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Subrings which are closed with respect to taking the inverse ☆

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Abstract

Let S be a subring of the ring R. We investigate the question of whether $S \cap U(R) = U(S)$ holds for the units. In many situations our answer is positive. There is a special emphasis on the case when R is a full matrix ring and S is a structural subring of R defined by a reflexive and transitive relation.

Keywords: Group of units in a ring; Subring of a ring; Chain conditions; PI-ring; Matrix ring; Structural matrix ring; Adjoint matrix

1. Introduction

Throughout the paper a ring R means a ring with identity, and all subrings inherit the identity. The group of units in R is denoted by $U(R)$ and the centre of R is denoted by $Z(R)$.

In general, if S is a subring of the ring R and $x \in S$ is an invertible element in R, then x^{-1} need not be in S. The aim of this paper is to investigate the question of whether $S \cap U(R) = U(S)$ holds for a subring $S \subseteq R$. For a structural matrix subring of a full matrix ring this question

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was raised by Johan Meyer. A similar problem for the additive subgroups of a division ring was considered in [3].

In Section 2 first we impose certain chain conditions on S or on R to derive that $S \cap U(R) = U(S)$. Then we combine the chain conditions with the assumption that R is a PI-ring. Using the prime ideals of R we formulate a reduction theorem providing $S \cap U(R) = U(S)$. In Section 2 we also deal with the subrings of a full matrix ring (over a Noetherian or a PI-ring).

Section 3 is devoted to the study of the structural matrix subring $M_n(\theta, R)$ of the full matrix ring $M_n(R)$ defined by a reflexive and transitive relation θ on the set $\{1, 2, \dots, n\}$. First we reformulate the general results of Section 2 to see that $M_n(\theta, R) \cap U(M_n(R)) = U(M_n(\theta, R))$ holds for various base rings R. Then we get the same equality for PI-rings. Finally we prove that $M_n(\theta, R)$ is closed with respect to taking the adjoint (note that the adjoint always exists, not as the inverse).

In proving our statements we use some classical and one recent theorem concerning PI-rings.

Section 4 contains an example (based on a classical construction of Jacobson) indicating that the Noetherian and the PI conditions play an adequate role in our development. Since any non-Dedekind-finite ring can appear as a base ring in our example, we can use the results of Section 3 to derive some more or less known statements about Dedekind-finite rings. The authors are grateful to Peter P. Pálfy for his help in Section 4.

2. Chain and PI conditions

It is known that a ring R is called strongly π-regular if for every $x \in R$ the DCC holds for the left ideals Rx^i, $i \geqslant 1$.

Proposition 2.1. *Let R be an arbitrary ring and let S be a strongly π-regular subring of R. If $x \in S$ is invertible in R, then $x^{-1} \in S$.*

Proof. The DCC for the left ideals Sx^i, $i \geqslant 1$, of S gives that $Sx^k = Sx^{k+1}$ for some $k \geqslant 1$. Thus

$$x^k = sx^{k+1}$$

for some $s \in S$, whence we obtain that $x^{-1} = s$ is in S. $\square$

Proposition 2.2. *Let R be a ring integral over a central subring $C \subseteq Z(R)$ and let $C \subseteq S \subseteq R$ be a subring. If $x \in S$ is invertible in R, then $x^{-1} \in S$.*

Proof. The integrality gives that

$$x^{-k} + c_{k-1}x^{-(k-1)} + \dots + c_1 x^{-1} + c_0 = 0$$

holds for $x^{-1} \in R$, where $k \geqslant 1$ and $c_{k-1}, \dots, c_1, c_0 \in C$. Thus

$$x^{-1} = -\big(c_{k-1} + c_{k-2}x + \dots + c_1 x^{k-2} + c_0 x^{k-1}\big)$$

is in S. $\square$

Proposition 2.3. *Let S be a subring of the ring R such that R is Noetherian as a left S-module. If $x \in S$ is invertible in R, then $x^{-1} \in S$.*

