}\begin{pmatrix}1\\ \overline{\beta}\end{pmatrix} &= g\begin{pmatrix}\beta_k\theta^k+\cdots+\beta_{m-1}\theta^{m-1}\\ \theta^k\end{pmatrix} - g\begin{pmatrix}1\\0\end{pmatrix}(\beta_k\theta^k+\cdots+\beta_{m-1}\theta^{m-1})\\ &= g\begin{pmatrix}\beta_k+\cdots+\beta_{m-1}\theta^{m-1-k}\\ 1\end{pmatrix}\theta^k\\ &= m_{\theta^k} f\begin{pmatrix}1\\ \alpha_1\theta+\cdots+\alpha_{k-1}\theta^{k-1}+\beta_k\theta^k+\cdots+\beta_{m-1}\theta^{m-1}\end{pmatrix}\theta^k\\ &= f\begin{pmatrix}1\\ \overline{\beta}\end{pmatrix}.\end{aligned}$$

Therefore $\hat{q}$ agrees with f on those generators $\left(\begin{smallmatrix}1\\ \beta_1\theta+\cdots+\beta_{m-1}\theta^{m-1}\end{smallmatrix}\right)$ with $(\beta_1, \dots, \beta_{k-1}) = (\alpha_1, \dots, \alpha_{k-1})$, and is zero on all other generators of covering submodules. Since there are $|F|^{k-1}$ such functions, by adding we obtain a function $q_\#$ which agrees with f on $\mathscr{C}_\#$ and is 0 off $\mathscr{C}_\#$.

For $k = 1$ the situation is somewhat easier. There is no need to partition the generators of the covering modules of $\mathscr{C}_\#$. For this case we use $\left[\begin{smallmatrix}0&0\\ \theta&0\end{smallmatrix}\right]e_\#$ and the h defined above, where e_μ is the idempotent determined by $\mathscr{C}_\mu$, i.e., $e_\mu(x) = x$ if $x \in \mathscr{C}_\mu$ and $e_\mu(x) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ if $x \notin \mathscr{C}_\mu$, $\mu \in \widehat{F}$. Thus for each $g \in M_R(R^2)$, $\hat{q} = g(\left[\begin{smallmatrix}0&0\\ \theta&0\end{smallmatrix}\right]e_\# + h) - gh \in I$. For $x \notin \mathscr{C}_\#$, $\hat{q}(x) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$. Further, $\hat{q}(\frac{1}{\overline{\beta}}) = g(\left(\begin{smallmatrix}0\\ \theta\end{smallmatrix}\right) + \left(\begin{smallmatrix}\overline{\beta}\\ 0\end{smallmatrix}\right)) - g\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\overline{\beta} = g\left(\begin{smallmatrix}\beta_1\theta+\beta_2\theta^2+\cdots+\beta_{m-1}\theta^{m-1}\\ \theta\end{smallmatrix}\right) - g\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\overline{\beta}$. Define $g(\mathscr{C}_\#) = \{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\}$ and

$$\begin{aligned}g\begin{pmatrix}\alpha_0+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}\\ 1\end{pmatrix} &= g\begin{pmatrix}\alpha_0+\alpha_1\theta+\cdots+\alpha_{m-2}\theta^{m-2}\\ 1\end{pmatrix}\\ &= m_\theta f\begin{pmatrix}1\\ \alpha_0\theta+\alpha_1\theta^2+\cdots+\alpha_{m-2}\theta^{m-1}\end{pmatrix}.\end{aligned}$$

As above one verifies that $g \in M_R(R^2)$ and that $\hat{q}$ agrees with f on $\mathscr{C}_\#$.

In a similar manner one constructs q_α, $\alpha \in F$, which agrees with f on $\mathscr{C}_\alpha$ and is 0 off $\mathscr{C}_\alpha$. Then $f = \sum_{\beta\in\widehat{F}} q_\beta \in I$, and so the proof of Theorem 2.4 is complete when F is finite.

