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The Minimum Number of Idempotent Generators of an Upper Triangular Matrix Algebra

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We prove that the minimum number $\nu = \nu(\mathscr{U}_m(R))$ such that the $m \times m$ upper triangular matrix algebra $\mathscr{U}_m(R)$ over an arbitrary commutative ring R can be generated as an R-algebra by ν idempotents, is given by

$$\nu(\mathscr{U}_m(R)) = \begin{cases} \lceil \log_2 m \rceil + 1, & \text{if } m = 2, 3, 4; \\ \lceil \log_2 m \rceil, & \text{if } m \geq 5. \end{cases}$$

In order to prove the result mentioned above, we show that $\nu(R^{(m)}) = \lceil \log_2 m \rceil$ for every $m \geq 2$, where $R^{(m)}$ denotes the direct sum of m copies of R. The latter result corrects an error by N. Krupnik (*Comm. Algebra* **20**, 1992, 3251–3257).

In [5] an internal characterization of structural matrix rings in terms of a set of matrix units associated with a partial order relation was obtained, and in [6] this characterization was used to recognize certain subrings of full matrix rings as structural matrix rings. However, the mentioned characterization is rather technical, and the search for a new internal characterization of upper triangular matrix rings, and of structural matrix rings in general, which is our long term goal, is the origin of this paper.

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Krupnik showed in [2] that for $n \geq 3$ the full matrix algebra $\mathbb{M}_n(F)$ over an arbitrary field F can be generated as an F-algebra by 3 idempotents. Krupnik also found the minimum number of idempotents needed to generate direct sums of full matrix algebras over F. In particular, in [2, Theorem 5] Krupnik asserts that $m-1$ is the minimum number of idempotents needed to generate the F-algebra $F^{(m)}$, where $F^{(m)}$ denotes the direct sum of m copies of F ($\cong \mathbb{M}_1(F)$), F an arbitrary infinite field. This assertion is incorrect. In Theorem 2 of the sequel we show that $m-1$ must be replaced by $\lceil \log_2 m \rceil$. In fact, this holds for every commutative ring R. We use this result, together with the fact that $R^{(m)}$ is a homomorphic image of $\mathscr{U}_m(R)$, the $m \times m$ upper triangular matrix algebra over R, to determine the minimum number of idempotents needed to generate $\mathscr{U}_m(R)$ as an R-algebra.

We wish to mention that idempotents in full matrix rings over commutative von Neumann regular rings were studied in their own right in [1]. Furthermore, the problem of finding the minimum number of idempotent generators of an algebra is also important in operator theory. See, for example, [3, 4]. In [4] it is shown that for each finitely generated Banach algebra $\mathscr{B}$ there is a number n_0 so that the full $n \times n$ matrix algebra $\mathbb{M}_n(\mathscr{B})$ can be generated by three idempotents whenever $n \geq n_0$.

Every algebra will be assumed to have an identity element, and subalgebras inherit the identity. Throughout the sequel R will be a commutative (and associative) ring (with identity).

Let $m \geq 2$. We denote the direct sum of m copies of R by $R^{(m)}$, and we consider $R^{(m)}$ as an algebra (over R). For $k = 1, \ldots, m$, the element of $R^{(m)}$ with 1 in position k and zeros elsewhere is denoted by $e_k^{(m)}$.

Let $n \geq 1$. In Lemma 1 we show that $R^{(2^n)}$ can be generated (as an algebra) by n idempotents; more particularly, by the following n idempotents. For $i = 1, \ldots, n$ we set

$$u_i^{(2^n)} := \sum_{j=0}^{2^{i-1}-1} \sum_{k=1+j\cdot 2^{n-i+1}}^{j\cdot 2^{n-i+1}+2^{n-i}} e_k^{(2^n)}. \tag{1}$$

For example, for $n = 3$ this means that

$$u_1^{(8)} = (1,1,1,1,0,0,0,0), \qquad u_2^{(8)} = (1,1,0,0,1,1,0,0),$$

$$u_3^{(8)} = (1,0,1,0,1,0,1,0).$$

Another way of viewing the $u_i^{(2^n)}$'s is the following. First set $u_1^{(2)} := e_1^{(2)} = (1,0)$. Then, for $n \geq 1$, decompose $R^{(2^{n+1})}$ as $R^{(2^n)} \oplus R^{(2^n)}$, set $u_1^{(2^{n+1})} := (1_{R^{(2^n)}}, 0_{R^{(2^n)}})$, and for $j = 2, \ldots, n+1$, set $u_j^{(2^{n+1})} := (u_{j-1}^{(2^n)}, u_{j-1}^{(2^n)})$.

