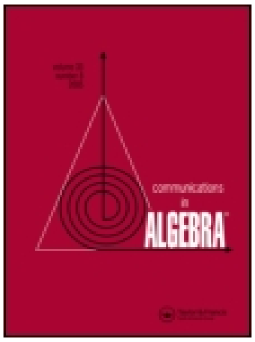

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SEMIGROUP GRADINGS OF FULL MATRIX RINGS

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ABSTRACT

Let F be a field, and let $A = M_n(F)$ be the algebra of all $n \times n$ matrices over F. For each finite semigroup S, we describe all gradings $A = \bigoplus_{s \in S} A_s$ of A by S such that all standard matrix units e_{ij} of A are homogeneous elements of A.

Let F be a field. The aim of this paper is to investigate semigroup gradings of matrix algebras with entries in F. Matrix rings play crucial role in describing structure of rings. Semigroup-graded rings include as special cases many other ring constructions (see, for instance, [6–9]). An interesting example of a semigroup grading of a matrix algebra was given in [10] by J. Wedderburn, who showed that, for a field F of characteristic zero, the full matrix algebra $M_n(F)$ can be graded by a rectangular band so that all homogenious components are isomorphic to the field F.

The general problem of describing all semigroup gradings of a full matrix algebra was posed by E. Zelmanov (see [7]). A grading is said to be *good* if all the matrix units e_{ij} are homogeneous elements. Good gradings were studied in the group-graded case in [2] and in a different setting in [3] and [4], where they were constructed from weight functions on the complete graph Γ on n points, using the fact that $M_n(F)$ is a quotient of the path algebra of the quiver Γ. If G is a group, then all good G-gradings of the F-algebra $M_n(F)$ have been described in [2]. It is known that in several natural cases all grading are isomorphic to good ones. For example, if the group G is torsion free, then [2], Corollary 1.5, tells us that each grading is isomorphic to a good grading. In this paper, for each finite semigroup S, we give a complete description of all good gradings of the full matrix algebra $M_n(F)$ by S.

Let S be a semigroup. An associative ring R is said to be S-*graded*, if $R = \bigoplus_{s \in sS} R_s$ is a direct sum, and $R_s R_t \subseteq R_{st}$ for all $s, t \in S$. If R is an F-algebra, we say that R is an S-graded F-algebra if R is an S-graded ring such that all the homogeneous components R_s are F-vector subspaces of R.

We use standard concepts of semigroup theory following [5]. Let us recall a few definitions. If S is a semigroup, then S^1 stands for S with identity adjoined.

Suppose that G is a group, I and Λ are nonempty sets, and $P = [p_{\lambda i}]$ is a $(\Lambda \times I)$-matrix with entries $p_{\lambda i} \in G$ for all $\lambda \in \Lambda$, $i \in I$. The *Rees matrix semigroup* $M(G; I, \Lambda; P)$ over G with *sandwich-matrix* P consists of all triples $(g; i, \lambda)$, where $i \in I$, $\lambda \in \Lambda$, and $g \in G$, with multiplication defined by the rule

$$(g_1; i_1, \lambda_1)(g_2; i_2, \lambda_2) = (g_1 p_{\lambda_1 i_2} g_2; i_1, \lambda_2).$$

Denote by G^0 the group G with zero adjoined. Now suppose that $Q = [q_{\lambda i}]$ is a $(\Lambda \times I)$-matrix with entries $q_{\lambda i} \in G^0$. Then the *Rees matrix semigroup* $M^0(G; I, \Lambda; P)$ over G^0 with *sandwich-matrix* Q consists of zero 0 and all triples $(g; i, \lambda)$, for $i \in I$, $\lambda \in \Lambda$, and $g \in G^0$, where all triples $(0, i, \lambda)$ are identified with 0, and multiplication is defined by the rule

$$(g_1; i_1, \lambda_1)(g_2; i_2, \lambda_2) = (g_1 p_{\lambda_1 i_2} g_2; i_1, \lambda_2).$$

A semigroup is *completely simple* if it has no proper ideals and has an idempotent minimal with respect to the partial order $e \leq f \Leftrightarrow e = ef = fe$. It is called *completely* 0-*simple* if it has no proper nonzero ideals and has a minimal nonzero idempotent. It is well known that every completely simple semigroup is isomorphic to a Rees matrix semigroup $M(G; I, \Lambda; P)$ over a group G, and every completely 0-simple semigroup is isomorphic to a Rees matrix semigroup $M^0(G; I, \Lambda; P)$ over a group G with zero adjoined. Besides, $M^0(G; I, \Lambda; P)$ is completely 0-simple if and only if each row and every column of P has at least one nonzero entry (see [5]).

