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The Minimum Number of Idempotent Generators of a Complete Blocked Triangular Matrix Algebra

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Let R be a complete blocked triangular matrix algebra over an infinite field F. Assume that R is not an upper triangular matrix algebra or a full matrix algebra. We prove that the minimum number $\nu = \nu(R)$ such that R can be generated as an F-algebra by ν idempotents, is given by

$$\nu(R) = \begin{cases} 3 & \text{if } m_1 \leq 8, \\ \lceil \log_2 m_1 \rceil & \text{if } m_1 > 8, \end{cases}$$

where m_1 is the number of 1×1 diagonal blocks of R. We also show that R can be generated as an F-algebra by two elements, and if $m_1 = 0$, R can be generated by an idempotent and a nilpotent element.

Key Words: idempotents; complete blocked triangular matrix algebras.

1. INTRODUCTION

Denote by $\mathbb{M}_n^m(F)$ the direct sum of m copies of the full $n \times n$ matrix algebra over a field F. Krupnik showed in [2] that the minimum number $\nu(\mathbb{M}_n^m(F))$ of idempotents needed to generate $\mathbb{M}_n^m(F)$ as an F-algebra over an infinite field F is given by

$$\nu\big(\mathbb{M}_n^m(F)\big) = \begin{cases} 2 & \text{if } n = 2, \\ 3 & \text{if } n \geq 3. \end{cases}$$

Also, for each finitely generated Banach algebra A there is a number n_0 such that the algebra $\mathbb{M}_n(A)$ can be generated by three idempotents whenever $n \geq n_0$ (see [3]). In [1] it was shown that the minimum number

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$\nu(\mathbb{U}_n(S))$ of idempotents needed to generate the $n \times n$ upper triangular matrix algebra $\mathbb{U}_n(S)$ as an S-algebra over an arbitrary commutative ring S, is given by

$$\nu(\mathbb{U}_n(S)) = \begin{cases} \lceil \log_2 n \rceil + 1 & \text{if } n = 2, 3, 4, \\ \lceil \log_2 n \rceil & \text{if } n \geq 5. \end{cases}$$

Consider the subalgebra,

$$\mathbb{M}_{r_1, r_2, \ldots, r_t}(F) := \begin{bmatrix} \mathbb{M}_{r_1}(F) & \mathbb{M}_{r_1 \times r_2}(F) & \cdots & \mathbb{M}_{r_1 \times r_t}(F) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbb{M}_{r_{t-1} \times r_t}(F) \\ 0 & \cdots & 0 & \mathbb{M}_{r_t}(F) \end{bmatrix}, \quad (1)$$

of the full matrix algebra $\mathbb{M}_n(F)$ $(= \mathbb{M}_{n \times n}(F))$ over a field F, where $r_1 + r_2 + \cdots + r_t = n$. As in [4], we call $\mathbb{M}_{r_1, r_2, \ldots, r_t}(F)$ a complete blocked triangular matrix algebra. For $X \in \mathbb{M}_{r_1, r_2, \ldots, r_k}(F)$ we call the submatrix of X corresponding to the position of $\mathbb{M}_{r_i \times r_j}(F)$ in (1) the (i, j)th block of X and denote it by $X_{[i,j]}$. If $i = j$ (respectively, $i \neq j$), then we call $X_{[i,j]}$ the ith diagonal (respectively, a non-diagonal) block of X.

In this paper we show that if R is a complete blocked triangular matrix algebra over an infinite field and if R is not an upper triangular matrix algebra or a full matrix algebra, then

$$\nu(R) = \begin{cases} 3 & \text{if } m_1 \leq 8, \\ \lceil \log_2 m_1 \rceil & \text{if } m_1 > 8, \end{cases}$$

where m_1 is the number of 1×1 diagonal blocks of R. In fact, this result holds for any field with at least $g + 1$ elements, where g is the maximum number of diagonal blocks of any specific size greater than or equal to 2.

We wish to point out that we cannot in general rearrange the diagonal blocks in $\mathbb{M}_{r_1, r_2, \ldots, r_t}(F)$ to group together the diagonal blocks of the same size, since, for example, the complete blocked triangular matrix algebras

$$\mathbb{M}_{2,3}(F) = \begin{bmatrix} F & F & F & F & F \\ F & F & F & F & F \\ 0 & 0 & F & F & F \\ 0 & 0 & F & F & F \\ 0 & 0 & F & F & F \end{bmatrix}$$

and

$$\mathbb{M}_{3,2}(F) = \begin{bmatrix} F & F & F & F & F \\ F & F & F & F & F \\ F & F & F & F & F \\ 0 & 0 & 0 & F & F \\ 0 & 0 & 0 & F & F \end{bmatrix}$$

are not isomorphic. In fact, with $\mathscr{T}$ denoting the Jacobson radical, we have that

$$\mathscr{T}(\mathbb{M}_{2,3}(F)) = \begin{bmatrix} 0 & 0 & F & F & F \\ 0 & 0 & F & F & F \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathscr{T}(\mathbb{M}_{3,2}(F)) = \begin{bmatrix} 0 & 0 & 0 & F & F \\ 0 & 0 & 0 & F & F \\ 0 & 0 & 0 & F & F \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and so it is clear that $\mathscr{T}(\mathbb{M}_{2,3}(F))$ is a left $\mathbb{M}_{2,3}(F)$-module of rank 3, whereas $\mathscr{T}(\mathbb{M}_{3,2}(F))$ is a left $\mathbb{M}_{3,2}(F)$-module of rank 2.

For an F-algebra R we denote by $\mu(R)$ the minimum number of elements needed to generate R as an F-algebra. Thus if $\mu(R) = \mu$, then R is a quotient of the polynomial F-algebra in μ non-commuting variables. In the second section of this paper we determine $\mu(R)$, where R is a direct sum of full matrix algebras over a field, and in the final section we determine $\mu(R)$ and $\nu(R)$, for a complete blocked triangular matrix algebra R.