Alternatively, one could use the following approach in the finite case. For $\alpha \in F$, define $p_\alpha\left(\begin{smallmatrix}\alpha+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}\\ 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\ \alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}\end{smallmatrix}\right)$ and $p_\alpha(x) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ for $x \notin \mathscr{C}_\alpha$. For each $g' \in N$, $q' = [g'(\left[\begin{smallmatrix}0&0\\ \theta^k&0\end{smallmatrix}\right] + h) - g'h]p_\alpha \in I$. For $x \notin \mathscr{C}_\alpha$, $q'(x) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$, and $q'\left(\begin{smallmatrix}\alpha+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}\\ 1\end{smallmatrix}\right) = g'\left(\begin{smallmatrix}\alpha_k\theta^k+\cdots+\alpha_{m-1}\theta^{m-1}\\ \theta^k\end{smallmatrix}\right) -$

$g'\binom{1}{0}(\alpha_k\theta^k + \cdots + \alpha_{m-1}\theta^{m-1})$. Define $g'(\mathscr{C}_\#) = \{\binom{0}{0}\}$ and

$$g'\begin{pmatrix}\beta_0 + \beta_1\theta + \cdots + \beta_{m-1}\theta^{m-1}\\ 1\end{pmatrix}$$

$$= \cdots = g'\begin{pmatrix}\beta_0 + \beta_1\theta + \cdots + \beta_{m-k-1}\theta^{m-k-1}\\ 1\end{pmatrix}$$

$$= m_{\theta^k}f\begin{pmatrix}\alpha + \alpha_1\theta + \cdots + \alpha_{k-1}\theta^{k-1} + \beta_0\theta^k + \cdots + \beta_{m-k-1}\theta^{m-1}\\ 1\end{pmatrix},$$

where we have partitioned the generators $\binom{\alpha+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}}{1}$ of the covering submodules in $\mathscr{C}_\alpha$ by using the k-tuples $(\alpha, \alpha_1, \ldots, \alpha_{k-1})$. One shows that g' is well-defined and continuing obtains a function which agrees with f on $\mathscr{C}_\alpha$ and is zero off $\mathscr{C}_\alpha$.

Suppose now F is infinite, and let $\delta_k\colon F^k \to F$ be a bijection. We again start with $\mathscr{C}_\#$, where as above we let $\overline{\alpha} = \alpha_1\theta + \cdots + \alpha_{m-1}\theta^{m-1}$. Define $h'\colon R^2 \to R^2$ by $h'\binom{1}{\overline{\alpha}} = \binom{\delta_k(\alpha_1,\ldots,\alpha_k)\theta^k+\alpha_{k+1}\theta^{k+1}+\cdots+\alpha_{m-1}\theta^{m-1}}{0}$ and $h'(x) = \binom{0}{0}$, $x \notin \mathscr{C}_\#$. As above one shows that $h' \in M_R(R^2)$. Thus for each $g \in N$, $t_\# = g(e_\# + h') - gh' \in I$. For $x \notin \mathscr{C}_\#$, $t_\#(x) = \binom{0}{0}$. For $x = \binom{1}{\overline{\alpha}}$, $t_\#(x) = g(\binom{0}{\theta^k} + \binom{\delta_k(\alpha_1,\ldots,\alpha_k)\theta^k+\alpha_{k+1}\theta^{k+1}+\cdots+\alpha_{m-1}\theta^{m-1}}{0}) - gh'(x)$. Define $g(\mathscr{C}_\#) = \{\binom{0}{0}\}$ and $g\binom{\beta_0+\beta_1\theta+\cdots+\beta_{m-1}\theta^{m-1}}{1} = \cdots = g\binom{\beta_0+\cdots+\beta_{m-1-k}\theta^{m-1-k}}{1} = m_{\theta^k}f\binom{1}{\mu_1\theta+\mu_2\theta^2+\cdots+\mu_k\theta^k+\beta_1\theta^{k+1}+\cdots+\beta_{m-k-1}\theta^{m-1}}$, where $\delta_k(\mu_1, \ldots, \mu_k) = \beta_0$.