Let I and J be ideals of a semigroup S such that $J \subseteq I$. The Rees quotient semigroup I/J is called a *factor* of S. In the case where $J = \emptyset$, we assume that $I/J = I$. Take any element s in S, put $I = S^1 s S^1$ and denote by J the set of all elements which generate principal ideals properly contained in I. Then J is also an ideal of S, and I/J is called a *principal factor* of S.

Dealing with principal factors of a semigroup we shall often use the following two basic facts. Recall that a *null* semigroup or a semigroup with *zero multiplication* is a semigroup G with zero 0 such that $S^2 = 0$. Each principal factor of a semigroup G is either simple, or 0-simple, or a null semigroup ([5], Proposition 3.1.5). In addition, all periodic simple or 0-simple semigroups are completely simple or completely 0-simple, respectively ([5], Theorem 3.2.11).

For any factor I/J of S we identify all elements of $I \setminus J$ with their images in I/J, and say that all elements of $I \setminus J$ belong to I/J.

For every completely 0-simple factor $Q = M(G^0; I, \Lambda; P)$ of S, denote by T_Q the set of all triples $(\varphi_I, \varphi_\Lambda, \varphi_G)$, where $\varphi_I : \{1, 2, \ldots, n\} \to I$ and $\varphi_\Lambda : \{1, 2, \ldots, n\} \to \Lambda$ are functions such that $p_{\varphi_\Lambda(i)\varphi_I(i)} \neq 0$ for all $i = 1, \ldots, n$, and $\varphi_G : \{1, 2, \ldots, n-1\} \to G$ is an arbitrary function.

A semigroup S may also have one completely simple factor $Q = M(G; I, \Lambda; P)$, which then coincides with the least ideal of S. In this case we denote by T_Q the set of all triples $(\varphi_I, \varphi_\Lambda, \varphi_G)$, where $\varphi_I : \{1, 2, \ldots, n\} \to I$, $\varphi_\Lambda : \{1, 2, \ldots, n\} \to \Lambda$, and $\varphi_G : \{1, 2, \ldots, n-1\} \to G$ are arbitrary functions.

Fix a triple $\tau = (\varphi_I, \varphi_\Lambda, \varphi_G) \in T_Q$ and an element $s = (g; i, \lambda) \in Q$. Denote by $U_s^{Q\tau}$ the set of all pairs (k, ℓ) such that $1 \leq k < \ell \leq n$ and

$$g = \varphi_G(k) p_{\varphi_\Lambda(k+1)\varphi_I(k+1)} \varphi_G(k+1) p_{\varphi_\Lambda(k+2)\varphi_I(k+2)} \cdots \varphi_G(\ell - 1), \tag{1}$$

$$i = \varphi_I(k), \tag{2}$$

$$\lambda = \varphi_\Lambda(\ell). \tag{3}$$

Let $D_s^{Q\tau}$ be the set of all pairs (k, ℓ) such that $1 \leq \ell < k \leq n$ and

$$g = p^{-1}_{\varphi_\Lambda(k)\varphi_I(k)} \varphi_G^{-1}(k-1) p^{-1}_{\varphi_\Lambda(k-1)\varphi_I(k-1)} \varphi_G^{-1}(k-2) \cdots p^{-1}_{\varphi_\Lambda(\ell)\varphi_I(\ell)}, \tag{4}$$

$$i = \varphi_I(k), \tag{5}$$

$$\lambda = \varphi_\Lambda(\ell). \tag{6}$$

Let $C_s^{Q\tau}$ be the set of all pairs (k, k) such that $1 \leq k \leq n$ and

$$g = p^{-1}_{\varphi_\Lambda(k)\varphi_I(k)}, \tag{7}$$
$$i = \varphi_I(k), \tag{8}$$
$$\lambda = \varphi_\Lambda(k). \tag{9}$$

