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Lie solvability and the identity $[x_1, y_1][x_2, y_2] \cdots [x_q, y_q] = 0$ in certain matrix algebras

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A B S T R A C T

We study the interplay between the polynomial identities $[\ldots[[x_1, x_2], x_3], \ldots, x_{q+1}] = 0$, $[x_1, y_1][x_2, y_2] \cdots [x_q, y_q] = 0$ and $[[x_1, y_1], [x_2, y_2]] = 0$ in general, and in certain matrix algebras, which are closely related to structural matrix algebras, in particular. We show that if such a matrix algebra $\mathcal{A}$ satisfies the identity $[[x_1, y_1], [x_2, y_2]] = 0$ (in particular, if $\mathcal{A}$ satisfies the identity $[[x_1, x_2], x_3] = 0$), then $\mathcal{A}$ satisfies the identity $[x_1, y_1][x_2, y_2] = 0$, and we prove a decomposition theorem for such subalgebras of upper triangular matrix algebras satisfying the Engel condition, and hence for such Lie nilpotent subalgebras.

An in depth analysis of the maximum dimension of a subalgebra of the full $n \times n$ matrix algebra $M_n(F)$ over a field F satisfying the identity $[x_1, y_1]\,[x_2, y_2] \cdots [x_q, y_q] = 0$ (as obtained in [5]) leads to a more detailed formula for this maximum dimension, as well to a precise way of obtaining a class of examples of F-subalgebras of $M_n(F)$ having

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this dimension, and finally to determining exactly when a simplified version of the mentioned formula can be used.

1. Notation

Throughout this paper, R, $J(R)$ and F denote a (not necessarily commutative) ring with identity, its Jacobson radical, and a field, respectively, and $[x, y] = xy - yx$ denotes the additive commutator of elements $x, y \in R$.

For a positive integer n, we use $M_n(R)$ for the full matrix ring of all $n \times n$ matrices over R, and I_n for the $n \times n$ identity matrix. By $e_{i,j}$ we denote the matrix with 1 in position (i, j) and 0 elsewhere, and by $A_{i,j}$ (or $(A)_{i,j}$ in case of possible confusion) the entry of a matrix A in position (i, j). We use $U_n(R)$ for the subring of $M_n(R)$ comprising all the upper triangular matrices, and $U_n^\star(R)$ for the subring of $U_n(R)$ consisting of all the matrices A in $U_n(R)$ with constant main diagonal, i.e., $A_{1,1} = \cdots = A_{n,n}$ for every $A \in U_n^\star(R)$.

If R is commutative, then $M_n(R)$ is an R-algebra, and the mentioned subrings are R-subalgebras.

2. Introduction

Various known results about, and many kinds of rings and algebras satisfying, the polynomial identities (PIs)

$$\begin{array}{cc} [\ldots[[x_1, x_2], x_3], \ldots, x_{q+1}] = 0, & [x_1, y_1][x_2, y_2] \cdots [x_q, y_q] = 0 \\ \text{and} & [[x_1, y_1], [x_2, y_2]] = 0 \end{array} \tag{1}$$

have been the inspiration for this paper.

Firstly, the identity

$$[x_1, y_1]\,[x_2, y_2] \cdots [x_q, y_q] = 0 \tag{2}$$

features prominently in numerous papers, e.g., [10], [1], [5] and [13]. Mal'tsev proved in [10] that all the polynomial identities of $U_q(F)$ are consequences of only one identity, namely the identity in (2). For an explicit form of a finite set of generators of an ideal of identities of the algebra $U_q^\star(R)$ over a commutative integral domain R, see [11].

Secondly, a Cayley–Hamilton trace identity for 2×2 matrices over Lie solvable (of index 2) rings was obtained in [13], where the R-subalgebra $U_3^\star\big(U_3^\star(R)\big)$ of $U_9^\star(R)$ (for any commutative ring R) was exhibited as an example of an algebra satisfying the (Lie solvable of index 2) identity $[[x_1, y_1], [x_2, y_2]] = 0$, but neither the (Lie nilpotent of index 2) identity $[[x_1, x_2], x_3] = 0$, nor the identity $[x_1, y_1][x_2, y_2] = 0$ (see (2) with $q = 2$).

Thirdly, the maximum dimension of an F-subalgebra of $M_n(F)$ satisfying the identity in (2) was studied in [5]. The examples of such F-subalgebras of $M_n(F)$ mentioned in [5] reminds one very strongly, in at least two ways, of the typical examples of F-subalgebras of $M_n(F)$ in [17] with maximum dimension satisfying the (Lie nilpotent of index q) identity

$$[\dots[[x_1, x_2], x_3], \dots, x_{q+1}] = 0. \tag{3}$$

The first resemblance is the fact that both classes of F-subalgebras of $M_n(F)$ are, modulo the blocks on the main diagonal, structural matrix algebras, in the sense that one has total freedom in the blocks in the "strictly upper triangular part", which constitutes the Jacobson radical.