LEMMA 1. *The set* $\{u_1^{(2^n)}, \ldots, u_n^{(2^n)}\}$ *is a set of idempotent generators of* $R^{(2^n)}$ *for every* $n \geq 1$.

Proof. If $n = 1$, then $u_1^{(2)} = e_1^{(2)} = (1,0)$, which, together with $1_{R^{(2)}} = (1,1)$, clearly generates $R^{(2)}$.

We proceed by induction on n. Therefore, suppose that the set $\{u_1^{(2^n)}, \dots, u_n^{(2^n)}\}$ generates $R^{(2^n)}$ for some $n \geq 1$. Let $\mathscr{A}$ denote the subalgebra of $R^{(2^{n+1})}$ generated by $\{u_1^{(2^{n+1})}, \dots, u_{n+1}^{(2^{n+1})}\}$. Then $(u_{j-1}^{(2^n)}, 0_{R^{(2^n)}}) = u_1^{(2^{n+1})} u_j^{(2^{n+1})} \in \mathscr{A}$ and $(0_{R^{(2^n)}}, u_{j-1}^{(2^n)}) = (1_{R^{(2^{n+1})}} - u_1^{(2^{n+1})}) u_j^{(2^{n+1})} \in \mathscr{A}$ for $j = 2, \dots, n+1$, which by the induction hypothesis completes the proof. ∎

We remark that the algebra $R^{(m)}$, for $m \geq 2$, cannot be generated by less than $\lceil \log_2 m \rceil$ idempotents. Indeed, for $m \geq 3$, let $p_1, \dots, p_{\lceil \log_2 m \rceil - 1}$ be any $\lceil \log_2 m \rceil - 1$ idempotents in $R^{(m)}$. Denote by P the set comprising 1 and all products of the elements $p_1, \dots, p_{\lceil \log_2 m \rceil - 1}$. Obviously, $|P| \leq 2^{\lceil \log_2 m \rceil - 1} < m$. The subalgebra $\mathscr{A}_m$ of $R^{(m)}$ generated by $\{p_1, \dots, p_{\lceil \log_2 m \rceil - 1}\}$ is the R-submodule of $\mathscr{A}_m$ generated by P, and so its rank is less than m. Therefore, $\mathscr{A}_m$ is a proper subalgebra of $R^{(m)}$.

Furthermore, if $m \leq 2^n$, then $R^{(m)}$ is a homomorphic image of $R^{(2^n)}$ via

$$\pi_{2^n, m} : (x_1, \dots, x_m, \dots, x_{2^n}) \mapsto (x_1, \dots, x_m),$$

and so it follows from Lemma 1 that $R^{(m)}$, for $2^{n-1} < m \leq 2^n$, can be generated by $n = \lceil \log_2 m \rceil$ idempotents. Thus we have proved the following result.

THEOREM 2. *Let R be a commutative ring, and let $m \geq 2$. The minimum number $\nu = \nu(R^{(m)})$ such that $R^{(m)}$ can be generated as an R-algebra by ν idempotents is $\lceil \log_2 m \rceil$, and $\pi_{2^{\lceil \log_2 m \rceil}, m}(\{u^{(2^{\lceil \log_2 m \rceil})}, \dots, u_{\lceil \log_2 m \rceil}^{(2^{\lceil \log_2 m \rceil})}\})$ is a set of idempotent generators of $R^{(m)}$.*

For example, with $m = 7$ we have $\lceil \log_2 m \rceil = 3$, and so with $u_1^{(8)}$, $u_2^{(8)}$, and $u_3^{(8)}$ as in the paragraph following (1), it follows from Theorem 2 that

$$\{(1,1,1,1,0,0,0), (1,1,0,0,1,1,0), (1,0,1,0,1,0,1)\}$$

is a set of idempotent generators of $R^{(7)}$.

Next we consider the $m \times m$ upper triangular matrix algebra $\mathscr{U}_m(R)$ (as an R-algebra).