Proof. The ACC for the S-submodules

$$H_k = \sum_{i=1}^{k} Sx^{-i}, \quad k \geqslant 1,$$

of the left S-module ${}_S R$ gives that

$$\sum_{i=1}^{k} Sx^{-i} = \sum_{i=1}^{k+1} Sx^{-i}$$

for some $k \geqslant 1$. Thus

$$x^{-(k+1)} = s_1 x^{-1} + s_2 x^{-2} + \cdots + s_k x^{-k}$$

with $s_1, s_2, \ldots, s_k \in S$, whence right multiplication by x^k gives that

$$x^{-1} = s_1 x^{k-1} + s_2 x^{k-2} + \cdots + s_{k-1} x + s_k$$

is in S. $\square$

Theorem 2.4. *Let R be a prime PI-ring such that $Z(R)$ is Noetherian. If $Z(R) \subseteq S \subseteq R$ is a subring, then $S \cap U(R) = U(S)$.*

Proof. A theorem of Formanek (see p. 109 in vol. II of [4]) ensures that R is a Noetherian $Z(R)$-module. The condition $Z(R) \subseteq S$ ensures that an S-submodule of the left S-module ${}_S R$ is a $Z(R)$-submodule of R, whence we obtain that R is Noetherian as a left S-module. Thus Proposition 2.3 can be applied to the pair of rings $S \subseteq R$. $\square$

Theorem 2.5. *Let R be a prime PI-ring such that $Z(R)$ is Noetherian. If S is a subring of $M_n(R)$ such that $\{rI \mid r \in Z(R)\} \subseteq S$, then $S \cap U(M_n(R)) = U(S)$.*

Proof. Since $M_n(R)$ is also a prime PI-ring (see p. 110 in vol. II of [4]) with Noetherian centre

$$Z\big(M_n(R)\big) = \big\{rI \bigm| r \in Z(R)\big\} \cong Z(R),$$

the application of Theorem 2.4 gives the desired equality. $\square$

Theorem 2.6. *Let R be a left Noetherian ring. If S is a subring of $M_n(R)$ such that $\{rI \mid r \in R\} \subseteq S$, then $S \cap U(M_n(R)) = U(S)$.*

Proof. Since $M_n(R)$ is a free left R-module (of rank n^2), $M_n(R)$ is Noetherian as a left R-module. The condition $\{rI \mid r \in R\} \subseteq S$ ensures that an S-submodule of ${}_S M_n(R)$ is an R-submodule of ${}_R M_n(R)$, whence we obtain that $M_n(R)$ is Noetherian as a left S-module. Thus Proposition 2.3 can be applied to the pair of rings $S \subseteq M_n(R)$. $\square$

Theorem 2.7. *Let $P_i \lhd R$, $1 \leqslant i \leqslant t$, be a finite collection of ideals of the ring R such that the intersection $P_1 \cap P_2 \cap \cdots \cap P_t$ is a nil ideal. For a subring $S \subseteq R$ consider the subring $S/P_i = \{s + P_i \mid s \in S\} \subseteq R/P_i$ of the factor ring R/P_i. If $(S/P_i) \cap U(R/P_i) = U(S/P_i)$ for all $i \in \{1, 2, \ldots, t\}$, then $S \cap U(R) = U(S)$.*

Proof. Take an element $x \in S \cap U(R)$. Since $x + P_i \in S/P_i$, our assumption gives that the inverse $(x + P_i)^{-1} = x^{-1} + P_i$ is in S/P_i. Thus $x^{-1} + P_i = s_i + P_i$ for some $s_i \in S$. In view of

$$1 - xs_i = x\left(x^{-1} - s_i\right) \in P_i,$$

we obtain that

$$(1 - xs_1)(1 - xs_2)\cdots(1 - xs_t) \in P_1 P_2 \cdots P_t \subseteq P_1 \cap P_2 \cap \cdots \cap P_t$$

is nilpotent. Clearly,

$$(1 - xs_1)(1 - xs_2)\cdots(1 - xs_t) = 1 - xs$$

for some $s \in S$, whence

$$0 = (1 - xs)^k = 1 + \binom{k}{1}(-xs) + \cdots + \binom{k}{k}(-xs)^k$$

follows for some integer $k \geqslant 1$. Consequently

$$x^{-1} = \binom{k}{1}s - \binom{k}{2}s(xs) + \cdots + (-1)^{k+1}\binom{k}{k}s(xs)^{k-1} \in S. \qquad \square$$