2. GENERATORS FOR $\oplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)$

In the remainder of this paper, we denote by F a field that is not necessarily infinite. In the first result we determine $\mu(\mathbb{M}_n^m(F))$.

For $n \geq 2$ and $\alpha \in F \setminus \{0\}$ denote by $P_{n,\alpha}$, Q_n, and R_n the following idempotents in $\mathbb{M}_n(F)$,

$$P_{n,\alpha} := (\alpha - 1)e_{1,2} + \sum_{1 \leq 2k-1 \leq n} e_{2k-1,2k-1} + \sum_{2 \leq 2k \leq n} e_{2k-1,2k}$$

$$= \begin{bmatrix} 1 & \alpha & & & \\ & 0 & 0 & & \\ & & 1 & 1 & \\ & & & \ddots & \ddots \end{bmatrix};$$

$$Q_n := \sum_{2 \leq 2k \leq n} e_{2k,2k} + \sum_{3 \leq 2k+1 \leq n} e_{2k,2k+1}$$

$$= \begin{bmatrix} 0 & 0 & & & \\ & 1 & 1 & & \\ & & 0 & 0 & \\ & & & \ddots & \ddots \end{bmatrix};$$

$$R_n := e_{n,1} + e_{n,n} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Here $e_{i,j}$ denotes the (i,j)th matrix unit, i.e., the matrix with 1 in position (i,j) and zeros elsewhere.

For $\alpha_1, \ldots, \alpha_m$ in $F \setminus \{0\}$ we define the following idempotents in $\mathbb{M}_n^m(F)$:

$$P_n^{\alpha_1,\ldots,\alpha_m} := \left(P_{n,\alpha_1}, \ldots, P_{n,\alpha_m}\right), \qquad Q_n^m := (Q_n, \ldots, Q_n) \quad \text{and}$$

$$R_n^m := (R_n, \ldots, R_n).$$

THEOREM 2.1. *Assume $|F| \geq m+1$ and $n > 1$. Then $\mu(\mathbb{M}_n^m(F)) = 2$. (Thus the multiplicative identity and two other elements generate $\mathbb{M}_n^m(F)$ as an F-algebra.) More specific, if $\alpha_1, \ldots, \alpha_m$ are distinct non-zero elements of F, then $P_n^{\alpha_1,\ldots,\alpha_m} + Q_n^m$ and R_n^m generate $\mathbb{M}_n^m(F)$ as an F-algebra. Also, $\mathbb{M}_n^m(F)$ can be generated by an idempotent and a nilpotent element.*

Proof. Because $\mathbb{M}_n^m(F)$ is not commutative, $\mathbb{M}_n^m(F)$ cannot be generated by a single element. To show that $P_n^{\alpha_1,\ldots,\alpha_m} + Q_n^m$ and R_n^m generate $\mathbb{M}_n^m(F)$, simply use the argument in part (3) of Theorem 5 in [2]. Finally, if we denote by I the multiplicative identity of $\mathbb{M}_n^m(F)$, then $P_n^{\alpha_1,\ldots,\alpha_m} + Q_n^m - I$ is nilpotent. ∎

Let n_j and m_j for $j = 1, \ldots, k$ be positive integers with $n_1 < n_2 < \cdots < n_k$. In the following two results we determine $\mu(\bigoplus_{j=1}^k \mathbb{M}_{n_j}^{m_j}(F))$. First we establish some notation. We denote by π_{n_i}: $\bigoplus_{j=1}^k \mathbb{M}_{n_j}^{m_j}(F) \to M_{n_i}^{m_i}(F)$ the projection map from $\bigoplus_{j=1}^k \mathbb{M}_{n_j}^{m_j}(F)$ onto $\mathbb{M}_{n_i}^{m_i}(F)$, by ι_{n_i} the injection in the opposite direction, and by id: $\bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F) \to \bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)$ the identity map.

LEMMA 2.2. *Define $g \in \mathbb{Z}$ as*

$$g := \begin{cases} \max\{m_1, \ldots, m_k\} & \text{if } n_1 > 1, \\ \max\{m_2, \ldots, m_k\} & \text{if } k > 1 \text{ and } n_1 = 1. \end{cases}$$

Assume that $|F| \geq g + 1$. Let $\alpha_i \in F \setminus \{0\}$ for $1 \leq i \leq g$ such that $\alpha_{i_1} \neq \alpha_{i_2}$ if $i_1 \neq i_2$.

(a) *Assume that $n_1 > 1$.*

Define $P, Q, R \in \bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)$ as

$$P := \left(P_{n_1}^{\alpha_1, \ldots, \alpha_{m_1}}, \ldots, P_{n_k}^{\alpha_1, \ldots, \alpha_{m_k}}\right),$$

$$Q := \left(Q_{n_1}^{m_1}, \ldots, Q_{n_k}^{m_k}\right),$$

$$R := \left(R_{n_1}^{m_1}, \ldots, R_{n_k}^{m_k}\right).$$

Then $P + Q$ and R generate $\bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)$ as an F-algebra.