The second similarity is the fact that the algebras in [17] are subalgebras of the algebras in [5], "hinting" or "suggesting" that the identity in (3) may perhaps imply the identity in (2) for such F-subalgebras of $M_n(F)$.

It is vitally important here to stress the fact that for the R-subalgebra $U_3^\star\big(U_3^\star(R)\big)$ of $U_9^\star(R)$ mentioned earlier one does not have total freedom in the strictly upper triangular blocks, in the sense that the entries in some positions in these blocks are "linked" or "tied" to one another; stated equivalently, even as far as the strictly upper triangular blocks in $U_3^\star\big(U_3^\star(R)\big)$ are concerned, $U_3^\star\big(U_3^\star(R)\big)$, contrary to the mentioned examples in [5] and [17], is not a structural matrix algebra.

In Section 3 the mentioned examples, together with other examples, are used to study in detail the interplay between the three identities in (1) in general. The resemblance mentioned above is pursued in Section 4, where these identities are studied in structural matrix algebras and in certain F-subalgebras of $M_n(F)$ having the "structural" form as far as the Jacobson radical is concerned. A decomposition theorem for such F-subalgebras of $U_n(F)$ satisfying the Engel condition (which is a weaker condition than Lie nilpotency) as a direct sum of F-subalgebras of the form $U_{n_i}^\star(F)$ is proved in Section 5.

We devote Section 6 to a thorough analysis of precisely constructing the examples of F-subalgebras of $M_n(F)$ satisfying the identity in (2). This gives rise to a more detailed formula for the maximum dimension of an F-subalgebra of $M_n(F)$ satisfying the identity in (2). Another upper, more simplified, bound for the maximum dimension of an F-subalgebra of $M_n(F)$ satisfying the identity in (2) was obtained in [5]. In Section 7 we determine exactly when these two upper bounds coincide, and we conclude the paper by stating an open problem in Section 8.

3. Pertinent examples involving the three polynomial identities

In this section we want to present some facts which show occurring relations between certain F-subalgebras of $M_n(F)$ which will be the main object of our interest in Section 4.

Our approach is directly related to some well known examples and should be seen as justification for the development of the sequel.

Define inductively the Lie central and Lie derived series of R as follows:

$$\mathfrak{C}^0(R) = R,\ \mathfrak{C}^{q+1}(R) = [\mathfrak{C}^q(R), R] \text{ (central series)},$$
$$\mathfrak{D}^0(R) = R,\ \mathfrak{D}^{q+1}(R) = [\mathfrak{D}^q(R), \mathfrak{D}^q(R)] \text{ (derived series)}.$$

We say that R is Lie nilpotent (respectively, Lie solvable) of index q (for short, R is Ln_q; respectively, R is Ls_q) if $\mathfrak{C}^q(R) = 0$ (respectively, $\mathfrak{D}^q(R) = 0$). It is evident that R is Ln_q or Ls_q if and only if R satisfies the corresponding polynomial identities, and that if R is Ln_q, then R is Ls_q, or, for short, $\mathrm{Ln}_q \ \Rightarrow\ \mathrm{Ls}_q$; in particular,

$$\mathrm{Ln}_2 \ \Rightarrow\ \mathrm{Ls}_2. \tag{4}$$

By [18, Theorem 2.2], for any integer $q \geq 3$, if a ring R is Ln_q, then

$$\mathfrak{C}^{q-1}(R) \cdot \mathfrak{C}^{q-1}(R) = 0. \tag{5}$$

However, the m-generated ($m \geq 4$) Grassmann algebra

$$E^{(m)} = F\langle v_1, \ldots, v_m :\ v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i \leq j \leq m\rangle$$

over a field F (with $4 \neq 0$) is used in [18, Remark 2.3] to show that (5) is not true for $q = 2$, i.e., although the m-generated ($m \geq 4$) Grassmann algebra satisfies the Lie nilpotent of index 2 identity $[[x_1, x_2], x_3] = 0$, it does not satisfy the identity $[x_1, y_1][x_2, y_2] = 0$.

If a ring R satisfies the identity in (2) for some $q \geq 1$, then we say that R is D_q (this identity is studied extensively by Domokos in [5]). In the above vein, it follows trivially that if R is D_{2^m}, then R is Ls_{m+1}; in particular,

$$\mathrm{D}_2 \ \Rightarrow\ \mathrm{Ls}_2. \tag{6}$$

It is worth noting (see [18, Proposition 2.1(1)] or [8]) that if a ring is Ln_2, then, although it need not be D_2, it does satisfy the weaker identity $[x_1, y_1][x_1, y_2] = 0$.