For $m \geq 3$ we consider the (two-sided) ideal $\mathscr{I}_m$ of $\mathscr{U}_m(R)$ generated by the set

$$\left\{E_{i,j}^{(m)} \mid 1 \leq i \leq m-2, i+2 \leq j \leq m\right\}.$$

Here $E_{i,j}^{(m)}$ denotes the (i,j)th matrix unit in $\mathscr{U}_m(R)$, i.e., the matrix with 1 in position (i,j) and zeros elsewhere. We denote the main diagonal by $D_0^{(m)}$ and the ith diagonal above the main diagonal by $D_i^{(m)}$, $i = 1, \dots, m-1$. Then the set above comprises the matrix units with zeros on the two diagonals $D_0^{(m)}$ and $D_1^{(m)}$, and so $\mathscr{I}_m$ is the ideal of $\mathscr{U}_m(R)$ comprising all the matrices in $\mathscr{U}_m(R)$ with zeros on $D_0^{(m)}$ and $D_1^{(m)}$. For example, for

$m = 4$ we have that

$$\mathscr{I}_4 = \begin{bmatrix} 0 & 0 & R & R \\ & 0 & 0 & R \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}.$$

Our first aim now is to show (in Lemma 3) that, for $n \geq 3$, the quotient algebra $\mathscr{U}_{2^n}(R)/\mathscr{I}_{2^n}$ can be generated by n idempotents.

For $i = 1, \ldots, n$ we denote the diagonal matrix in $\mathscr{U}_{2^n}(R)$ with (j, j)th entry equal to the jth entry of $u_i^{(2^n)}$ (as in (1)), $j = 1, \ldots, 2^n$, by $U_i^{(2^n)}$, i.e., $U_i^{(2^n)}$ is the image of $u_i^{(2^n)}$ under the natural injection $\iota_{R^{(2^n)}, \mathscr{U}_{2^n}(R)} : R^{(2^n)} \hookrightarrow \mathscr{U}_{2^n}(R)$. For example, for $n = 3$ and $i = 2$ this means that we construct $U_2^{(8)}$ by placing $u_2^{(8)}$ on the main diagonal, i.e.,

$$U_2^{(8)} := \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & \bigcirc & \\ & & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{bmatrix}.$$

(Since we deal throughout the sequel with upper triangular matrices, we leave the part below the main diagonal vacant, implying that that part consists entirely of zeros.)

Using permutations together with the natural injection mentioned in the previous paragraph, we will in the next paragraph construct three idempotent matrices generating $\mathscr{U}_8(R)/\mathscr{I}_8$. Then in Lemma 3 we use, among others, induction and the ideas involved in the proof of Lemma 1 to obtain n idempotent generators for $\mathscr{U}_{2^n}(R)/\mathscr{I}_{2^n}$ for $n \geq 4$. Next, since $\mathscr{I}_{2^n}$ is a nilpotent ideal of $\mathscr{U}_{2^n}(R)$, we proceed in Lemma 4 and Corollary 5 by lifting the n idempotents in $\mathscr{U}_{2^n}(R)/\mathscr{I}_{2^n}$ to n idempotents in $\mathscr{U}_{2^n}(R)$ which generate $\mathscr{U}_{2^n}(R)$.

Consider the elements $u_1^{(8)} = (1, 1, 1, 1, 0, 0, 0, 0)$, $1_{R^{(8)}} - u_2^{(8)} = (0, 0, 1, 1, 0, 0, 1, 1)$, and $u_3^{(8)} = (1, 0, 1, 0, 1, 0, 1, 0)$ of $R^{(8)}$. By Lemma 1 the set $\{u_1^{(8)}, 1_{R^{(8)}} - u_2^{(8)}, u_3^{(8)}\}$ generates $R^{(8)}$. Let $\sigma \in S_8$ (the symmetric group on 8 symbols) be the permutation $(2\ 6\ 4)(3\ 5\ 7)$, and let $\tau : R^{(8)} \to R^{(8)}$ be the map

$$(x_1, \ldots, x_8) \mapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(8)}).$$

Then

$$\left\{u_2^{(8)}, \tau\left(1_{R^{(8)}} - u_2^{(8)}\right), u_3^{(8)}\right\} \text{ generates } R^{(8)}, \tag{2}$$

since $\tau(u_1^{(8)}) = (1,1,0,0,1,1,0,0) = u_2^{(8)}$ and $\tau(u_3^{(8)}) = u_3^{(8)}$. Note that

$$\tau\left(1_{R^{(8)}} - u_2^{(8)}\right) = (0,1,1,0,1,0,0,1). \tag{3}$$

Next we set

$$Y^{(8)} := U_2^{(8)} + E_{2,3}^{(8)} + E_{4,5}^{(8)} + E_{6,7}^{(8)} = \begin{bmatrix} 1 & 0 & & & & & & \bigcirc \\ & 1 & 1 & & & & & \\ & & 0 & 0 & & & & \\ & & & 0 & 1 & & & \\ & & & & 1 & 0 & & \\ & & & & & 1 & 1 & \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}. \tag{4}$$