(b) *Assume that $n_1 = 1$ and $k \geq 2$. Denote by 0_n^m the additive identity of $\mathbb{M}_n^m(F)$. Define $P, Q, R \in \bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)$ as*

$$P := \left(0_{n_1}^{m_1}, P_{n_2}^{\alpha_1, \ldots, \alpha_{m_2}}, \ldots, P_{n_k}^{\alpha_1, \ldots, \alpha_{m_k}}\right),$$

$$Q := \left(0_{n_1}^{m_1}, Q_{n_2}^{m_2}, \ldots, Q_{n_k}^{m_k}\right),$$

$$R := \left(0_{n_1}^{m_1}, R_{n_2}^{m_2}, \ldots, R_{n_k}^{m_k}\right).$$

Let $A_1, \ldots, A_l$ be elements of $\bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)$ such that:

(i) *$\pi_{n_1}(A_1), \ldots, \pi_{n_1}(A_l)$ generate $\mathbb{M}_{n_1}^{m_1}(F)$ $(= F^{m_1})$ as an F-algebra;*

(ii) *$P + Q, R \in \{(\mathrm{id} - \iota_{n_1}\pi_{n_1})(A_1), \ldots, (\mathrm{id} - \iota_{n_1}\pi_{n_1})(A_l)\}$.*

Then $A_1, \ldots, A_l$ generate $\bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)$ as an F-algebra.

Proof. We only prove (b). A similar argument will work for (a). Denote by S the subalgebra of $\bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)$ generated by $A_1, \ldots, A_l$, and by I the identity element of $\bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)$. We first show that $i_{n_k}(\mathbb{M}_{n_k}^{m_k}(F)) \subseteq S$. By (ii), $P + Q = (\mathrm{id} - \iota_{n_1}\pi_{n_1})(A_r)$ for some A_r, $1 \leq r \leq l$. Therefore the construction of P and Q implies that $\pi_{n_k}(A_r - I)^{n_k - 1}$ equals

$$\left(\begin{bmatrix} 0 & \ldots & 0 & \alpha_1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \cdots & 0 & \alpha_{m_k} \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}\right)$$

$$= (\alpha_1 e_{1, n_k}, \ldots, \alpha_{m_k} e_{1, n_k}),$$

and $\pi_{n_i}(A_r - I)^{n_k - 1} = 0$ for $1 < i < k$. We now construct a matrix B in S such that $\pi_{n_1}(B) = 0$ and $\pi_{n_i}(B) = \pi_{n_i}(A_r - I)^{n_k - 1}$ if $1 < i \leq k$. To this end, note that $\pi_{n_i}(A_r - I)^{n_k} = 0$ for $1 < i \leq k$. Say, $\pi_{n_1}(A_r - I)^{n_k} = (a_1, \dots, a_{m_1})$, with $a_1, \dots, a_{m_1} \in F$. For all i with $a_i \neq 0$ let b_i be arbitrary in F, and let $b_i = 0$ if $a_i = 0$. For $s = 1, \dots, m_1$, let

$$c_s = \begin{cases} a_s^{-1} b_s & \text{if } a_s \neq 0, \\ 0 & \text{if } a_s = 0. \end{cases}$$

Since $\pi_{n_1}(A_1), \dots, \pi_{n_1}(A_l)$ generate F^{m_1}, there is a matrix C in S with $\pi_{n_1}(C) = (c_1, \dots, c_{m_1})$. Then $\pi_{n_1}((A_r - I)^{n_k} C) = (b_1, \dots, b_{m_1})$ and $\pi_{n_i}((A_r - I)^{n_k} C) = 0$ for $1 < i \leq k$. Since $(A_r - I)^{n_k} C \in S$ and since b_i are arbitrary in F if $a_i \neq 0$, it follows that if A is any matrix in $\bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)$ with zeros in the positions where $(A_r - I)^{n_k}$ has zeros, then $A \in S$. Therefore $\iota_{n_1} \pi_{n_1}(A_r - I)^{n_k - 1} \in S$. Consequently, with $B := (A_r - I)^{n_k - 1} - \iota_{n_1} \pi_{n_1}(A_r - I)^{n_k - 1}$, we have $B \in S$, $\pi_{n_1}(B) = 0$, and $\pi_{n_i}(B) = \pi_{n_i}(A_r - I)^{n_k - 1}$ if $1 < i \leq k$. Hence, $\pi_{n_i}(B) = 0$ if $1 \leq i < k$, and $\pi_{n_k}(B) = (\alpha_1 e_{1, n_k}, \dots, \alpha_{m_k} e_{1, n_k})$.

By (ii) and Theorem 2.1 we have $\pi_{n_k}(S) = \mathbb{M}_{n_k}^{m_k}(F)$, and so if a is an arbitrary element of F and $1 \leq u, v \leq n_k$, then there are matrices D_1 and D_2 in S with $\pi_{n_k}(D_1) = (a \alpha_1^{-1} e_{u,1}, 0, \dots, 0)$ and $\pi_{n_k}(D_2) = (e_{n_k, v}, 0, \dots, 0)$. Hence $\pi_{n_i}(D_1 B D_2) = 0$ if $1 \leq i < k$ and $\pi_{n_k}(D_1 B D_2) = (a e_{u,v}, 0, \dots, 0)$. Since a, u, and v are arbitrary, $\iota_{n_k}(\mathbb{M}_{n_k}^{m_k}(F)) \subseteq S$. Now consider $(A_r - I)^{n_{k-1} - 1}$ and repeat the above argument to eventually obtain that $\iota_{n_2}(\mathbb{M}_{n_2}^{m_2}(F)) + \cdots + \iota_{n_k}(\mathbb{M}_{n_k}^{m_k}(F)) \subseteq S$. Finally, (i) completes the proof. ∎

THEOREM 2.3. *Let $g := \max\{m_1, \dots, m_k\}$ and assume that $|F| \geq g + 1$. Then*

$$\mu\Big(\bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)\Big) = \begin{cases} 1 & \text{if } n_k = 1 \text{ (and thus } k = 1) \text{ and } m_k\ (= m_1) > 1, \\ 2 & \text{if } n_k > 1. \end{cases}$$

Also, if $n_1 > 1$, then $\bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)$ can be generated (as an F-algebra) by an idempotent and a nilpotent element.