If $\gamma = \gamma_0 + \gamma_1\theta + \cdots + \gamma_{m-1-k}\theta^{m-1-k} + \cdots + \gamma_{m-1}\theta^{m-1}$ and $\binom{\gamma}{1}\theta^l = \binom{\beta}{1}\theta^l$, then $(\gamma_0, \gamma_1, \ldots, \gamma_{m-l-1}) = (\beta_0, \beta_1, \ldots, \beta_{m-l-1})$. If $l \le k$, then $m - l - 1 \ge m - k - 1$ and so $g\binom{\gamma}{1}\theta^l = g\binom{\beta}{1}\theta^l$. If $l \ge k+1$, then $g\binom{\beta}{1} = m_{\theta^k}f\binom{1}{\mu_1\theta+\cdots+\mu_k\theta^k+\beta_1\theta^{k+1}+\cdots+\beta_{m-l-1}\theta^{m-l-1+k}} + \sigma_l\theta^{m-l} + \cdots + \sigma_{k+1}\theta^{m-k-1}$, where $\sigma_{k+1}, \ldots, \sigma_l \in R^2$, and

$$g\binom{\gamma}{1} = m_{\theta^k}f\begin{pmatrix}1\\ \nu_1\theta + \cdots + \nu_k\theta^k + \gamma_1\theta^{k+1} + \cdots + \gamma_{m-l-1}\theta^{m-l-1+k}\end{pmatrix}$$
$$+ \sigma_l'\theta^{m-l} + \cdots + \sigma_{k+1}'\theta^{m-k-1},$$

where $\sigma_{k+1}', \ldots, \sigma_l' \in R^2$ and $\delta_k(\nu_1, \ldots, \nu_k) = \gamma_0$. Since $\gamma_0 = \beta_0$, $(\nu_1, \ldots, \nu_k) = (\mu_1, \ldots, \mu_k)$ and $g\binom{\beta}{1}\theta^l = g\binom{\gamma}{1}\theta^l$. Hence $g \in N$.

Further,

$$t_{\#}\begin{pmatrix}1\\ \bar{\alpha}\end{pmatrix} = g\begin{pmatrix}\delta_k(\alpha_1, \dots, \alpha_k)\theta^k + \alpha_{k+1}\theta^{k+1} + \cdots + \alpha_{m-1}\theta^{m-1}\\ \theta^k\end{pmatrix} - gh'\begin{pmatrix}1\\ \bar{\alpha}\end{pmatrix}$$

$$= g\begin{pmatrix}\delta_k(\alpha_1, \dots, \alpha_k) + \alpha_{k+1}\theta + \cdots + \alpha_{m-1}\theta^{m-1-k}\\ 1\end{pmatrix}\theta^k - 0$$

$$= m_{\theta^k} f\begin{pmatrix}1\\ \alpha_1\theta + \cdots + \alpha_k\theta^k + \alpha_{k+1}\theta^{k+1} + \cdots + \alpha_{m-1}\theta^{m-1}\end{pmatrix}\theta^k$$

$$= f\begin{pmatrix}1\\ \bar{\alpha}\end{pmatrix}.$$

Thus $t_{\#}$ agrees with f on $\mathscr{C}_{\#}$ and is zero off $\mathscr{C}_{\#}$.