On the other hand, we draw the reader's attention to the fact that the identity $[[x_1, y_1], [x_1, y_2]] = 0$ implies the "seemingly stronger" (Lie solvable of index 2) identity $[[x_1, y_1], [x_2, y_2]] = 0$ (see [13, page 2582]).

For our purposes, we want to see the k-generated ($k \geq 4$) Grassmann algebra, first of all, as showing that if an algebra is Ln_2, then it is not necessarily D_2, or for short, in general,

$$\mathrm{Ln}_2 \ \not\Rightarrow\ \mathrm{D}_2. \tag{7}$$

Observe that the 2×2 upper triangular F-algebra $U_2(F)$ is D_2, but it is not Ln_2, and so, for F-algebras $\mathcal{A}$, we also have, in general,

$$\mathrm{D}_2 \;\not\Rightarrow\; \mathrm{Ln}_2. \tag{8}$$

Moreover, note that it is even possible for an F-subalgebra $\mathcal{A}$ of $U_n^\star(F)$ to be D_2 (and hence Ls_2) without $\mathcal{A}$ being Ln_2. To wit, let $\mathcal{A} = U_4^\star(F)$. By [17, Remark 33] the maximum dimension of an Ln_2 F-subalgebra of $M_n(F)$ is exactly $\left\lfloor \frac{n^2}{3} \right\rfloor + 1$, which, for $n = 4$, is equal to 6. On the other hand, $\mathcal{A}$ is an F-algebra of the type on page 157 in [5] (see also (2) in Example 1 below), with $q = 2$, and so $\mathcal{A}$ is D_2. However, $\dim_F(\mathcal{A}) = 7$, and so being an F-subalgebra of $M_4(F)$, $\mathcal{A}$ is not Ln_2. Hence, the answer to Question 2 in [17, Section 9], relating to [17, Theorem 24], is no.

As far as (7) is concerned, we now consider the following typical examples of F-subalgebras of $M_n(F)$ which are Ln_{q-1}, considered in [17], and typical examples of F-subalgebras of $M_n(F)$ which are D_q, considered in [5].

Example 1. Let $n \geq 1$ and let $n_1, n_2, \ldots, n_q$ be positive integers such that $n_1 + \cdots + n_q = n$, for some $q \leq n$. For these integers we consider two classes of F-subalgebras. Firstly, we work with the F-subalgebra

$$\mathcal{A} = FI_n + \left\{ \begin{bmatrix} A_1 & A_{(1,2)} & \cdots & A_{(1,q)} \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{(q-1,q)} \\ 0 & \cdots & 0 & A_q \end{bmatrix} \right\} \tag{9}$$

of $U_n^\star(F)$, where for every i, $A_i = M_{n_i}(\{0\})$, and $A_{(j,k)} = M_{n_j \times n_k}(F)$ for all j and k such that $1 \leq j < k \leq q$. The class of algebras $\mathcal{A}$ is considered in [17], where they are called algebras of $n \times n$ matrices over F of type $(n_1, n_2, \ldots, n_q)$. They are basic examples of F-subalgebras of $M_n(F)$ which are Ln_{q-1}. In fact, if

$$n_1 = n_2 = \cdots = n_{q-r} = \left\lfloor \frac{n}{q} \right\rfloor, \qquad \text{and} \qquad n_{q-r+1} = \cdots = n_q = \left\lfloor \frac{n}{q} \right\rfloor + 1$$

where, from the Division Algorithm, r is the (unique) non-negative integer in the equation

$$n = q \left\lfloor \frac{n}{q} \right\rfloor + r, \quad 0 \leq r < q, \tag{10}$$

then $\mathcal{A}$ is an Ln_{q-1} F-subalgebra of $\mathbb{M}_n(F)$ with maximum dimension.

Now, we consider the F-subalgebra

$$\mathcal{D} = \left\{ \begin{bmatrix} D_1 & D_{(1,2)} & \cdots & D_{(1,q)} \\ 0 & D_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & D_{(q-1,q)} \\ 0 & \cdots & 0 & D_q \end{bmatrix} \right\} \tag{11}$$

of $M_n(F)$, where D_i is a commutative F-subalgebra of $M_{n_i}(F)$ for every i, and as before, $D_{j,k} = M_{n_j \times n_k}(F)$ for all j and k such that $1 \le j < k \le q$. These F-algebras $\mathcal{D}$ (see [5]) are D_q (see (2)). In Section 6 we provide detailed (more detailed than in [5]) results of how, using $\mathcal{D}$, one can obtain F-subalgebras of $M_n(F)$ of this form with maximum dimension.