Finally, put

$$P_s^{Q\tau} = U_s^{Q\tau} \cup C_s^{Q\tau} \cup D_s^{Q\tau} \tag{10}$$

Theorem 1. *Let F be a field, n a positive integer, and let S be a finite semigroup. Then there exists a one-to-one correspondence between all good S-gradings of the matrix F-algebra $M_n(F)$ and the union of all sets T_Q, where Q runs over all simple and 0-simple principal factors of S. Namely, for each simple or 0-simple principal factor $Q = M(G; I, \Lambda; P)$ or $Q = M^0(G; I, \Lambda; P)$ of S, every triple $\tau = (\varphi_I, \varphi_\Lambda, \varphi_G) \in T_Q$ determines a grading*

$$M_n(F) = \bigoplus_{s \in S} R_s^{(\tau)}, \tag{11}$$

where

$$R_s^{(\tau)} = \begin{cases} \sum_{(k,\ell) \in P_s^{Q\tau}} Fe_{k\ell} & \text{if } s = (g; i, \lambda) \in Q \setminus \{0\}, \\ 0 & \text{if } s \notin Q \setminus \{0\}. \end{cases} \tag{12}$$

Next, we describe all S such that $M_n(F)$ has a good grading as an F-algebra. We say that a good grading is *trivial*, if S has an idempotent e such that the only nonzero homogeneous component of the grading is R_e.

Theorem 2. *The F-algebra $M_n(F)$ has a nontrivial good S-grading if and only if S contains a homomorphic image of the B_n defined by generators $x_{i,j}$, $1 \leq i, j \leq n$ and relations $x_{i,i}^2 = x_{i,i}$, $x_{i,j}x_{j,k} = x_{i,k}$, for all $1 \leq i, j, k \leq n$.*

The semigroup B_n is infinite, it has arbitrarily large homomorphic images, and therefore it is impossible to replace it by any finite semigroup in the theorem above. For convenience of further reference we combine Proposition 3.1.5 and Theorem 3.2.11 of [5], mentioned above, in the following lemma.

Lemma 3. *Every principal factor of a periodic semigroup is completely simple, or completely 0-simple, or a null semigroup.*

Let G be a group, $Q = M(G; I, \Lambda; P)$, and let $i \in I$, $\lambda \in \Lambda$. Then we put

$$Q_\lambda = \{(g;i,\lambda) \mid g \in G, i \in I\},$$
$$Q_i = \{(g;i,\lambda) \mid g \in G, \lambda \in \Lambda\},$$
$$Q_{i\lambda} = \{(g;i,\lambda) \mid g \in G\}.$$

In the case where $Q = M^0(G;I,\Lambda;P)$ we include zero in all of these sets, *i.e.*, put

$$Q_\lambda = \{0\} \cup \{(g;i,\lambda) \mid g \in G, i \in I\},$$
$$Q_i = \{0\} \cup \{(g;i,\lambda) \mid g \in G, \lambda \in \Lambda\},$$
$$Q_{i\lambda} = \{0\} \cup \{(g;i,\lambda) \mid g \in G\}.$$

We need a few facts which immediately follow from the definition of a Rees matrix semigroup.

Lemma 4. *Let G be a group, and let $Q = M(G;I,\Lambda;P)$ or $Q = M^0(G;I,\Lambda;P)$. Then, for all $i \in I$, $\lambda \in \Lambda$,*

(i) *the sets Q_λ are minimal nonzero right ideals of Q;*
(ii) *the sets Q_i are minimal nonzero left ideals of Q;*
(iii) *the set $Q_{i\lambda}$ is a left ideal of Q_i and a right ideal of Q_λ;*
(iv) *if $p_{\lambda i} \neq 0$, then $Q_{i\lambda}$ is a maximal subgroup of Q;*
(v) *if $p_{\lambda i} = 0$, then $Q_{i\lambda}^2 = 0$.*

Proof of Theorem 1. First, we take any simple or 0-simple factor $Q = M(G;I,\Lambda;P)$ or $Q = M^0(G;I,\Lambda;P)$ of S, and any triple $\tau = (\varphi_I, \varphi_\Lambda, \varphi_G) \in T_Q$, and show that it defines a good grading. Consider the additive subgroups $R_s^{(\tau)}$ of $M_n(F)$ introduced in (12).