If we call the map

$$r \mapsto \begin{bmatrix} r & & \bigcirc \\ & \ddots & \\ & & r \end{bmatrix}$$

the *scalar matrix injection*, then for $n \geq 3$ we construct $Y^{(2^{n+1})}$ as the sum of $E_{2^n,2^n+1}^{(2^{n+1})}$ and the image

$$\begin{bmatrix} Y^{(2^n)} & 0 \\ & Y^{(2^n)} \end{bmatrix}$$

of $Y^{(2^n)}$ under the scalar matrix injection of $\mathscr{U}_{2^n}(R)$ into $\mathscr{U}_{2^{n+1}}(R)$.

We also set

$$Z^{(8)} := \iota_{R^{(8)},\mathscr{U}_8(R)}\left(\tau\left(1_{R^{(8)}} - u_2^{(8)}\right)\right) + E_{1,2}^{(8)} + E_{3,4}^{(8)} + E_{5,6}^{(8)} + E_{7,8}^{(8)}.$$

Then by (3),

$$Z^{(8)} = \begin{bmatrix} 0 & 1 & & & & & & \\ & 1 & 0 & & & & \bigcirc & \\ & & 1 & 1 & & & & \\ & & & 0 & 0 & & & \\ & & & & 1 & 1 & & \\ & & & & & 0 & 0 & \\ & & & & & & 0 & 1 \\ & & & & & & & 1 \end{bmatrix}. \tag{5}$$

For $n \geq 4$ we construct $Z^{(2^n)}$ as the image

$$\begin{bmatrix} Z^{(8)} & & \bigcirc \\ & \ddots & \\ & & Z^{(8)} \end{bmatrix}$$

of $Z^{(8)}$ under the scalar matrix injection $\mathcal{U}_8(R)$ into $\mathcal{U}_{2^n}(R)$. Equivalently, for $n \geq 3$, $Z^{(2^{n+1})}$ may be viewed as the image

$$\begin{bmatrix} Z^{(2^n)} & 0 \\ & Z^{(2^n)} \end{bmatrix}$$

of $Z^{(2^n)}$ under the scalar matrix injection of $\mathcal{U}_{2^n}(R)$ into $\mathcal{U}_{2^{n+1}}(R)$.

Before we state Lemma 3, recall that

$$U_3^{(8)} := \begin{bmatrix} 1 & & & & & & \bigcirc & \\ & 0 & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & & \\ & & & & 1 & & & \\ & & & & & 0 & & \\ & & & & & & 1 & \\ & & & & & & & 0 \end{bmatrix}. \tag{6}$$

For $n \geq 3$ and for a matrix $X^{(2^n)}$ in $\mathcal{U}_{2^n}(R)$ we write $\overline{X}^{(2^n)}$ for its image in $\mathcal{U}_{2^n}(R)/\mathcal{I}_{2^n}$, and we write $\overline{X}^{(2^n)}$ as a matrix with nonzero elements of R allowed only on $D_0^{(2^n)}$ and $D_1^{(2^n)}$.

LEMMA 3. *The set $\{\overline{Y}^{(8)}, \overline{Z}^{(8)}, \overline{U}_3^{(8)}\}$ is a set of idempotent generators of the quotient algebra $\mathcal{U}_8(R)/\mathcal{I}_8$, and $\{\overline{Y}^{(2^n)}, \overline{Z}^{(2^n)}, \overline{U}_n^{(2^n)}, \overline{U}_1^{(2n)}, \ldots, \overline{U}_{n-3}^{(2^n)}\}$ is a set of idempotent generators of $\mathcal{U}_{2^n}(R)/\mathcal{I}_{2^n}$ for every $n \geq 4$.*

Proof. Direct verification shows that the mentioned matrices are idempotent.