Theorem 2.8. *Let R be a ring with ACC on ideals and $S \subseteq R$ be a subring such that $(S/P) \cap U(R/P) = U(S/P)$ for all prime ideals $P \lhd R$. Then $S \cap U(R) = U(S)$.*

Proof. The ACC ensures that the prime radical of R is a finite intersection of prime ideals (see p. 364 in vol. I of [4]):

$$\mathrm{rad}(R) = P_1 \cap P_2 \cap \cdots \cap P_t.$$

Since $\mathrm{rad}(R)$ is nil, we can apply Theorem 2.7 to get the desired equality. $\square$

In the rest of this section we shall make use of a Lie nilpotent R of index m as the underlying ring in $M_n(R)$, in other words a ring R satisfying the identity

$$\big[\big[\big[\ldots\big[[x_1, x_2], x_3\big], \ldots\big], x_m\big], x_{m+1}\big] = 0,$$

with $[x, y] = xy - yx$. The following theorem can easily be obtained from Proposition 4.1 and Theorem 4.2 in [5]:

Theorem 2.T. *If R is a ring satisfying the identity*

$$\big[\big[\big[\dots\big[[x_1,x_2],x_3\big],\dots\big],x_m\big],x_{m+1}\big]=0$$

and $A\in M_n(R)$, then a left Cayley–Hamilton identity

$$\lambda_d A^d+\lambda_{d-1}A^{d-1}+\dots+\lambda_1 A+\lambda_0 I=0$$

holds for A, with $d=n^m$, $\lambda_d\in\mathbb{Z}\setminus\{0\}$ and $\lambda_i\in R$, $0\leqslant i\leqslant d$.

Theorem 2.9. *Let R be a ring such that $\mathbb{Z}\setminus\{0\}\subseteq U(R)$ and R satisfies the identity*

$$\big[\big[\big[\dots\big[[x_1,x_2],x_3\big],\dots\big],x_m\big],x_{m+1}\big]=0.$$

If S is a subring of $M_n(R)$ such that $\{rI\mid r\in R\}\subseteq S$, then $S\cap U(M_n(R))=U(S)$.

Proof. If $A\in S\cap U(M_n(R))$, then Theorem 2.T provides a left Cayley–Hamilton identity for A^{-1} of the form

$$\gamma_d A^{-d}+\gamma_{d-1}A^{-(d-1)}+\dots+\gamma_1 A^{-1}+\gamma_0 I=0.$$

Since $\gamma_d\in\mathbb{Z}\setminus\{0\}$ is in $U(R)$, right multiplication by A^{d-1} and then left multiplication by γ_d^{-1} gives that

$$A^{-1}=-\gamma_d^{-1}\big(\gamma_{d-1}I+\gamma_{d-2}A+\dots+\gamma_1 A^{d-2}+\gamma_0 A^{d-1}\big)$$

is in S. □

Corollary 2.10. *Let R be a commutative ring. If S is a subring of $M_n(R)$ such that $\{rI\mid r\in R\}\subseteq S$, then $S\cap U(M_n(R))=U(S)$.*

Proof. We have $d=n$ and $\gamma_n=1$ in the classical Cayley–Hamilton identity. □

3. Structural matrix rings

The class of structural matrix rings has been studied extensively, see for example, [1] and [2]. For a reflexive and transitive binary relation θ on the set $\{1,2,\dots,n\}$, the structural matrix subring $M_n(\theta,R)$ of the full matrix ring $M_n(R)$ is defined as follows:

$$M_n(\theta,R)=\big\{[a_{i,j}]\in M_n(R)\;\big|\;a_{i,j}=0\text{ if }(i,j)\notin\theta\big\}.$$

Henceforth θ is a reflexive and transitive binary relation on $\{1,2,\dots,n\}$. In the next three theorems we collect the consequences of Theorems 2.6, 2.9 and Corollary 2.10.