Proof. Note that if α is an element in F^m with distinct non-zero components, then it follows from the Vandermonde determinant that $\alpha, \alpha^2, \dots, \alpha^m$ are linearly independent over F. For the case $n_k > 1$, use Lemma 2.2 and the fact that a single element cannot generate a non-commutative algebra, to determine $\mu(\bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F))$. Note that if $n_1 > 1$, then $P + Q - I$ is nilpotent and R is idempotent, where P, Q, and R are defined as in Lemma 2.2(a). Also, from Lemma 2.2 it follows that $P + Q - I$ and R generate $\bigoplus_{j=1}^{k} \mathbb{M}_{n_j}^{m_j}(F)$. ∎

3. GENERATORS FOR $\mathbb{M}_{r_1, r_2, \dots, r_t}(F)$

In the first result we obtain a lower bound for $\nu(\mathbb{M}_{r_1, \dots, r_t}(F))$.

THEOREM 3.1. *Let m_1 be the number of 1×1 diagonal blocks of $\mathbb{M}_{r_1, \dots, r_t}(F)$, and assume that $M_{r_1, \dots, r_t}(F)$ is not an upper triangular or full matrix algebra. Then*

$$\nu(R) \geq \begin{cases} 3 & \text{if } m_1 \leq 8, \\ \lceil \log_2 m_1 \rceil & \text{if } m_1 > 8. \end{cases}$$

Proof. By [5], every algebra generated by two idempotents satisfies the standard polynomial identity

$$\sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} = 0,$$

of degree 4, where x_1, x_2, x_3, and x_4 are non-commutating indeterminates. However, $\mathbb{M}_{r_1, \dots, r_t}(f)$ does not satisfy the mentioned polynomial identity, which can be seen by using $e_{1,1}$, $e_{1,2}$, $e_{2,2}$, and $e_{2,3}$.

Assume that $m_1 \geq 1$. From Theorem 2 in [1] we have that $\nu(F^{m_1}) \geq \lceil \log_2 m_1 \rceil$. Since there is an F-algebra epimorphism from $\mathbb{M}_{r_1, \dots, r_t}(F)$ onto F^{m_1}, $\nu(\mathbb{M}_{r_1, \dots, r_t}(f)) \geq \max\{3, \lceil \log_2 m_1 \rceil\}$. ∎

For $A \in \mathbb{M}_n(F)$ we denote by $D_1(A) \in F^n$ the main diagonal of A, and by $D_2(A) \in F^{n-1}$ the superdiagonal (diagonal right above the main diagonal) of A. If we do not refer to a specific matrix, then we only write D_1 and D_2, respectively.

Recall that $r_1 + \cdots + r_t = n$ in $\mathbb{M}_{r_1, \dots, r_t}(F)$. Let N^2 be the vector space over F generated by $e_{i,j} \in \mathbb{M}_{r_1, \dots, r_t}(F)$, where $i + 2 \leq j \leq n$; i.e., N^2 comprises all the matrices in $\mathbb{M}_{r_1, \dots, r_t}(F)$ with zeros on and below D_2. In the following result we show that if S is a subalgebra of $\mathbb{M}_{r_1, \dots, r_t}(F)$ and if the natural F-linear map from S to $\mathbb{M}_{r_1, \dots, r_t}(F)/N^2$ is onto, then $S = \mathbb{M}_{r_1, \dots, r_t}(F)$.

LEMMA 3.2. *Let R be a ring with subring S and let N be a subgroup of $(R, +)$ such that $x^l = 0$ for all $x \in N$, for some fixed positive integer l. Also assume that $N \supseteq N^2$, where N^2 is the subgroup of $(R, +)$ generated by $\{xy | x, y \in N\}$. Then $S = R$ if the natural group homomorphism from S to R/N^2 is onto.*

Proof. The proof is similar to the proof of Lemma 4 in [1]. ∎

In the following result we specialize Lemma 3.2 to obtain a criterion for a subalgebra of $\mathbb{M}_{r_1, \dots, r_t}(F)$ to be equal to $\mathbb{M}_{r_1, \dots, r_t}(F)$.

THEOREM 3.3. *Let S be a subalgebra of $\mathbb{M}_{r_1,\ldots,r_t}(F)$, with $t > 1$, such that the projection from S to $\bigoplus_{j=1}^t \mathbb{M}_{r_j}(F)$ is onto. If for each integer i, with $1 \le i \le t-1$, there is a matrix A in S such that the entry in the bottom left corner of $A_{[i,i+1]}$ is non-zero and all the entries in the last row of $A_{[i,i]}$ and all the entries in the first column of $A_{[i+1,i+1]}$ are zero, then $S = \mathbb{M}_{r_1,\ldots,r_t}(F)$.*

Proof. Since the projection from S to $\bigoplus_{j=1}^t \mathbb{M}_{r_j}(F)$ is onto, there is a matrix B in S such that the entry in the bottom right corner of $B_{[i,j]}$ is 1 and all the other entries in all the diagonal blocks of B are 0. From the assumption on A it is clear that all the diagonal blocks of B^2A are zero and that the entry in the bottom left corner of $(B^2A)_{[i,i+1]}$ is the only non-zero entry in $D_2(B^2A)$. In particular, B^2A is strictly upper triangular and $D_2(B^2A)$ has precisely one non-zero entry, viz. in the bottom left corner of $(B^2A)_{[i,i+1]}$.