Now, let

$$\mathcal{A}' = FI_{n_1+n_2} + \begin{bmatrix} A_1 & A_{(1,2)} \\ 0 & A_2 \end{bmatrix} \subseteq U^{\star}_{n_1+n_2}(F),$$

where A_1, A_2 and $A_{(1,2)}$ are the matrices appearing in (9). It is easy to see that $\mathcal{A}'$ is a commutative F-subalgebra of $M_{n_1+n_2}(F)$ (and hence $\mathcal{A}'$ is Ln_q and D_q for every $q \ge 1$), and so by referring to $\mathcal{D}$, we conclude that $\mathcal{A}$, apart from being Ln_{q-1}, as mentioned earlier, is also D_{q-1}.

For the special case $q = 3$, the typical F-subalgebra $\mathcal{A}$ of $M_n(F)$ considered above is Ln_2, and by the foregoing arguments, $\mathcal{A}$ is also D_2. This shows that, as far as these particular F-subalgebras $\mathcal{A}$ of $M_n(F)$ are concerned, the situation seems to differ from the general case (7), as the mentioned m-generated Grassmann algebra shows.

In fact, the m-generated Grassmann algebra also shows that an Ln_2 F-subalgebra of a full matrix algebra over F need not be D_2, as is evident from the following example.

Example 2. Since $E^{(m)}$ is a 2^m-dimensional vector space over F, it follows that the F-algebra of all linear transformations of $E^{(m)}$ is isomorphic to the full matrix algebra $M_{2^m}(F)$. Also, the map $a \mapsto \mathcal{L}_a,\ a \in E^{(m)}$, is a monomorphism, where $\mathcal{L}_a$ is the “left multiplication” linear transformation $\mathcal{L}_a(x) = ax,\ x \in E^{(m)}$. (See, also, [12].)

On a positive note, for a large class of R-subalgebras of $M_n(R)$, R any commutative ring, we will show in Section 4 that the implication $\mathrm{Ln}_2 \Rightarrow \mathrm{D}_2$ does hold.

Example 3. By [13, Corollary 2.2], for any commutative ring R, the subring $U_3^{\star}(U_3^{\star}(R))$ of $U_9^{\star}(R)$ is Ls_2, but it is neither Ln_2 nor D_2, and so we have, in general,

$$\mathrm{Ls}_2 \quad \not\Rightarrow \quad \mathrm{Ln}_2 \ \text{ or } \ \mathrm{D}_2. \tag{12}$$

At this stage we point out that there is an error in the sentence preceding [13, Theorem 2.1], where the authors attempt to show that if R is a non-commutative ring, then $U_3^\star(R)$ is not D_2. Namely, the authors consider $e_{2,2}$ as an element of $U_3^\star(R)$, which is not true. All the other results in [13] are correct, and fortunately the resulting gap in [13] can be fixed by showing that, for an arbitrary ring R (not necessarily commutative), $U_3^\star(U_3^\star(R))$ is not D_2. This can be done as follows:

For $e_{1,2},\ e_{2,3},\ I_3 \in U_3^\star(R)$ consider the matrices

$$X = \begin{bmatrix} e_{1,2} & 0 & 0 \\ 0 & e_{1,2} & 0 \\ 0 & 0 & e_{1,2} \end{bmatrix}, \qquad Y = \begin{bmatrix} e_{2,3} & 0 & 0 \\ 0 & e_{2,3} & 0 \\ 0 & 0 & e_{2,3} \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & I_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_3 \\ 0 & 0 & 0 \end{bmatrix}$$

in $U_3^\star(U_3^\star(R))$. Then

$$[X, Y]\,[U, V] = \begin{bmatrix} 0 & 0 & e_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$$

(with $e_{1,3} \in U_3^\star(R)$).

4. The polynomial identities in structural and related R-subalgebras of $M_n(R)$, R a commutative ring

In this section we will consider structural R-subalgebras of $M_n(R)$, R a commutative ring, and R-subalgebras of $M_n(R)$ such that their Jacobson radicals are 'structural'.

A structural matrix ring over a (not necessarily commutative) ring R is a subring of full the matrix ring $M_n(R)$ consisting of all matrices having zero in certain prescribed positions and any elements of R in the other positions. To be more precise, recall that the class of structural matrix rings or incidence rings has been studied extensively, see for example, [3], [4], [9], [16], [19] and [20]. For a reflexive and transitive binary relation θ on the set $\{1, 2, ..., n\}$, the structural matrix subring $M_n(\theta, R)$ of $M_n(R)$ is defined as follows:

$$M_n(\theta, R) = \{A \in M_n(R) \mid A_{i,j} = 0 \text{ if } (i,j) \notin \theta\}.$$

Note that if, for any ordered pair (i, j), there is a matrix A in a structural matrix ring $M_n(\theta, R)$ such that $A_{i,j} \neq 0$, then $Re_{i,j} \subseteq M_n(\theta, R)$, i.e.,

$$\pi_{i,j}\big(M_n(\theta, R)\big) \neq \{0\} \quad \Rightarrow \quad Re_{i,j} \subseteq M_n(\theta, R) \tag{13}$$