Let us verify that $M_n(F)$ is a direct sum of the $R_s^{(\tau)}$, where $s \in S$. Since $M_n(F) = \sum_{1 \le k,\ell \le n} Fe_{k\ell}$ and each nonzero component $R_s^{(\tau)}$ equals $\sum_{(k,\ell) \in P_s^{Q\tau}} Fe_{k\ell}$ for some $s = (g;i,\lambda) \in Q \setminus \{0\}$, it suffices to show that the set of all pairs (k,ℓ) such that $1 \le k,\ell \le n$ is a disjoint union of the sets $P_s^{Q\tau}$ for all $s \in Q \setminus \{0\}$.

Take any pair (k,ℓ), where $1 \le k,\ell \le n$, and suppose that $(k,\ell) \in P_s^{Q\tau}$, for some $s = (g;i,\lambda) \in Q \setminus \{0\}$.

If $k < \ell$, then by the definition (k,ℓ) belongs to $P_s^{Q\tau}$ if and only if it belongs to $U_s^{Q\tau}$, which is equivalent to (4), (5) and (6). Since k,ℓ and $\tau = (\varphi_I, \varphi_\Lambda, \varphi_G) \in T_Q$ are fixed, conditions (5) and (6) uniquely determine i and λ, respectively. After that condition (4) gives us g. Therefore s is uniquely determined by the pair (k,ℓ).

If $k = \ell$ or $k > \ell$, then we use the definitions of $C_s^{Q\tau}$ and $D_s^{Q\tau}$, respectively, and conditions (7), (8), (9) or (1), (2), (3) show that s is uniquely determined again. Thus $M_n(F) = \bigoplus_{s \in S} R_s^{(\tau)}$ is a direct sum.

Next take any $s, s' \in S$. If $s \notin Q \setminus \{0\}$ or $s' \notin Q \setminus \{0\}$, then $R_s = 0$ or $R_{s'} = 0$ and so $R_s R_{s'} \subseteq R_{ss'}$.

Suppose that $s = (g; i, \lambda) \in Q \setminus \{0\}$ and $s' = (g'; i', \lambda') \in Q \setminus \{0\}$. Consider $R_s^{(\tau)} = \sum_{(k,\ell)\in P_s^{Q\tau}} Fe_{k\ell}$ and $R_{s'}^{(\tau)} = \sum_{(k,\ell)\in P_{s'}^{Q\tau}} Fe_{k\ell}$. In order to show that $R_s R_{s'} \subseteq R_{ss'}$ it suffices to pick any $(k, \ell) \in P_s^{Q\tau}$ and $(k', \ell') \in P_{s'}^{Q\tau}$ and verify that $e_{k\ell}e_{k'\ell'} \in R_{ss'}^{(\tau)}$.

If $\ell \neq k'$, then $e_{k\ell}e_{k'\ell'} = 0$, and so we may assume that $\ell = k'$. Then $e_{k\ell}e_{k'\ell'} = e_{k\ell'}$ and we need to verify that $(k, \ell') \in P_{ss'}^{Q\tau}$.