Again, since the projection from S to $\bigoplus_{j=1}^t \mathbb{M}_{r_j}(F)$ is onto, thus it follows that for any position in any diagonal block or any position on D_2 there is a matrix in S with a non-zero entry in that particular position and 0 elsewhere in the diagonal blocks and on D_2. The desired result now follows from the paragraph preceding Lemma 3.2. ∎

COROLLARY 3.4. *Let S be a subalgebra of $\mathbb{M}_{r_1,\ldots,r_t}(F)$. Assume that $t > 1$ and that the projection from S to $\bigoplus_{j=1}^t M_{r_j}(F)$ is onto. Also assume that there is a matrix A in S for each integer i with $1 \le i \le t-1$ such that* (I) *or* (II) *holds*:

(I) *There is a non-zero entry in position (r,s) in $A_{[i,i+1]}$, for some r and s, and the rth row of $A_{[i,i]}$ and sth column of $A_{[i+1,i+1]}$ are zero.*

(II) *There is a non-zero entry in position $(r,1)$ of $A_{[i,i+1]}$, for some r. If this non-zero entry is in the bottom left corner of $A_{[i,i+1]}$, then the first entry is the only possible non-zero entry in the first column of $A_{[i+1,i+1]}$. Otherwise all entries in the first column of $A_{[i+1,i+1]}$ are equal to zero. In both cases the last entry is the only possible non-zero entry in the rth row of $A_{[i,i]}$. There is also a matrix B in S such that the entry in the bottom right corner of $B_{[i,i]}$ or the entry in the top left corner of $B_{[i+1,i+1]}$ is non-zero, but not both are non-zero. All other entries in the last row of $B_{[i,i]}$ and first column of $B_{[i+1,i+1]}$ are zero. The entry on the superdiagonal next to the last row of $B_{[i,i]}$ is also zero.*

Proof. Under the above hypothesis, $S = \mathbb{M}_{r_1,\ldots,r_t}(F)$. Let $1 \le i \le t-1$, and assume that the ith diagonal block is $l \times l$.

We first consider case (I). There is a matrix C in S with 1 in position (l,r) of $C_{[i,i]}$ and 0 elsewhere in all the diagonal blocks, and there is a matrix D in S with 1 in position $(s,1)$ of $D_{[i+1,i+1]}$ and 0 elsewhere in all the diagonal blocks. Then CAD, with A as in (I), has the properties of the matrix A in the formulation of Theorem 3.3

Now for case (II). If $r = l$, then $BA - AB$, with A as in (II), has the properties of the matrix A in Theorem 3.3. If $r \neq l$, then with C as in the previous paragraph and A as in (II), CA has a non-zero entry in the bottom left corner of $(CA)_{[i,i+1]}$, the last entry is the only possible non-zero entry in the last row of $(CA)_{[i,i]}$ and all the entries in the first column of $(cA)_{[i+1,i+1]}$ are 0. Then $BCA - CAB$ has the properties of the matrix A in Theorem 3.3. ∎

In the next result we use Theorem 3.3 to show that $\nu(\mathbb{M}_{r_1,\dots,r_t}(F)) = 3$ if $\mathbb{M}_{r_1,\dots,r_t}(F)$ has no 1×1 diagonal blocks. Since this special case is technically a lot less demanding, we consider it separately.

Let $A \in \mathbb{M}_{r_1,\dots,r_t}(F)$ with components i and $i+1$ of $D_1(A)$ equal to a and c, respectively, and component i of $D_2(A)$ equal to b. Then we denote by A_{*i} the tuple (a, b, c).

THEOREM 3.5. *Assume that* $|F| \geq g + 1$, *where* g *is a positive integer such that* $\mathbb{M}_{r_1,\dots,r_t}(F)$ *has at most* g *diagonal* $r_i \times r_i$ *blocks on the main diagonal for* $1 \leq i \leq t$. *Also assume that* $r_i > 1$ *for* $1 \leq i \leq t$. *Then* $\nu(\mathbb{M}_{r_1,\dots,r_t}(F)) = 3$.

Proof. We define idempotents $U^{[1]}$, $U^{[2]}$, and $U^{[3]}$ of $\mathbb{M}_{r_1,\dots,r_t}(F)$ and denote by S the subalgebra of $\mathbb{M}_{r_1,\dots,r_t}(F)$ generated by $U^{[1]}$, $U^{[2]}$, and $U^{[3]}$.

$$\text{Let } U^{[3]} = \begin{cases} R_{r_j} & \text{on } U^{[3]}_{[j,j]} \\ 0 & \text{on } U^{[3]}_{[i,j]} \text{ if } i \neq j. \end{cases}$$

Recall that $P_{n,\alpha}$, Q_n, and R_n were defined in Section 2. Construct idempotents $U^{[1]}$ and $U^{[2]}$ with the following properties:

(a) $U^{[i]}_{[j,j]} = P_{r_j,\alpha}$ or Q_{r_j} for some $\alpha \in F \setminus \{0\}$, $i = 1, 2$, and $j = 1, 2, \dots, t$.

(b) $U^{[1]}_{[j,j]} + U^{[2]}_{[j,j]} = P_{r_j,\alpha} + Q_{r_j}$ for $j = 1, 2, \dots, t$.

(c) The α's are distinct for diagonal blocks of the same size.

(d) If the ith component of the superdiagonal of matrices in $\mathbb{M}_{r_1,\dots,r_t}(F)$ is not part of a diagonal block, then $\{U^{[1]}_{*i} U^{[2]}_{*i}\} = \{(1,1,0), (0,0,1)\}$.

From Lemma 2.2 it follows that the projection from S to $\oplus_{j=1}^{t} \mathbb{M}_{r_j}(F)$ is onto. Since all the components of $D_2(U^{[1]} + U^{[2]} - I)$ are non-zero, where I is the multiplicative identity, we have from Theorem 3.3 that $S = \mathbb{M}_{r_1,\dots,r_t}(F)$. ∎

In the next result we determine the result corresponding to Theorem 2.3 for $\mathbb{M}_{r_1,\ldots,r_t}(F)$.