(here $\pi_{i,j}$ is the natural projection onto the (i,j)-entry). It can be shown (see [3, page 1386] or [16, page 5604]) that, for some k, there are positive integers $n_1, \ldots, n_k$ such that $n_1 + \cdots + n_k = n$ and $M_n(\theta, R)$ is (isomorphic to) a block(ed) triangular matrix ring

$$\begin{bmatrix} M_{n_1}(R) & M_{n_1\times n_2}(X_{(1,2)}) & \cdots & M_{n_1\times n_k}(X_{(1,k)}) \\ 0 & M_{n_2}(R) & \ddots & \vdots \\ \vdots & \ddots & \ddots & M_{n_{k-1}\times n_k}(X_{(k-1,k)}) \\ 0 & \cdots & 0 & M_{n_k}(R) \end{bmatrix}, \tag{14}$$

where $X_{(i,j)} = \{0\}$ or $X_{(i,j)} = R$ for all i, j with $1 \le i < j \le k$. (See also [4].) By, e.g., [20, Theorem 2.7],

$$J\big(M_n(\theta,R)\big) \cong \begin{bmatrix} M_{n_1}\big(J(R)\big) & M_{n_1\times n_2}(X_{(1,2)}) & \cdots & M_{n_1\times n_k}(X_{(1,k)}) \\ 0 & M_{n_2}\big(J(R)\big) & \ddots & \vdots \\ \vdots & \ddots & \ddots & M_{n_{k-1}\times n_k}(X_{(k-1,k)}) \\ 0 & \cdots & 0 & M_{n_k}\big(J(R)\big) \end{bmatrix},$$

implying that the quotient ring $M_n(\theta,R)/J\big(M_n(\theta,R)\big)$ is (isomorphic to) a direct sum of full matrix rings:

$$M_n(\theta,R)/J\big(M_n(\theta,R)\big) \cong M_{n_1}\big(R/J(R)\big) \oplus \cdots \oplus M_{n_k}\big(R/J(R)\big). \tag{15}$$

At this point we note that complete block triangular matrix rings $M_n(\theta, F)$ over a field F (that is the case when $X_{(i,j)} = F$ for all i, j with $1 \le i < j \le k$ in the block triangular matrix ring $M_n(\theta, R)$ above) feature prominently in [6], where it is proved that $Id\big(M_n(\theta,F)\big) = Id\big(M_{n_1}(F)\big)\cdots Id\big(M_{n_k}(F)\big)$. Here, $Id(\mathcal{A})$ denotes the set of all polynomial identities of $\mathcal{A}$ (for an algebra $\mathcal{A}$), which is a two-sided ideal of the free (associative) algebra $F\langle X\rangle$ of polynomials in the non-commuting indeterminates $x \in X$ (for a set X). In fact, $Id(\mathcal{A})$ is an ideal invariant under all endomorphisms of $F\langle X\rangle$.

Considering (6) and (12), and observing that $U_3^\star\big(U_3^\star(R)\big)$ in Example 3 is not a structural subring of $M_9(R)$, we prove now that the converse of (6) holds for the class of structural matrix rings over a commutative ring. In the light of (4) this shows that the implication $\mathrm{Ln}_2 \Rightarrow \mathrm{D}_2$ holds for this class of matrix rings.

Theorem 4. *Let $\mathcal{A}$ be a structural matrix subring of $M_n(R)$, R a commutative ring. Without loss of generality, we may assume that $\mathcal{A}$ is a block triangular matrix ring as in (14). If $\mathcal{A}$ satisfies the polynomial identity $[[x_1,y_1],[x_2,y_2]] = 0$ (in particular, if $\mathcal{A}$ is* Ln_2*), then $\mathcal{A} \subseteq U_n(R)$ and $\mathcal{A}$ satisfies the identity $[x_1,y_1][x_2,y_2] = 0$.*

Proof. We first show that $\mathcal{A} \subseteq U_n(R)$. By (15), there are positive integers $n_1, \ldots, n_k$ such that $n_1 + \cdots + n_k = n$ and

$$\mathcal{A}/J(\mathcal{A}) = M_{n_1}\big(R/J(R)\big) \oplus \cdots \oplus M_{n_k}\big(R/J(R)\big).$$

Assuming that $\mathcal{A}$ is Ls_2, then so is $\mathcal{A}/J(\mathcal{A})$, which implies that $n_1 = \cdots = n_k = 1$ (otherwise $[[e_{1,1}, e_{1,2}], [e_{2,2}, e_{2,1}]] = e_{1,1} - e_{2,2} \neq 0$). Hence, $\mathcal{A} \subseteq U_n(R)$, and so

$$[\mathcal{A}, \mathcal{A}] \subseteq J(\mathcal{A}), \tag{16}$$

which means that there is nothing to prove for $n = 1, 2$.