First, we consider the case where $k < \ell < \ell'$. Since $ss' = (gp_{\lambda i'}g'; i, \lambda')$, condition (2) given for s implies the same condition for ss', and condition (3) for s' yields the same condition for ss'. Now conditions (1) for s and s' give us the following equalities:

$$g = \varphi_G(k)p_{\varphi_\Lambda(k+1)\varphi_I(k+1)}\varphi_G(k+1)p_{\varphi_\Lambda(k+2)\varphi_I(k+2)} \cdots \varphi_G(\ell - 1),$$
$$g' = \varphi_G(k')p_{\varphi_\Lambda(k'+1)\varphi_I(k'+1)}\varphi_G(k'+1)p_{\varphi_\Lambda(k'+2)\varphi_I(k'+2)} \cdots \varphi_G(\ell' - 1).$$

Combining these with conditions (2) and (3) we get

$$\begin{aligned} gp_{\lambda i'}g' = {} & \varphi_G(k)p_{\varphi_\Lambda(k+1)\varphi_I(k+1)} \cdots \varphi_G(\ell - 1)p_{\varphi_\Lambda(\ell)\varphi_I(k')} \\ & \times \varphi_G(k')p_{\varphi_\Lambda(k'+1)\varphi_I(k'+1)} \cdots \varphi_G(\ell' - 1). \end{aligned}$$

Thus condition (1) for ss' is satisfied, too. This means that $(k, \ell') \in P_{ss'}^{Q\tau}$.

Other cases are similar, and we omit them. Thus $R_s R_{s'} \subseteq R_{ss'}$, and so $M_n(F) = \bigoplus_{s\in S} R_s^{(\tau)}$ is an S-graded ring.

As we have shown above, for every $1 \leq i, j \leq n$, e_{ij} is a homogeneous element of $M_n(F)$, and so this grading is good.

Conversely, take any good grading $M_n(F) = \bigoplus_{s\in S} R_s$. For each $1 \leq i$, $j \leq n$, there exists an element $s_{ij} \in S$ such that $e_{ij} \in R_{s_{ij}}$. We need to find a simple or 0-simple factor Q of S and a triple $\tau = (\varphi_I, \varphi_\Lambda, \varphi_G) \in T_Q$ which defines the same grading, *i.e.*, $R_s = R_s^{(\tau)}$ for all $s \in S$.

Since every unit e_{ij} generates the whole ring $M_n(F)$ as an ideal, it follows that all elements s_{ij} generate the same principal ideal of S. Denote by Q the principal factor of S containing all the s_{ij}. All s_{ii} are idempotents, whence [5], Theorem 3.2.11, shows that Q is completely simple or completely 0-simple. We consider only the case where $Q = M^0(G; I, \Lambda; P)$ is completely 0-simple, since the second case is similar.

For $1 \leq i \leq n$, let

$$s_{ii} = (g_{ii}; \varphi_I(i), \varphi_\Lambda(i)), \tag{13}$$

where $g_{ii} \in G$. This defines functions $\varphi_I : \{1, 2, \ldots, n\} \to I$ and $\varphi_\Lambda : \{1, 2, \ldots, n\} \to \Lambda$. Since s_{ii} is an idempotent, it follows that $g_{ii}p_{\varphi_\Lambda(i)\varphi_I(i)}g_{ii} = g_{ii}$; whence

$$g_{ii} = p^{-1}_{\varphi_\Lambda(i)\varphi_I(i)}g_{ii}. \tag{14}$$

Therefore $p_{\varphi_\Lambda(i)\varphi_I(i)} \neq 0$, as required in the definition of T_Q.

Take any $i \in \{1, \ldots, n-1\}$, and suppose that $e_{i,i+1} = (\varphi_G(i); \zeta(i), \eta(i))$. This defines a function

$$\varphi_G : \{1, 2, \ldots, n-1\} \to G.$$

The equality $e_{ii}e_{i,i+1} = e_{i,i+1}$ implies $\zeta(i) = \varphi(i)$, and $e_{i,i+1}e_{i+1,i+1} = e_{i,i+1j}$ implies $\eta(i+1) = \varphi(i+1)$. Thus

$$e_{i,i+1} = (\varphi_G(i); \varphi_I(i), \varphi_\Lambda(i+1)). \tag{15}$$

Next we verify that the triple $\tau = (\varphi_G, \varphi_I, \varphi_\Lambda)$ determines the same good grading of $M_n(F)$.

Take any $s \in S$ and consider the component R_s. We are going to prove that it satisfies (12).