Theorem 3.1. *If R is a left Noetherian ring and $A\in M_n(\theta,R)$ is invertible in $M_n(R)$, then $A^{-1}\in M_n(\theta,R)$.*

Theorem 3.2. *Let R be a Lie nilpotent ring such that $\mathbb{Z} \setminus \{0\} \subseteq U(R)$. If $A \in M_n(\theta, R)$ is invertible in $M_n(R)$, then $A^{-1} \in M_n(\theta, R)$.*

Theorem 3.3. *If R is a commutative ring and $A \in M_n(\theta, R)$ is invertible in $M_n(R)$, then $A^{-1} \in M_n(\theta, R)$.*

For arbitrary rings R and T, let $\mu : R \longrightarrow T$ be a ring homomorphism with $\mu(1) = 1$ and consider the induced ring homomorphism

$$\mu_n : M_n(R) \longrightarrow M_n(T).$$

Then the containment $\mu_n(M_n(\theta, R)) \subseteq M_n(\theta, T)$ is obvious. In addition, if μ is injective and $\mu_n(A) \in M_n(\theta, T)$, then $A \in M_n(\theta, R)$.

Let θ' and θ'' be reflexive and transitive binary relations on the sets $\{1, 2, \dots, n\}$ and $\{1, 2, \dots, m\}$ respectively. Then it is evident that

$$M_n\big(\theta', M_m(\theta'', R)\big) \cong M_{nm}(\bar{\theta}, R)$$

for every ring R, where $\bar{\theta}$ is the reflexive and transitive binary relation on the set $\{1, 2, \dots, nm\}$ defined by

$$\bar{\theta} = \big\{(i, j) \,\big|\, \big(\lceil i/m \rceil, \lceil j/m \rceil\big) \in \theta' \text{ and } (i_{(m)}, j_{(m)}) \in \theta''\big\},$$

where $\lceil \cdot \rceil$ denotes the ceiling and

$$i_{(m)} \equiv i \pmod m \quad \text{and} \quad 1 \leqslant i_{(m)} \leqslant m,$$
$$j_{(m)} \equiv j \pmod m \quad \text{and} \quad 1 \leqslant j_{(m)} \leqslant m.$$

We are now in a position to state the following result.

Theorem 3.4. *Let C be a Noetherian subring of $Z(R)$ such that R is a PI-algebra over C. If $A \in M_n(\theta, R)$ and A is invertible in $M_n(R)$, then A^{-1} is in $M_n(\theta, R)$.*

Proof. Let $A = [a_{i,j}]$, $A^{-1} = [b_{i,j}]$ and consider the C-subalgebra

$$D = C\langle a_{i,j}, b_{i,j} \mid 1 \leqslant i, j \leqslant n\rangle$$

of R. Since R is PI over C, the same holds for D. The theorem of Razmyslov–Kemer–Braun (see p. 151 in vol. II of [4]) ensures that the upper nilradical $N = \mathrm{Nil}(D)$ of the affine (finitely generated) C-algebra D is nilpotent: $N^k = \{0\}$ for some integer $k \geqslant 1$. We know that D/N can be embedded in a full matrix ring $M_m(E)$ over a commutative ring E (see p. 98 in vol. II of [4]). Thus $\varphi = \mu \circ \varepsilon$ induces a ring homomorphism

$$\varphi_n : M_n(D) \longrightarrow M_n\big(M_m(E)\big),$$

where $\varepsilon : D \longrightarrow D/N$ is the natural surjection and $\mu : D/N \longrightarrow M_m(E)$ is our embedding. Since A is invertible in $M_n(D)$, $\varphi_n(A)$ is invertible in $M_n(M_m(E))$. The argument above gives that