Next, let $\mathcal{A}$ be an Ls_2 structural matrix subring of $U_n(R)$ for some $n \geq 3$, and assume by induction that the result is true for Ls_2 structural matrix subrings of $U_k(R)$, $k < n$.

Since every matrix A in $\mathcal{A}$ is upper triangular, there are an $(n-1) \times (n-1)$ upper triangular matrix $\overline{A}$ and a $1 \times (n-1)$ matrix N_A such that

$$A = \left[\begin{array}{c|c} A_{1,1} & N_A \\ \hline 0 & \overline{A} \end{array}\right] \tag{17}$$

It is not hard to see that

$$\overline{\mathcal{A}} := \{\overline{A} : A \in \mathcal{A}\} \tag{18}$$

is an Ls_2 structural subring of $U_{n-1}(R)$, and so by the induction hypothesis $\overline{\mathcal{A}}$ is D_2. Hence, for $A, B, C, D \in \mathcal{A}$, it follows from the induction hypothesis that

$$[A, B]\,[C, D] = \left[\begin{array}{c|c} 0 & N_{[A,B][C,D]} \\ \hline 0 & [\overline{A}, \overline{B}]\,[\overline{C}, \overline{D}] \end{array}\right] = \left[\begin{array}{c|c} 0 & N_{[A,B][C,D]} \\ \hline 0 & 0 \end{array}\right].$$

Suppose that for some $A, B, C, D \in \mathcal{A}$ we have $N_{[A,B][C,D]} \neq 0$, which means that $\mathcal{A}$ is not D_2. Then $(N_{[A,B][C,D]})_{1,t} \neq 0$ for some t such that $1 \leq t \leq n-1$, i.e., $([A, B]\,[C, D])_{1,t+1} \neq 0$ (keep in mind that $N_{[A,B][C,D]}$ is a $1 \times (n-1)$ matrix, and $[A, B]\,[C, D]$ is an $n \times n$ matrix). Since

$$(N_{[A,B][C,D]})_{1,t} = \sum_{s=1}^{t+1} [A, B]_{1,s} \cdot [C, D]_{s,t+1}\,,$$

it follows from (16) that

$$(N_{[A,B][C,D]})_{1,t} = \sum_{s=2}^{t} [A, B]_{1,s} \cdot [C, D]_{s,t+1} \neq 0. \tag{19}$$

Since $\mathcal{A}$ is assumed to be Ls_2, we clearly have

$$N_{[C,D][A,B]} = N_{[A,B][C,D]}, \tag{20}$$

and so

$$(N_{[C,D][A,B]})_{1,t} = (N_{[A,B][C,D]})_{1,t} \neq 0. \tag{21}$$

Let C' and D' be the matrices obtained form C and D, respectively, such that

$$(C')_{i,t+1} = (D')_{i,t+1} = 0 \text{ for } i = 1, \dots, t, \tag{22}$$

and all other entries in C' and D' are the same as in C and D, respectively. As $\mathcal{A}$ is structural, it follows that $C', D' \in \mathcal{A}$. Consequently, by (22), for any $s = 2, \dots, t$ we have

$$\begin{aligned}[C', D']_{s,t+1} &= \left(\sum_{z=1}^{t} C'_{s,\,z} D'_{z,t+1}\right) + C'_{s,t+1} D'_{t+1,t+1} \\ &\quad - \left(\sum_{w=1}^{t} D'_{s,w} C'_{w,t+1}\right) - D'_{s,t+1} C'_{t+1,t+1} = 0.\end{aligned}$$

Thus

$$(N_{[A,B][C',D']})_{1,t} = \sum_{s=2}^{t} [A, B]_{1,s} \cdot [C', D']_{s,t+1} = 0. \tag{23}$$

On the other hand,

$$(N_{[C',D'][A,B]})_{1,t} = \sum_{s=2}^{t} [C', D']_{1,s} \cdot [A, B]_{s,t+1}\,,$$

and for any $s = 2, \dots, t$ we have

$$\begin{aligned}[C', D']_{1,s} = (C'D' - D'C')_{1,s} &= \sum_{g=1}^{s} (C'_{1,g} D'_{g,s} - D'_{1,g} C'_{g,s}) \\ &= \sum_{g=1}^{s} (C_{1,g} D_{g,s} - D_{1,g} C_{g,s}) = [C, D]_{1,s}\,.\end{aligned}$$