THEOREM 3.6. *Assume that* $|F| \geq g+1$, *where* g *is a positive integer such that* $\mathbb{M}_{r_1,\ldots,r_t}(F)$ *has at most* g *diagonal* $r_i \times r_i$ *blocks if* $r_i \geq 2$. *Also assume that* $\mathbb{M}_{r_1,\ldots,r_t}(F)$ *is not equal to* F^m, $m \geq 1$. *Then* $\mu(\mathbb{M}_{r_1,\ldots,r_t}(F)) = 2$. *Also, if there are no* 1×1 *diagonal blocks, then* $\mathbb{M}_{r_1,\ldots,r_t}(F)$ *can be generated (as an F-algebra) by an idempotent and a nilpotent element.*

Proof. This follows from Lemma 2.2, Theorem 2.3, the proof of Theorem 2.3, Theorem 3.3, and the proof of Theorem 3.5. ∎

Now we are in a position to use Corollary 3.4 to prove the main result.

THEOREM 3.7. *Assume that* $|F| \geq g+1$, *where* g *is a positive integer such that* $\mathbb{M}_{r_1,\ldots,r_t}(F)$ *has at most* g *diagonal* $r_i \times r_i$ *blocks if* $r_i \geq 2$, *for* $1 \leq i \leq t$. *Also assume that* $\mathbb{M}_{r_1,\ldots,r_t}(F)$ *is not an upper triangular matrix algebra or a full matrix algebra. Then*

$$\nu\left(\mathbb{M}_{r_1,\ldots,r_t}(F)\right) = \begin{cases} 3 & \text{if } m_1 \leq 8, \\ \lceil \log_2 m_1 \rceil & \text{if } m_1 > 8, \end{cases}$$

where m_1 *is the number of* 1×1 *diagonal blocks of* R.

Proof. From Theorem 3.5 we may assume that $\mathbb{M}_{r_1,\ldots,r_t}(F)$ has at least one 1×1 diagonal block.

Recall from [1] that $\nu(\mathbb{U}_m(F))$, where $\mathbb{U}_m(F)$ is the $m \times m$ upper triangular matrix algebra, is given by

$$\nu\left(\mathbb{U}_m(F)\right) = \begin{cases} \lceil \log_2 m \rceil + 1 & \text{if } m = 2,3,4, \\ \lceil \log_2 m \rceil & \text{if } m \geq 5. \end{cases}$$

In [1] it was shown that it is possible to construct idempotents $U_1^{(m)}, \ldots, U_{\lceil \log_2 m \rceil}^{(m)}$ (or $U_1^{(m)}, \ldots, U_{\lceil \log_2 m \rceil + 1}^{(m)}$ if $m = 2,3,4$) in $\mathbb{U}_m(F)$ in such a way that:

(i) $D_1(U_1^{(m)}, \ldots, D_1(U_{\lceil \log_2 m \rceil}^{(m)})$ (or $D_1(U_1^{(m)}), \ldots, D_1(U_{\lceil \log_2 m \rceil + 1}^{(m)})$ if $m = 2,3,4$) generate F^m.

(ii) If $m > 2$, then $U_3^{(m)}$ is a diagonal matrix with $1,0,1,0,\ldots$ on the main diagonal.

(iii) The ith component, for all i with $1 \leq i \leq m-1$, of at least one of $D_2(U_1^{(m)})$ or $D_2(U_2^{(m)})$ is non-zero.

Now assume that we have idempotents $U_i^{(m)}$ with the above stated three properties. From Corollary 3.4(II) we have that the idempotents $U_i^{(m)}$ generate $\mathbb{U}_m(F)$ if $m > 2$.

Let

$$l = \begin{cases} 3 & \text{if } m_1 \le 8, \\ \lceil \log_2 m_1 \rceil & \text{if } m_1 > 8. \end{cases}$$

From Theorem 3.1, $\nu(\mathbb{M}_{r_1,\ldots,r_t}(F)) \ge l$. We now construct idempotents $V^{[1]},\ldots,V^{[l]}$ in $\mathbb{M}_{r_1,\ldots,r_t}(F)$ that generate $\mathbb{M}_{r_1,\ldots,r_t}(F)$. Let $\mathcal{T}$ be the nilpotent ideal in $\mathbb{M}_{r_1,\ldots,r_t}(F)$ consisting of all elements with all diagonal blocks equal to zero (thus $\mathcal{T}$ is the Jacobson radical of $\mathbb{M}_{r_1,\ldots,r_t}(F)$). Then since idempotents in $\mathbb{M}_{r_1,\ldots,r_t}(F)/\mathcal{T}^2$ can be lifted to idempotents in $\mathbb{M}_{r_1,\ldots,r_t}(F)$, it is sufficient to construct the elements $V^{[i]}$ so that their images in $\mathbb{M}_{r_1,\ldots,r_t}(F)/\mathcal{T}^2$ are idempotent.

Denote by S the subalgebra generated by idempotents $V^{[1]},\ldots,V^{[l]}$ with the following properties:

(a) If $m_1 = 1$, then the 1×1 diagonal block of all the idempotents are equal to 0. Otherwise, we place $D_1(U_i^{(m_1)})$ on the 1×1 diagonal blocks of $V^{[i]}$ for $1 \le i \le \lceil \log_2 m_1 \rceil$ (or $\lceil \log_2 m_1 \rceil + 1$ if $m = 2, 3, 4$), and we place $(1, 0, 1, 0, 1, \ldots)$ on the 1×1 diagonal blocks of $V^{[3]}$ even if $m_1 = 2$.

(b) $V^{[i]}_{[j,j]} = P_{r_j,\alpha}$ or Q_{r_j} for some $\alpha \in F \setminus \{0\}$ if $r_j \ge 2$, $i = 1, 2$.

(c) $V^{[1]}_{[j,j]} + V^{[2]}_{[j,j]} = P_{r_j,\alpha} + Q_{r_j}$ if $r_j \ge 2$, and the α's are distinct for diagonal blocks of the same size.