We next show that there is a function $\hat{t}_{\#}$ in I which agrees with f off $\mathscr{C}_{\#}$ and is zero on $\mathscr{C}_{\#}$. This will imply that $f = t_{\#} + \hat{t}_{\#} \in I$. To this end let $\delta_{k+1}\colon F^{k+1} \to F$ be a bijection, let $\alpha = \alpha_0 + \alpha_1\theta + \cdots + \alpha_{m-1}\theta^{m-1}$ and define $h''\colon R^2 \to R^2$ by $h''(\mathscr{C}_{\#}) = \{\binom{0}{0}\}$ while $h''\binom{\alpha}{1} = \binom{\delta_{k+1}(\alpha_0, \dots, \alpha_k)\theta^k + \alpha_{k+1}\theta^{k+1} + \cdots + \alpha_{m-1}\theta^{m-1}}{0}$. One finds that $h'' \in N$. Let $\widehat{E}_{\#} = \left[\begin{smallmatrix}0 & 0\\ 0 & \theta^k\end{smallmatrix}\right]$ (id. $-e_{\#}$). Then $\widehat{E}_{\#}\binom{\alpha}{1} = \binom{0}{\theta^k}$ and $\widehat{E}_{\#}(\mathscr{C}_{\#}) = \{\binom{0}{0}\}$. Since $\widehat{E}_{\#} \in I$, for each $g \in N$, $\hat{t}_{\#} = g(\widehat{E}_{\#} + h'') - gh''$ is in I. For $x \in \mathscr{C}_{\#}$, $\hat{t}_{\#}(x) = \binom{0}{0}$ and for

$$x = \begin{pmatrix}\alpha\\ 1\end{pmatrix},\ \hat{t}_{\#}\begin{pmatrix}\alpha\\ 1\end{pmatrix} = g\left(\begin{pmatrix}0\\ \theta^k\end{pmatrix} + h''\begin{pmatrix}\alpha\\ 1\end{pmatrix}\right) - gh''\begin{pmatrix}\alpha\\ 1\end{pmatrix}$$

$$= g\begin{pmatrix}\delta_{k+1}(\alpha_0, \dots, \alpha_k) + \alpha_{k+1}\theta + \cdots + \alpha_{m-1}\theta^{m-1-k}\\ 1\end{pmatrix}\theta^k$$

$$- g\begin{pmatrix}1\\ 0\end{pmatrix}(\delta_{k+1}(\alpha_0, \dots, \alpha_k)\theta^k + \alpha_{k+1}\theta^{k+1} + \cdots + \alpha_{m-1}\theta^{m-1}).$$

Again we define $g(\mathscr{C}_{\#}) = \{\binom{0}{0}\}$ and

$$g\begin{pmatrix}\gamma\\ 1\end{pmatrix} = g\begin{pmatrix}\gamma_0 + \gamma_1\theta + \cdots + \gamma_{m-1}\theta^{m-1}\\ 1\end{pmatrix}$$

$$= \cdots = g\begin{pmatrix}\gamma_0 + \gamma_1\theta + \cdots + \gamma_{m-1-k}\theta^{m-1-k}\\ 1\end{pmatrix}$$

$$= m_{\theta^k} f\begin{pmatrix}c_0 + c_1\theta + \cdots + c_k\theta^k + \gamma_1\theta^{k+1} + \cdots + \gamma_{m-1-k}\theta^{m-1}\\ 1\end{pmatrix},$$

where $\delta_{k+1}(c_0, c_1, \ldots, c_k) = \gamma_0$. As above, $g \in N$ and $\hat{t}_{\#}\binom{\alpha}{1} = f\binom{\alpha}{1}$. Thus $f = t_{\#} + \hat{t}_{\#} \in I$, and the proof of Theorem 2.4 is complete.

4. Applications

In this final section we apply the above characterization of the ideals of N to determine the radicals $J_\nu(N)$ of N and the quotient structures $N/J_\nu(N)$, $\nu = 0, 1, 2$.