We now use induction on n, starting with $n = 3$, i.e., with $2^n = 2^3 = 8$. First note that for every matrix

$$\overline{X}^{(8)} := \begin{bmatrix} x_{1,1} & x_{1,2} & & & & & & \\ & x_{2,2} & x_{2,3} & & & & \bigcirc & \\ & & x_{3,3} & x_{3,4} & & & & \\ & & & x_{4,4} & x_{4,5} & & & \\ & & & & x_{5,5} & x_{5,6} & & \\ & & & & & x_{6,6} & x_{6,7} & \\ & & & & & & x_{7,7} & x_{7,8} \\ & & & & & & & x_{8,8} \end{bmatrix}$$

in $\mathscr{U}_8(R)/\mathscr{I}_8$, we have that

$$\overline{U}_3^{(8)}\overline{X}^{(8)} - \overline{X}^{(8)}\overline{U}_3^{(8)}$$
$$= \begin{bmatrix} 0 & x_{1,2} & & & & & & \\ & 0 & -x_{2,3} & & & & \bigcirc & \\ & & 0 & x_{3,4} & & & & \\ & & & 0 & -x_{4,5} & & & \\ & & & & 0 & x_{5,6} & & \\ & & & & & 0 & -x_{6,7} & \\ & & & & & & 0 & x_{7,8} \\ & & & & & & & 0 \end{bmatrix}. \quad (7)$$

By (3)–(6) the main diagonals of $\overline{Y}^{(8)}$, $\overline{Z}^{(8)}$, and $\overline{U}_3^{(8)}$ are

$$u_2^{(8)}, \qquad \tau\left(1_{R^{(8)}} - u_2^{(8)}\right), \qquad \text{and} \qquad u_3^{(8)},$$

respectively, and so by (2) it follows from matrix multiplication in an upper triangular matrix algebra that, for $i = 1, \dots, 8$, there is a matrix $\overline{V}_i^{(8)}$ in the subalgebra $\mathscr{B}$ of $\mathscr{U}_8(R)/\mathscr{I}_8$ generated by $\{\overline{Y}^{(8)}, \overline{Z}^{(8)}, \overline{U}_3^{(8)}\}$ with 1 in position (i, i) and zeros elsewhere on $D_0^{(8)}$. Therefore (7) implies that

$$\overline{V}_i^{(8)}\left(\overline{U}_3^{(8)}\overline{X}^{(8)} - \overline{X}^{(8)}\overline{U}_3^{(8)}\right) = \pm x_{i,i+1}\overline{E}_{i,i+1}^{(8)}.$$

Hence, since for $i = 1, \dots, 7$, $\overline{Y}^{(8)}$ or $\overline{Z}^{(8)}$ has 1 in position $(i, i+1)$, it follows that $\overline{E}_{i,i+1}^{(8)} \in \mathscr{B}$. We conclude that $\mathscr{B} = \mathscr{U}_8(R)/\mathscr{I}_8$, i.e., $\mathscr{U}_8(R)/\mathscr{I}_8$ can be generated by $\{\overline{Y}^{(8)}, \overline{Z}^{(8)}, \overline{U}_3^{(8)}\}$.

Next, suppose that $\{\overline{Y}^{(2^n)}, \overline{Z}^{(2^n)}, \overline{U}_n^{(2^n)}, \overline{U}_1^{(2^n)}, \dots, \overline{U}_{n-3}^{(2^n)}\}$ generates $\mathscr{U}_{2^n}(R)/\mathscr{I}_{2^n}$ for some $n \geq 4$. It will be clear that our arguments, which will show that

$$\left\{\overline{Y}^{(2^{n+1})}, \overline{Z}^{(2^{n+1})}, \overline{U}_{n+1}^{(2^{n+1})}, \overline{U}_1^{(2^{n+1})}, \dots, \overline{U}_{n-2}^{(2^{n+1})}\right\} \text{ generates } \mathscr{U}_{2^{n+1}}(R)/\mathscr{I}_{2^{n+1}}, \quad (8)$$

cater for the transition from $n = 3$ to $n = 4$ as well. By the induction hypothesis, the arguments regarding the scalar matrix injection (preceding the statement of Lemma 3), the construction of $Y^{(2^{n+1})}$ and $Z^{(2^{n+1})}$, and the spirit of the proof of Lemma 1 it follows that the set