(d) Assume that the ith position on the super diagonal of $\mathbb{M}_{r_1,\ldots,r_t}(F)$ is not part of a diagonal block. Then if this position is next to a 1×1 diagonal block and above a 1×1 diagonal block, at least one of $D_2(V^{[1]})$ or $D_2(V^{[2]})$ is non-zero. If this position (on the superdiagonal) is next to a 1×1 diagonal block and above a diagonal block of size at least 2×2, it is zero if the 1×1 diagonal block to the left of this position is 0 in both $V^{[1]}$ and $V^{[2]}$ or 1 in both $V^{[1]}$ and $V^{[2]}$, otherwise $\{V^{[1]}_{*i}, V^{[2]}_{*i}\} = \{(1,1,0),(0,0,1)\}$. If this position (on the superdiagonal) is next to a diagonal block of size at least 2×2 and above a diagonal block of size at least 2×2, then $\{V^{[1]}_{*i}, V^{[2]}_{*i}\} = \{(1,1,0),(0,0,1)\}$. Finally, if this partition is next to a diagonal block of size at least 2×2 and above a 1×1 diagonal block, the ith components of $D_2(V^{[1]})$ and $D_2(V^{[2]})$ are both zero.

(e) We construct $V^{[3]}$ in the following way. On $V^{[3]}_{[j,j]}$ we place R_{r_j} if $r_j \ge 2$. Now suppose that the ith position on the superdiagonal is next to the jth diagonal block, $r_j \ge 2$ and $r_{j+1} = 1$. Then $V^{[3]}_{*i} = (1,1,0)$ or $(1,-1,1)$. In the case where $V^{[3]}_{*i} = (1,-1,1)$, the first row of block $(j, j+1)$ is equal to 1. Now suppose that the ith position on the super-diagonal is next to the jth diagonal block, that $r_j = 1$, $r_{j+1} \ge 2$, and that the jth diagonal block is equal to 0 in both $V^{[1]}$ and $V^{[2]}$, or equal to 1 in both $V^{[1]}$ and $V^{[2]}$. In this situation $V^{[3]}_{*\,i} = (1,1,0)$ or $(0,1,0)$, and the last entry in the block $(j, j+1)$ is equal to 1 if $V^{[3]}_{*\,i} = (0,1,0)$. All other entries in $V^{[3]}$ in superdiagonal blocks are zero.

(f) The only non-zero entries in $V^{[i]}$ for $i \geq 4$ are the 1×1 diagonal blocks.

First note that from Lemma 2.2 it follows that the projection from S to $\oplus_{j=1}^{t} \mathbb{M}_{r_j}(F)$ is onto. To complete the proof, we use Corollary 3.4. Assume that the ith component of the superdiagonal is not part of a diagonal block.

If this position (on the superdiagonal) is next to a 1×1 diagonal block and above a 1×1 diagonal block, we let $A = V^{[1]}$ if the ith component of $D_2(V^{[1]})$ is non-zero, $A = V^{[2]}$ otherwise, and $B = V^{[3]}$ in Corollary 3.4(II).

Now assume that this position on the superdiagonal is next to a 1×1 diagonal block and above a block of size at least 2×2. If $\{V^{[1]}_{*\,i}, V^{[2]}_{*\,i}\} = \{(1,1,0),(0,0,10\}$, $\{A, B\} = \{V^{[1]}, V^{[2]}\}$ in Corollary 3.4(II). Next we assume that $\{V^{[1]}_{*\,i}, V^{[2]}_{*\,i}\} = \{(1,0,1),(1,0,0)\}$. Let m be a positive integer such that $(V^{[1]}V^{[2]})^{m}_{[j+1,\,j+1]} = 0$. In this case $A = (V^{[1]}V^{[2]})^m V^{[3]}$ and $B = (V^{[1]}V^{[2]})^m$ in Corollary 3.4(II). Finally, if $\{V^{[1]}_{*\,i}, V^{[2]}_{*\,i}\} = \{(0,0,1),(0,0,0)\}$, we replace $V^{[1]}, V^{[2]}$ by $I - V^{[1]}, I - V^{[2]}$ in the previous argument.

If this position (on the superdiagonal) is next to and above a diagonal block of size at least 2×2, we let $A = V^{[1]} + V^{[2]} - I$ in Corollary 3.4(I).

Finally assume that this position (on the superdiagonal) is next to the jth diagonal block, that $r_j \geq 2$ and that $r_{j+1} = 1$. Here we use Corollary 3.4(I). Without loss of generality assume that $V^{[1]}_{[j,\,j]} = P_{r_j,\,\alpha}$. First note that $(V^{[1]}V^{[2]})^{m}_{[j,\,j]} = 0$ for m large enough, and that the $(j+1, j+1)$th block of $(V^{[1]}V^{[2]})^m$ are equal to 1 if and only if the $(j+1, j+1)$th blocks of both $V^{[1]}$ and $V^{[2]}$ are equal to 1. Using this observation we might as well assume that the $(j+1, j+1)$th block of $(V^{[1]}V^{[2]})^i$, for $i \geq 1$, is equal to ep0. Thus by using the matrices $V^{[1]}V^{[2]}, V^{[1]}V^{[2]}V^{[1]}, V^{[1]}V^{[2]}V^{[1]}V^{[2]}, \ldots,$ we can obtain a matrix C with a 1 in the last position of the first row of block (j, j) and all other entries in blocks $(j, j), (j, j+1), (j+1, j+1)$ equal to zero. Also, by using the matrices $CV^{[3]}V^{[1]}, CV^{[3]}V^{[1]}V^{[2]}$, $CV^{[3]}V^{[1]}V^{[2]}V^{[1]}, CV^{[3]}V^{[1]}V^{[2]}V^{[1]}V^{[2]}, \ldots,$ we can obtain matrices with any desired first row in block (j, j), and all the other entries in blocks $(j, j), (j, j+1), (j+1, j+1)$ equal to 0. Denote by D a matrix with the first and last entry in the first row of block (j, j) equal to 1 and all other entries in blocks $(j, j), (j, j+1), (j+1, j+1)$ equal to zero. We set A equal to $CV^{[3]} - D$. ▮