Thus,

$$(N_{[C',D'][A,B]})_{1,t} = (N_{[C,D][A,B]})_{1,t}. \tag{24}$$

As $\mathcal{A}$ is Ls_2, we have, by (20),

$$N_{[A,B][C',D']} - N_{[C',D'][A,B]} = 0;$$

in particular,

$$(N_{[A,B][C',D']})_{1,t} - (N_{[C',D'][A,B]})_{1,t} = 0,$$

which together with (23) and (24) gives

$$(N_{[A,B][C,D]})_{1,t} = 0.$$

This contradicts (21). □

Analyzing the proof of Theorem 4 it is clear that we use only the fact that $J(\mathcal{A})$ has the 'structural' property. (In this regard, see also [2], where structural matrix bimodules are studied.) Therefore we propose the following:

Definition 5. An R-subalgebra $\mathcal{A}$ of $U_n(R)$, R a commutative ring, is said to have structural Jacobson radical if, whenever $\pi_{i,j}(\mathcal{A}) \neq \{0\}$ for $i < j$, then $Re_{i,j} \subseteq \mathcal{A}$.

Therefore we have another class of matrix rings for which the implication $\mathrm{Ln}_2 \Rightarrow \mathrm{D}_2$ holds:

Corollary 6. *If R is a commutative ring and $\mathcal{A}$ is an Ls_2 (in particular, if $\mathcal{A}$ is an Ln_2) R-subalgebra of $U_n(R)$ with structural Jacobson radical, then $\mathcal{A}$ is D_2.*

For the class of all F-subalgebras of $U_n(F)$ with structural Jacobson radical, Corollary 6 and the property of the 2×2 upper triangular F-algebra $U_2(F)$ mentioned between (7) and (8) can be summarized as follows as far as (4) and (6)–(8) are concerned:

$$\begin{array}{ccl} D_2 & \Leftrightarrow & Ls_2 \\ Ln_2 & \Rightarrow & Ls_2 \text{ (and } D_2) \\ D_2 & \not\Rightarrow & Ln_2. \end{array}$$

With respect to the property of the 2×2 upper triangular F-algebra $U_2(F)$ mentioned between (7) and (8), and the crucial role played by the Jacobson radical, being structural in Corollary 6, the above summary gives rise to the following question for subrings (obviously not structural) of upper triangular matrix rings:

Question 7. Does there exist a subring of $U_n(R)$ (for some ring R) which is Ln_2 but not D_2?

5. Decomposition of certain F-subalgebras of $U_n(F)$

Recall (see, for example, [14]) that a ring R is said to satisfy the Engel condition of index m if the identity

$$[\dots[[x,\overbrace{y],y],\dots,y}^{m \text{ times}}]=0,$$

holds in R. Obviously, any Ln_m ring satisfies the Engel condition of index m. A ring R is said to satisfy the Engel condition if it satisfies the Engel condition of index m for some $m \in \mathbb{N}$. It is shown in [17, Proposition 6] that every idempotent in a ring satisfying the Engel condition is central.

Now let $\mathcal{A}$ be an Ln_2 structural F-subalgebra of $M_n(F)$. Without loss of generality, we may assume that $\mathcal{A}$ is a block triangular matrix F-algebra as in (14), with $R = F$. Then by Theorem 4, $\mathcal{A} \subseteq U_n(F)$. As $\mathcal{A}$ is structural, we have $Fe_{k,k} \subseteq \mathcal{A}$ for all $k,\ k = 1,\dots,n$. Since $e_{k,k}$ is an idempotent, by [17, Proposition 6], it is central. If $\pi_{i,j}(\mathcal{A}) \neq \{0\}$ for some $i < j$, then again the fact that $\mathcal{A}$ is structural ensures that $e_{i,j} \in \mathcal{A}$, and so we have $e_{i,j} = e_{i,i}e_{i,j} = e_{i,j}e_{i,i} = 0$; a contradiction. (We note that one can also show that $e_{i,j} \notin \mathcal{A}$ if $i < j$, without using the Engel condition. To wit, if $e_{i,j} \in \mathcal{A}$ for some $i < j$, then $[e_{i,i},[e_{i,i},e_{i,j}]] = e_{i,j} \neq 0$, contradicting the assumption that $\mathcal{A}$ is Ln_2.) Thus $\pi_{i,j}(\mathcal{A}) = \{0\}$ for all $1 \le i < j \le n$, and so

$$\mathcal{A} = \left\{ \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix} : a_i \in F \text{ for all } i \right\},$$

i.e., $\mathcal{A}$ comprises precisely the $n \times n$ diagonal matrices over F.