$M_n(\theta, M_m(E)) \cong M_{nm}(\bar{\theta}, E)$. By assumption $A \in M_n(\theta, D)$, and so $\varphi_n(A) \in M_n(\theta, M_m(E))$ can be viewed as a matrix in $M_{nm}(\bar{\theta}, E)$ having an inverse in $M_{nm}(E)$. Theorem 3.3 shows that the inverse $(\varphi_n(A))^{-1}$, viewed as an $nm \times nm$ matrix over E, is in $M_{nm}(\bar{\theta}, E)$. As

$$\mu_n\big(\varepsilon_n\big(A^{-1}\big)\big) = \varphi_n\big(A^{-1}\big) = \big(\varphi_n(A)\big)^{-1},$$

we conclude that $\varepsilon_n(A^{-1}) \in M_n(\theta, D/N)$ (see the above observations preceding Theorem 3.4). Thus $\varepsilon(b_{i,j}) = b_{i,j} + N = 0$ holds in D/N for all $(i, j) \notin \theta$. Define an $n \times n$ matrix $W = [w_{i,j}]$ over N as follows:

$$w_{i,j} = \begin{cases} b_{i,j} & \text{if } (i, j) \notin \theta, \\ 0 & \text{if } (i, j) \in \theta. \end{cases}$$

Now $A^{-1} - W \in M_n(\theta, D)$ and

$$I - AW = A\big(A^{-1} - W\big) \in M_n(\theta, D),$$

whence $AW \in M_n(\theta, D)$ follows. Clearly, $W \in M_n(N)$ implies that $AW \in M_n(N)$ and hence $(AW)^k = 0$. In view of $A^{-1} - W = A^{-1}(I - AW)$, we obtain that

$$A^{-1} = \big(A^{-1} - W\big)(I - AW)^{-1} = \big(A^{-1} - W\big)\big(I + (AW) + \cdots + (AW)^{k-1}\big)$$

is in $M_n(\theta, D) \subseteq M_n(\theta, R)$. $\square$

We note that the final calculations in the above proof can be omitted by applying Theorem 2.7 to $M_n(\theta, D) \subseteq M_n(D)$ and $P_1 = M_n(N) \lhd M_n(D)$.

Corollary 3.5. *Let R be a PI-ring (a PI-algebra over $\mathbb{Z} \subseteq Z(R)$). If $A \in M_n(\theta, R)$ and A is invertible in $M_n(R)$, then A^{-1} is in $M_n(\theta, R)$.*

Recall that in case R is commutative, then a matrix $A = [a_{i,j}] \in M_n(R)$ is invertible if and only if $\det(A) \in U(R)$, in which case

$$A^{-1} = \big(\det(A)\big)^{-1} \operatorname{adj}(A).$$

For the classical adjoint matrix $\operatorname{adj}(A) = [b_{r,s}]$ we have

$$b_{r,s} = \sum_{\rho \in S_n,\, \rho(s)=r} \operatorname{sgn}(\rho) a_{1,\rho(1)} \cdots a_{s-1,\rho(s-1)} a_{s+1,\rho(s+1)} \cdots a_{n,\rho(n)},$$

where the sum is taken over all permutations ρ of the set $\{1, 2, \ldots, n\}$ with $\rho(s) = r$.

If R is an arbitrary ring (not necessarily commutative), then the preadjoint $A^* = [a^*_{r,s}] \in M_n(R)$ of $A = [a_{i,j}] \in M_n(R)$ was defined as follows in [5]:

$$a^*_{r,s} = \sum_{\tau,\rho} \operatorname{sgn}(\rho) a_{\tau(1),\rho(\tau(1))} \cdots a_{\tau(s-1),\rho(\tau(s-1))} a_{\tau(s+1),\rho(\tau(s+1))} \cdots a_{\tau(n),\rho(\tau(n))},$$

where the sum is taken over all permutations τ of the set $\{1,\dots,s-1,s+1,\dots,n\}$ and all permutations ρ of the set $\{1,2,\dots,n\}$ with $\rho(s)=r$. If R is commutative, then $A^*=(n-1)!\operatorname{adj}(A)$.