EXAMPLE 3.8. (Also see the example at the end of [1].) In this example we construct four idempotent generators for $\mathbb{M}_{1,1,3,1,1,2,2,1,1,3,1,2,3,1,1,1}(F)$, where F is a field with at least four elements. Let a, b, c be three distinct non-zero elements in F. Define the idempotents $V^{[1]}, V^{[2]}, V^{[3]}, V^{[4]}$ as follows (entries not indicated are equal to 0):

$$V^{[1]} := \left[\begin{array}{*{25}{c}}
1 \\
&1 \\
&&0&0&0 \\
&&0&1&1 \\
&&0&0&0 \\
&&&&&0 \\
&&&&&&0 \\
&&&&&&&1&a \\
&&&&&&&0&0 \\
&&&&&&&&&1&b \\
&&&&&&&&&0&0 \\
&&&&&&&&&&&1 \\
&&&&&&&&&&&&1&1 \\
&&&&&&&&&&&&&0&0&0 \\
&&&&&&&&&&&&&0&1&1 \\
&&&&&&&&&&&&&0&0&0 \\
&&&&&&&&&&&&&&&&0 \\
&&&&&&&&&&&&&&&&&1&c \\
&&&&&&&&&&&&&&&&&0&0 \\
&&&&&&&&&&&&&&&&&&&1&c&0 \\
&&&&&&&&&&&&&&&&&&&0&0&0 \\
&&&&&&&&&&&&&&&&&&&0&0&1 \\
&&&&&&&&&&&&&&&&&&&&&&0&1 \\
&&&&&&&&&&&&&&&&&&&&&&&1 \\
&&&&&&&&&&&&&&&&&&&&&&&&1
\end{array}\right]$$

$$V^{[2]} := \left[\begin{array}{*{25}{c}}
0&1 \\
&1 \\
&&1&a&0 \\
&&0&0&0 \\
&&0&0&1 \\
&&&&&1&1 \\
&&&&&&0 \\
&&&&&&&0&0&0 \\
&&&&&&&0&1&1 \\
&&&&&&&&&0&0 \\
&&&&&&&&&0&1 \\
&&&&&&&&&&&1&1 \\
&&&&&&&&&&&&0 \\
&&&&&&&&&&&&&1&b&0 \\
&&&&&&&&&&&&&0&0&0 \\
&&&&&&&&&&&&&0&0&1 \\
&&&&&&&&&&&&&&&&0 \\
&&&&&&&&&&&&&&&&&0&0&0 \\
&&&&&&&&&&&&&&&&&0&1&1 \\
&&&&&&&&&&&&&&&&&&&0&0&0 \\
&&&&&&&&&&&&&&&&&&&0&1&1 \\
&&&&&&&&&&&&&&&&&&&0&0&0 \\
&&&&&&&&&&&&&&&&&&&&&&1 \\
&&&&&&&&&&&&&&&&&&&&&&&0&1 \\
&&&&&&&&&&&&&&&&&&&&&&&&1
\end{array}\right]$$

$$V^{[3]} := \left[\begin{array}{*{25}{c}}
\boxed{1} \\
& \boxed{0} & 1 & 0 & 1 \\
& & 0 & 0 & 0 & 0 \\
& & 0 & 0 & 0 & 1 \\
& & 1 & 0 & 1 & 0 \\
& & & & & \boxed{1} \\
& & & & & & \boxed{0} & 1 & 1 \\
& & & & & & & 0 & 0 \\
& & & & & & & 1 & 1 \\
& & & & & & & & & 0 & 0 & 1 \\
& & & & & & & & & 1 & 1 & -1 \\
& & & & & & & & & & & \boxed{1} \\
& & & & & & & & & & & & \boxed{0} \\
& & & & & & & & & & & & & 0 & 0 & 0 & 1 & 1 \\
& & & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & & & 1 & 0 & 1 & -1 & -1 \\
& & & & & & & & & & & & & & & & \boxed{1} & 1 \\
& & & & & & & & & & & & & & & & & 0 & 0 \\
& & & & & & & & & & & & & & & & & 1 & 1 \\
& & & & & & & & & & & & & & & & & & & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & & & & & & & & & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & & & & & & & & & 1 & 0 & 1 & 1 \\
& & & & & & & & & & & & & & & & & & & & & & \boxed{0} \\
& & & & & & & & & & & & & & & & & & & & & & & \boxed{1} \\
& & & & & & & & & & & & & & & & & & & & & & & & \boxed{0}
\end{array}\right]$$

On the 1×1 diagonal blocks of $V^{[4]}$ we place $(1, 1, 1, 1, 1, 1, 1, 1, 0, 0)$. All other components of $V^{[4]}$ are equal to zero.

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REFERENCES

1. A. V. Kelarev, A. B. van der Merwe, and L. van Wyk, The minimum number of idempotent generators of an upper triangular matrix algebra, *J. Algebra* **205** (1998), 606–616.
2. N. Krupnik, Minimal number of idempotent generators of matrix algebras over arbitrary field, *Comm. Algebra* **20** (1992), 3251–3257.
3. N. Krupnik, S. Roch, and B. Silbermann, On C^*-algebras generated by idempotents, *J. Funct. Anal.* **137** (1996), 303–319.
4. M. S. Li and J. M. Zelmanowitz, Artinian rings with restricted primeness conditions, *J. Algebra* **124** (1989), 139–148.
5. S. Roch and B. Silbermann, Algebras generated by idempotents and the symbol calculus for singular integral operators, *Integral Equations Operator Theory* **11** (1988), 385–419.