$$\Big\{\overline{U}_1^{(2^{n+1})}\overline{Y}^{(2^{n+1})}\overline{U}_1^{(2^{n+1})}, \overline{U}_1^{(2^{n+1})}\overline{Z}^{(2^{n+1})}\overline{U}_1^{(2^{n+1})}, \overline{U}_1^{(2^{n+1})}\overline{U}_{n+1}^{(2^{n+1})}\overline{U}_1^{(2^{n+1})},$$
$$\overline{U}_1^{(2^{n+1})}\overline{U}_1^{(2^{n+1})}\overline{U}_1^{(2^{n+1})}, \dots, \overline{U}_1^{(2^{n+1})}\overline{U}_{n-2}^{(2^{n+1})}\overline{U}_1^{(2^{n+1})}\Big\}$$

generates the subalgebra

$$
\begin{array}{c}
\qquad\qquad\qquad 2^n \\
\qquad\qquad\qquad \downarrow \\
\left[\begin{array}{cccccc}
R & R & 0 & \cdots & 0 & \\
 & \ddots & \ddots & \ddots & \vdots & \\
 & & \ddots & \ddots & 0 & \bigcirc \\
 & & & \ddots & R & \\
 & & & & R & \\
 & & & & & \\
 & & & & & \bigcirc
\end{array}\right]
\end{array}
$$

of $\mathscr{U}_{2^{n+1}}(R)/\mathscr{I}_{2^{n+1}}$. Note that this subalgebra represents the image of $\mathscr{U}_{2^n}(R)/\mathscr{I}_{2^n}$ under the injection map $r \mapsto \left[\begin{smallmatrix} r & 0 \\ & 0 \end{smallmatrix}\right]$ of $\mathscr{U}_{2^n}(R)/\mathscr{I}_{2^n}$ into $\mathscr{U}_{2^{n+1}}(R)/\mathscr{I}_{2^{n+1}}$. Similarly, with $\overline{1}$ denoting the identity of $\mathscr{U}_{2^{n+1}}(R)/\mathscr{I}_{2^{n+1}}$, and with $\overline{W}^{(2^{n+1})} := \overline{1} - \overline{U}_1^{(2^{n+1})}$, it follows that

$$
\Big\{\overline{W}^{(2^{n+1})}\overline{Y}^{(2^{n+1})}\overline{W}^{(2^{n+1})}, \overline{W}^{(2^{n+1})}\overline{Z}^{(2^{n+1})}\overline{W}^{(2^{n+1})}, \overline{W}^{(2^{n+1})}\overline{U}_{n+1}^{(2^{n+1})}\overline{W}^{(2^{n+1})},
$$
$$
\overline{W}^{(2^{n+1})}\overline{U}_1^{(2^{n+1})}\overline{W}^{(2^{n+1})}, \dots, \overline{W}^{(2^{n+1})}\overline{U}_{n-2}^{(2^{n+1})}\overline{W}^{(2^{n+1})}\Big\}
$$

generates the subalgebra

$$
\begin{array}{c}
\qquad\qquad\qquad 2^n + 1 \\
\qquad\qquad\qquad \downarrow \\
\left[\begin{array}{cccccc}
 & & & & & \\
\bigcirc & & & & \bigcirc & \\
 & R & R & 0 & \cdots & 0 \\
 & & \ddots & \ddots & \ddots & \vdots \\
 & & & \ddots & \ddots & 0 \\
 & & & & \ddots & R \\
 & & & & & R
\end{array}\right]
\end{array}
$$

of $\mathscr{U}_{2^{n+1}}(R)/\mathscr{I}_{2^{n+1}}$.

Up to now we have shown that all the matrix units in $\mathscr{U}_{2^{n+1}}(R)/\mathscr{I}_{2^{n+1}}$, except possibly $\overline{E}_{2^n, 2^n+1}^{(2^{n+1})}$, are in $\mathscr{C}$, where $\mathscr{C}$ is the subalgebra of $\mathscr{U}_{2^{n+1}}(R)/\mathscr{I}_{2^{n+1}}$ generated by $\{\overline{Y}^{(2^{n+1})}, \overline{Z}^{(2^{n+1})}, \overline{U}_{n+1}^{(2^{n+1})}, \overline{U}_1^{(2^{n+1})}, \dots, \overline{U}_{n-2}^{(2^{n+1})}\}$.

Therefore $\overline{Y}^{(2^{n+1})} - \overline{E}^{(2^{n+1})}_{2^n, 2^n+1} \in \mathscr{C}$, and hence $\overline{E}^{(2^{n+1})}_{2^n, 2^n+1} \in \mathscr{C}$. Thus $\mathscr{C} = \mathscr{U}_{2^{n+1}}(R)/\mathscr{I}_{2^{n+1}}$, which establishes (8). ▮

The following result holds for every ring S, not necessarily commutative.