Suppose that $R_s \neq 0$ and choose a nonzero element $x \in R_s$. Since $x \in M_n(F)$, we get

$$x\left(\sum_{i=1}^{n} e_{ii}\right) = \left(\sum_{i=1}^{n} e_{ii}\right)x = x.$$

Hence there exists $1 \leq i, j \leq n$ such that $s_{ii}s = ss_{jj} = s$. It follows that s generates the same principal ideal as all the s_{ii} in S, and so $s \in Q \setminus \{0\}$. Thus $R_s = 0$ for all $s \notin Q \setminus \{0\}$.

Now fix any $s \in Q \setminus \{0\}$, say $s = (g; i, \lambda)$. We have to show that

$$R_s^{(\tau)} = \sum_{(k,\ell)\in P_s^{Q\tau}} Fe_{k\ell}.$$

Since all elements $e_{k\ell}$ are homogeneous, it suffices to prove that $e_{k\ell} \in R_s$ if and only if $(k, \ell) \in P_s^{Q\tau}$. Take any pair (k, ℓ) where $1 \leq k, \ell \leq n$.

First, consider the case where $k < \ell$. Then

$$e_{k\ell} = e_{k,k+1}e_{k+1,k+2} \cdots e_{\ell-1,\ell}.$$

This and (15) give us

$$s_{k\ell} = (h; \varphi_I(k), \varphi_\Lambda(\ell)),$$

where

$$h = \varphi_G(k)p_{\varphi_\Lambda(k+1)\varphi_I(k+1)} \cdots \varphi_G(\ell-2)p_{\varphi_\Lambda(\ell-1)\varphi_I(\ell-1)}\varphi_G(\ell-1).$$

Thus (k, ℓ) satisfies (1), (2), and (3), and so (k, ℓ) belongs to $U_s^{Q\tau} \subseteq P_s^{Q\tau}$.

Second, if $\ell < k$, then (15) and

$$e_{k\ell} = e_{k,k-1}e_{k-1,k-2} \cdots e_{\ell+1,\ell}$$

similarly show that all conditions (4), (5), and (6), and so (k, ℓ) belongs to $D_s^{Q\tau} \subseteq P_s^{Q\tau}$.

Finally, if $k = \ell$, then (14) implies (7) and (13) yields us (8) and (9). Thus (k, ℓ) belongs to $C_s^{Q\tau} \subseteq P_s^{Q\tau}$. This completes our proof. □

Proof of Theorem 2. Suppose that the F-algebra $M_n(F)$ has a good S-grading $M_n(F) = \bigoplus_{s \in S} R_s$. As usual this defines the degree function

$$\deg : \bigcup_{s \in S} R_s \to S.$$

For $i, j \in \{1, 2\}$, put $s_{i,j} = \deg(e_{i,j})$, *i.e.*, denote by $s_{i,j}$ an element of S such that $e_{i,j} \in R_{s_{i,j}}$. Denote by H the subsemigroup generated in S by all $s_{i,j}$ for $1 \leq i, j, k \leq n$. Consider the homomorphism

$$\varphi : B_n \to H$$

defined by

$$\varphi(x_{i,j}) = s_{i,j} \quad \text{for all } 1 \leq i, j \leq n.$$

For any $1 \leq i, j, k \leq n$, $e_{i,i}^2 = e_{i,i}$ and $e_{i,j} e_{j,k} = e_{i,k}$ imply $s_{i,i}^2 = s_{i,i}$, $s_{i,j} s_{j,k} = s_{i,k}$; whence $\varphi(x_{i,i}^2) = \varphi(x_{i,i})$ and $\varphi(x_{i,j} x_{j,k}) = \varphi(x_{i,k})$. Thus the hopmomorphism φ respects all relations of $B_n(F)$, and so it is well defined. Hence all properties in the theorem follow. □

All this theory could be formulated in a Hopf algebra language, since a graded algebra structure is just a comodule algebra structure over the group or semigroup algebra of the grading group or semigroup. Several interesting results, including results about the number of isomorphism types of gradings, are obtained using this language in [1]. We prefer the ring theory language in this paper.

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