Theorem 3.6. *If R is an arbitrary ring and $A\in M_n(\theta,R)$, then $A^*\in M_n(\theta,R)$.*

Proof. Let $1\leqslant r,s\leqslant n$, with $(r,s)\notin\theta$. We prove that $a^*_{r,s}=0$. Take a permutation τ of the set $\{1,\dots,s-1,s+1,\dots,n\}$ and a permutation ρ of the set $\{1,2,\dots,n\}$ with $\rho(s)=r$.

We claim that $(\tau(i),\rho(\tau(i)))\notin\theta$ for some $i\in\{1,\dots,s-1,s+1,\dots,n\}$. Suppose the contrary, that is $(j,\rho(j))\in\theta$ for all $j\in\{1,\dots,s-1,s+1,\dots,n\}$. Consider the cycle

$$\bigl(r,\rho(r),\dots,\rho^t(r)\bigr)$$

of the permutation ρ (of length $t+1$ say). Since $\rho(s)=r$, it follows that $\rho^t(r)=s$. The reflexivity of θ ensures that $r\neq s$, and so

$$\bigl(r,\rho(r)\bigr),\bigl(\rho(r),\rho^2(r)\bigr),\dots,\bigl(\rho^{t-1}(r),s\bigr)\in\theta.$$

The transitivity of θ implies that $(r,s)\in\theta$; a contradiction. Thus $a_{\tau(i),\rho(\tau(i))}=0$ for some $i\in\{1,\dots,s-1,s+1,\dots,n\}$. Consequently, each product

$$a_{\tau(1),\rho(\tau(1))}\cdots a_{\tau(s-1),\rho(\tau(s-1))}a_{\tau(s+1),\rho(\tau(s+1))}\cdots a_{\tau(n),\rho(\tau(n))}$$

in the summation for $a^*_{r,s}$ is zero, whence we obtain that $a^*_{r,s}=0$. □

Corollary 3.7. *If R is a commutative ring and $A\in M_n(\theta,R)$, then* $\operatorname{adj}(A)\in M_n(\theta,R)$.

Proof. Comparing the definitions of $\operatorname{adj}(A)$ and A^* and ignoring τ in the proof of Theorem 3.6 shows that $\operatorname{adj}(A)\in M_n(\theta,R)$. □

4. Dedekind-finite rings

A ring R is called Dedekind-finite if $xy=1$ implies $yx=1$ for all $x,y\in R$. The ring of linear transformations $\operatorname{Hom}_K(V,V)$ of a vector space V (over a field K) with a countably infinite basis $\{b_1,b_2,\dots,b_n,\dots\}\subseteq V$ is not Dedekind-finite. Define the linear transformations $\alpha,\beta:V\longrightarrow V$ on the elements of the given basis as $\alpha(b_i)=b_{i-1}$ for $i\geqslant 2$, $\alpha(b_1)=0$ and $\beta(b_i)=b_{i+1}$ for $i\geqslant 1$, then $\alpha\beta=1$ and $\beta\alpha\neq 1$. Note that the ring $\operatorname{Hom}_K(V,V)$ is not left (right) Noetherian and not PI.

The following example shows that we cannot drop the left (or right) Noetherian condition in Theorem 3.1 and the PI condition in Theorem 3.4.

Example 4.1. Let R be an arbitrary non-Dedekind-finite ring with elements $x,y\in R$ such that $xy=1$ and $yx\neq 1$. The inverse of the upper triangular 2×2 matrix

$$A=\begin{bmatrix} y & 1-yx\\ 0 & x\end{bmatrix}$$

over R is the lower triangular 2×2 matrix

$$A^{-1} = \begin{bmatrix} x & 0 \\ 1 - yx & y \end{bmatrix}.$$

Thus $A \in M_2(\theta, R)$ and $A^{-1} \notin M_2(\theta, R)$, where $\theta = \{(1,1), (1,2), (2,2)\}$.

In view of Theorems 3.1 and 3.4 the following corollaries can easily be obtained.

Corollary 4.2. *If R is a left Noetherian ring, then R is Dedekind-finite.*

Corollary 4.3. *If C is a Noetherian subring of $Z(R)$ such that R is a PI-algebra over C, then R is Dedekind-finite.*

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