LEMMA 4. *Let S be a ring with subring A and nilpotent ideal I. If the natural ring homomorphism $A \to S/I^2$ is an epimorphism, then $A = S$.*

Proof. Let N be the index of nilpotency of I. Then $I^N = \{0\} \subseteq A$. We use induction to show that $I^n \subseteq A$ for $n = N, N-1, \ldots, 2$. Suppose that $I^{n+1} \subseteq A$ for some $n+1 \leq N$. Since $A \to S/I^2$ is an epimorphism, it follows that $A + I^2 = S$. Therefore $(A + I^2) \cap I = I$. But $(A + I^2) \cap I = (A \cap I) + I^2$, and so $(A \cap I) + I^2 = I$. Raising both sides to the power n, we obtain, using the induction hypothesis, that $I^n \subseteq A$. Thus $I^2 \subseteq A$, and so $A = A + I^2 = S$. ▮

Let $m \geq 3$, and let $\{\overline{W}^{(m)}_1, \ldots, \overline{W}^{(m)}_k\}$ be a set of k idempotent generators of $\mathscr{U}_m(R)/\mathscr{I}_m$, for some k. Since $\mathscr{I}_m$ is a nilpotent ideal of $\mathscr{U}_m(R)$, we can lift $\overline{W}^{(m)}_i$ to an idempotent $W^{(m)}_i$ of $\mathscr{U}_m(R)$, $i = 1, \ldots, k$, and so, by letting $\mathscr{A}$ be the subalgebra of $\mathscr{U}_m(R)$ generated by $\{W^{(m)}_1, \ldots, W^{(m)}_k\}$, Lemma 4 implies that $\mathscr{A} = \mathscr{U}_m(R)$. We thus conclude from Lemma 3 and Lemma 4 that

COROLLARY 5. *The set $\{Y^{(8)}, Z^{(8)}, U^{(8)}_3\}$ is a set of idempotent generators of the algebra $\mathscr{U}_8(R)$, and $\{Y^{(2^n)}, Z^{(2^n)}, U^{(2^n)}_n, U^{(2^n)}_1, \ldots, U^{(2^n)}_{n-3}\}$ is a set of idempotent generators of $\mathscr{U}_{2^n}(R)$ for every $n \geq 4$.*

If $m' \leq m$, then $\mathscr{U}_{m'}(R)$ is a homomorphic image of $\mathscr{U}_m(R)$ via $\psi_{m,m'} : \mathscr{U}_m(R) \to \mathscr{U}_{m'}(R)$, with

$$\psi_{m,m'} : \begin{bmatrix} a_{1,1} & \cdots & a_{1,m'} & a_{1,m'+1} & \cdots & a_{1,m} \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & a_{m',m'} & a_{m',m'+1} & \cdots & a_{m',m} \\ & & & a_{m'+1,m'+1} & \cdots & a_{m'+1,m} \\ & & & & \ddots & \vdots \\ & & & & & a_{m,m} \end{bmatrix}$$

$$\mapsto \begin{bmatrix} a_{1,1} & \cdots & a_{1,m'} \\ & \ddots & \vdots \\ & & a_{m',m'} \end{bmatrix}.$$

We can now state our main result.

THEOREM 6. *Let R be a commutative ring, and let $m \geq 2$.*

(i) *The minimum number $\nu = \nu(\mathscr{U}_m(R))$ such that the $m \times m$ upper triangular matrix algebra $\mathscr{U}_m(R)$ over R can be generated as an R-algebra by ν*

idempotents, is given by

$$\nu(\mathscr{U}_m(R)) = \begin{cases} \lceil \log_2 m \rceil + 1, & \text{if } m = 2, 3, 4; \\ \lceil \log_2 m \rceil, & \text{if } m \geq 5. \end{cases}$$

(ii) *The set* $\{[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}], [\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}]\}$ *is a set of idempotent generators of* $\mathscr{U}_2(R)$, *and* $\psi_{8,m}(\{Y^{(8)}, Z^{(8)}, U_3^{(8)}\})$ *is a set of idempotent generators of* $\mathscr{U}_m(R)$ *for* $m = 3, \ldots, 8$.