From Theorem 2.1 and [7, Theorem 5.20], $J_\nu(N) = J_\nu(M_{R_1}(R_1^2)) \oplus \cdots \oplus J_\nu(M_{R_t}(R_t^2))$. If R_i is a PID, then $J_0(M_{R_i}(R_i^2)) = \{0\}$. If R_i is a PID, not a field, then $J_1(M_{R_i}(R_i^2)) = J_2(M_{R_i}(R_i^2)) = M_{R_i}(R_i^2)$, and if R_i is a field, then $J_1(M_{R_i}(R_i^2)) = J_2(M_{R_i}(R_i^2)) = \{0\}$. If R_i is a special PIR, then from the previous section we know that $M_{R_i}(R_i^2)$ has a unique maximal ideal $(A_1^2 : R_i^2) = (\{\binom{0}{0}\} : A_{m-1}^2)$. Moreover, A_{m-1}^2 is a type 2, $M_{R_i}(R_i^2)$-module, for if $\binom{x\theta^{m-1}}{y\theta^{m-1}} \in A_{m-1}^2$ then x and y are units in R (or zero), and so if $x \neq 0$ (say) then $\begin{bmatrix} rx^{-1} & 0 \\ sx^{-1} & 0 \end{bmatrix}\binom{x\theta^{m-1}}{y\theta^{m-1}} = \binom{r\theta^{m-1}}{s\theta^{m-1}}$ for an arbitrary $\binom{r\theta^{m-1}}{s\theta^{m-1}}$ in A_{m-1}^2. Therefore $J_2(N) \neq N$, so we have $J_0(M_{R_i}(R_i^2)) \subseteq J_1(M_{R_i}(R_i^2)) \subseteq J_2(M_{R_i}(R_i^2)) \subseteq (A_1^2 : R_i^2)$. On the other hand it is straightforward to verify that $(A_1^2 : R_i^2)$ is a nil ideal, so by [7, Theorem 5.37], $J_0(M_{R_i}(R_i^2)) \supseteq (A_1^2 : R_i^2)$. This proves the following result.

THEOREM 4.1.. *If R is a special PIR with $J(R) = \langle\theta\rangle$, then $J_\nu(M_R(R^2)) = (\langle\theta\rangle^2 : R^2)$, $\nu = 0, 1, 2$.*

Since $N/J_\nu(N) \cong M_{R_1}(R_1^2)/J_\nu(M_{R_1}(R_1^2)) \oplus \cdots \oplus M_{R_t}(R_t^2)/J_\nu(M_{R_t}(R_t^2))$, it remains to determine $M_{R_i}(R_i^2)/J_\nu(M_{R_i}(R_i^2))$ when R_i is a special PIR. This characterization is provided in the following result.

THEOREM 4.2. *Let R be a special PIR with $J(R) = \langle\theta\rangle$ and index of nilpotency m. Then $M_R(R^2)/J_\nu(M_R(R^2)) \cong M_{R/J(R)}(R/J(R))^2$, $\nu = 0, 1, 2$.*

PROOF. We know that every element of $(R/J(R))^2$ has a unique representative $\binom{\alpha+J(R)}{\beta+J(R)}$, where $\alpha, \beta \in F$. We define $\psi : M_R(R^2) \to M_{R/J(R)}(R/J(R))^2$ as follows: for $f \in M_R(R^2)$, $\psi(f)\binom{\alpha+J(R)}{\beta+J(R)} = f\binom{\alpha}{\beta} + J(R)^2$. If $\binom{\alpha+J(R)}{\beta+J(R)} = \binom{\gamma+J(R)}{\delta+J(R)}$, then $\alpha = \gamma$ and $\beta = \delta$, so $\psi(f)$ is well-defined. Furthermore

$\psi(f) \in M_{R/J(R)}(R/J(R))^2$, since $\psi(f)[\binom{\alpha+J(R)}{\beta+J(R)}(\gamma+J(R))] = f\binom{\alpha\gamma}{\beta\gamma}+J(R)^2 = f\binom{\alpha}{\beta}\gamma + J(R)^2 = \psi(f)\binom{\alpha+J(R)}{\beta+J(R)}(\gamma + J(R))$.