(iii) *For* $m \geq 9$, *a set of idempotent generators of* $\mathscr{U}_m(R)$ *is given by*

$$\psi_{2^{\lceil \log_2 m \rceil}, m}\left(\left\{Y^{(2^{\lceil \log_2 m \rceil})}, Z^{(2^{\lceil \log_2 m \rceil})}, U^{(2^{\lceil \log_2 m \rceil})}_{\lceil \log_2 m \rceil}, U^{(2^{\lceil \log_2 m \rceil})}_{1}, \ldots, U^{(2^{\lceil \log_2 m \rceil})}_{\lceil \log_2 m \rceil - 3}\right\}\right).$$

Proof. Since $R^{(m)}$ is a homomorphic image of $\mathscr{U}_m(R)$, it follows from Theorem 2 that $\nu(\mathscr{U}_m(R)) \geq \lceil \log_2 m \rceil$ if $m \geq 2$. Hence, since $\lceil \log_2 5 \rceil = 3$, we conclude from Lemma 3, Corollary 5, and the fact that $\mathscr{U}_{m'}(R)$ is a homomorphic image of $\mathscr{U}_m(R)$ if $m' \leq m$, that $\nu(\mathscr{U}_m(R)) = \lceil \log_2 m \rceil$ if $m \geq 5$.

Next we consider the cases $m = 3$ or $m = 4$. By [4] or [2, Theorem 6] every algebra generated by two idempotents satisfies the standard polynomial identity

$$\sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} = 0$$

of degree 4, with $x_1, \ldots, x_4$ noncommuting indeterminates. However, $\mathscr{U}_3(R)$ and $\mathscr{U}_4(R)$ do not satisfy the mentioned polynomial identity, which can be seen by using $E^{(m)}_{1,1}$, $E^{(m)}_{1,2}$, $E^{(m)}_{2,2}$, and $E^{(m)}_{2,3}$, $m = 3, 4$. Therefore Lemma 3 and Corollary 5 imply that $\nu(\mathscr{U}_m(R)) = 3 = \lceil \log_2 m \rceil + 1$ if $m = 3, 4$.

The preceding two paragraphs and the paragraph preceding Theorem 6 also cater for (iii) and the second part of (ii). Finally, $\mathscr{U}_2(R)$ can be generated by $\psi_{8,2}(\{Y^{(8)}, Z^{(8)}, U_3^{(8)}\})$. Since $\psi_{8,2}(Y^{(8)}) = 1_{\mathscr{U}_2(R)}$, it follows that $\mathscr{U}_2(R)$ is generated (as an R-algebra) by $\{[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}], [\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}]\}$. Furthermore, $\mathscr{U}_2(R)$ cannot be generated by less than two idempotents. Indeed, every subalgebra $\mathscr{E}$ of $\mathscr{U}_2(R)$ generated by less than 2 idempotents is commutative, and so the argument preceding Theorem 2 shows that $\mathscr{E}$ is an R-module of rank at most 2. However, $\mathscr{U}_2(R)$ is an R-module of rank 3. ∎

For example, for $m = 10$ the four idempotents

$$\begin{bmatrix} 1 & 0 & & & & & & & & \\ & 1 & 1 & & & & \bigcirc & & & \\ & & 0 & 0 & & & & & & \\ & & & 0 & 1 & & & & & \\ & & & & 1 & 0 & & & & \\ & & & & & 1 & 1 & & & \\ & & & & & & 0 & 0 & & \\ & & & & & & & 0 & 1 & \\ & & & & & & & & 1 & 0 \\ & & & & & & & & & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & & & & & & & & \\ & 1 & 0 & & & & \bigcirc & & & \\ & & 1 & 1 & & & & & & \\ & & & 0 & 0 & & & & & \\ & & & & 1 & 1 & & & & \\ & & & & & 0 & 0 & & & \\ & & & & & & 0 & 1 & & \\ & & & & & & & 1 & 0 & \\ & & & & & & & & 0 & 1 \\ & & & & & & & & & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & & & & & & & & & \\ & 0 & & & & & \bigcirc & & & \\ & & 1 & & & & & & & \\ & & & 0 & & & & & & \\ & & & & 1 & & & & & \\ & & & & & 0 & & & & \\ & & & & & & 1 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & & & & & & & & & \\ & 1 & & & & & \bigcirc & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ & & & & & 1 & & & & \\ & & & & & & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 0 & \\ & & & & & & & & & 0 \end{bmatrix}$$

generate $\mathscr{U}_{10}(R)$.

We finally wish to mention that we have used MAGMA extensively during this project.

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