It is clear that $\psi(f+g) = \psi(f) + \psi(g)$. Further, $\psi(fg)\binom{\alpha+J(R)}{\beta+J(R)} = fg\binom{\alpha}{\beta}+J(R)^2$, while $(\psi(f)\psi(g))\binom{\alpha+J(R)}{\beta+J(R)} = \psi(f)(g\binom{\alpha}{\beta}+J(R)^2)$. If $g\binom{\alpha}{\beta} = \binom{\alpha_0+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}}{\beta_0+\beta_1\theta+\cdots+\beta_{m-1}\theta^{m-1}}$, then $\psi(f)(g\binom{\alpha}{\beta} + J(R)^2) = f\binom{\alpha_0}{\beta_0} + J(R)^2$. But, as in Lemma 3.3, one finds $f\binom{\alpha_0+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}}{\beta_0+\beta_1\theta+\cdots+\beta_{m-1}\theta^{m-1}} = f\binom{\alpha_0}{\beta_0} + \sigma\theta$, $\sigma \in R^2$, so $f\binom{\alpha_0}{\beta_0} + J(R)^2 = f\binom{\alpha_0+\cdots+\alpha_{m-1}\theta^{m-1}}{\beta_0+\cdots+\beta_{m-1}\theta^{m-1}} + J(R)^2 = fg\binom{\alpha}{\beta} + J(R)^2$, i.e., $\psi(fg) = \psi(f)\psi(g)$.

We complete the proof by showing that ψ is onto and Ker $\psi = J_\nu(M_R(R^2))$. To show that ψ is onto, let $g \in M_{R/J(R)}(R/J(R))^2$. For $\binom{\alpha_0+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}}{\beta_0+\beta_1\theta+\cdots+\beta_{m-1}\theta^{m-1}} \equiv \binom{\alpha}{\beta}$ define $f\colon R^2 \to R^2$ by $f\binom{\alpha}{\beta} = \binom{\alpha_0'}{\beta_0'}$ where $g\binom{\alpha_0+J(R)}{\beta_0+J(R)} = \binom{\alpha_0'+J(R)}{\beta_0'+J(R)}$. If $\binom{\alpha_0+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}}{\beta_0+\beta_1\theta+\cdots+\beta_{m-1}\theta^{m-1}}\theta^l = \binom{\delta_0+\delta_1\theta+\cdots+\delta_{m-1}\theta^{m-1}}{\varepsilon_0+\varepsilon_1\theta+\cdots+\varepsilon_{m-1}\theta^{m-1}}\theta^l$, then $f\binom{\alpha_0+\alpha_1\theta+\cdots+\alpha_{m-1}\theta^{m-1}}{\beta_0+\beta_1\theta+\cdots+\beta_{m-1}\theta^{m-1}}\theta^l = f\binom{\delta_0+\delta_1\theta+\cdots+\delta_{m-1}\theta^{m-1}}{\varepsilon_0+\varepsilon_1\theta+\cdots+\varepsilon_{m-1}\theta^{m-1}}\theta^l$, so one finds that $f \in M_R(R^2)$. Moreover, $\psi(f)\binom{\alpha_0+J(R)}{\beta_0+J(R)} = f\binom{\alpha_0}{\beta_0} + J(R)^2 = \binom{\alpha_0'}{\beta_0'} + J(R)^2 = g\binom{\alpha_0+J(R)}{\beta_0+J(R)}$, and hence $\psi(f) = g$.

Finally, Ker $\psi = \{f \in M_R(R^2) \mid f\binom{\alpha}{\beta} \in J(R)^2$, for all $\alpha, \beta \in F\} = \{f \in M_R(R^2) \mid f\binom{x}{y} \in J(R)^2$ for all $x, y \in R\} = (J(R)^2 : R^2) = (\langle\theta\rangle^2 : R^2) = J_\nu(M_R(R^2))$.

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