We have proved the following result:

Proposition 8. *Let $\mathcal{A}$ be a structural matrix F-subalgebra of $M_n(F)$. Without loss of generality, we may again assume that $\mathcal{A}$ is a block triangular matrix F-algebra as in (14), with $R = F$. If $\mathcal{A}$ is Ln_2, then $\mathcal{A} \cong F^n$.*

The above proposition together with Example 1 and Corollary 6 justify studying the structure of F-subalgebras of $U_n(F)$ with structural Jacobson radical:

Theorem 9. *If $\mathcal{A}$ is an F-subalgebra of $U_n(F)$ (for some $n \ge 1$) with structural Jacobson radical such that $\mathcal{A}$ satisfies the Engel condition (in particular, if $\mathcal{A}$ is Lie nilpotent) then, for some $q \ge 1$, $\mathcal{A}$ is a direct sum*

$$\mathcal{A} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_q = \begin{array}{c} \begin{bmatrix} \mathcal{A}_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{A}_q \end{bmatrix} \end{array} \begin{array}{c} n_1 \\ \vdots \\ \vdots \\ n_q \end{array} \Bigg\} n$$

of F-subalgebras $\mathcal{A}_i$ of $U^\star_{n_i}(F)$ with structural Jacobson radicals, for some $n_i \geq 1$, $i = 1, 2, \dots, q$, such that $n_1 + \cdots + n_q = n$.

Proof. For $n = 1$ there is nothing to prove, since then we can take $q = 1$ and $n_1 = 1$. Hence, let $n \geq 2$, and suppose the result is true for all $k < n$. Let $A \in \mathcal{A}$ be as in (17), and let $\overline{\mathcal{A}}$ be as in (18). Then it follows readily that $\overline{\mathcal{A}}$ is an F-subalgebra of $U_{n-1}(F)$ satisfying the Engel condition, and $\overline{\mathcal{A}}$ has structural Jacobson radical (verification of the latter: $1 < i < j$ and $\pi_{i,j}(\overline{\mathcal{A}}) \neq \{0\} \Rightarrow \pi_{i,j}(\mathcal{A}) \neq 0 \Rightarrow Fe_{ij} \subseteq \mathcal{A} \Rightarrow Fe_{ij} \subseteq \overline{\mathcal{A}}$). By the induction hypothesis there is a $q \geq 1$ and $n_i \geq 1$, $i = 1, \dots, q$, such that

$$\overline{\mathcal{A}} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_q$$

for some F-subalgebras $\mathcal{A}_i$ of $U^\star_{n_i}(F)$ with structural Jacobson radicals, and $n_1 + \cdots + n_q = n - 1$.

First consider the case $q = 1$, in which case $\overline{\mathcal{A}}$ is an F-subalgebra of $U^\star_{n-1}(F)$ with structural Jacobson radical. Then $I_{n-1} \in \overline{\mathcal{A}}$. If $A_{1,1} = 1$ for every $A \in \mathcal{A}$ such that $\overline{A} = I_{n-1}$, then $\mathcal{A} \subseteq U^\star_n(F)$. Otherwise, there is an $A' \in \mathcal{A}$, with $A'_{1,1} \neq 1$, such that $\overline{A'} = I_{n-1}$, i.e.,

$$A' = \left[\begin{array}{c|c} A'_{1,1} & N_{A'} \\ \hline 0 & I_{n-1} \end{array}\right].$$

As $\mathcal{A}$ has structural Jacobson radical, we have that

$$\left[\begin{array}{c|c} 0 & N_{A'} \\ \hline 0 & 0 \end{array}\right] \in \mathcal{A},$$

and so $e_{11} = (A'_{1,1} - 1)^{-1}(A'_{1,1} - 1)e_{11} = (A_{1,1} - 1)^{-1}(A' - N_{A'} - I_n) \in \mathcal{A}$, implying that

$$Fe_{11} \subseteq \mathcal{A}. \tag{25}$$

If $\pi_{1,j}(\mathcal{A}) \neq \{0\}$ for some j, $2 \leq j \leq n$, then $e_{1j} \in \mathcal{A}$, since $\mathcal{A}$ has structural Jacobson radical. As e_{11} is an idempotent in $\mathcal{A}$, it follows from [17, Proposition 6] that e_{11} is central, and so $e_{1,j} = e_{1,1}e_{1,j} = e_{1,j}e_{1,1} = 0$; a contradiction. We conclude that $\pi_{1,j}(\mathcal{A}) = \{0\}$ for all $j = 2, \dots, n$, and so by (25),

$$\mathcal{A} = F \oplus \overline{\mathcal{A}} = \left[\begin{array}{c|c} F & 0 \\ \hline 0 & \overline{\mathcal{A}} \end{array}\right] = \left[\begin{array}{c|c} U^\star_1(F) & 0 \\ \hline 0 & \overline{\mathcal{A}} \end{array}\right]$$

(with $\overline{\mathcal{A}} \subseteq U^\star_{n-1}(F)$ and $1 + (n-1) = n$).

Next, consider the case $q \geq 2$. Then, since $\overline{\mathcal{A}} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_q \subseteq U^{\star}_{n_1}(F) \oplus \cdots \oplus U^{\star}_{n_q}(F)$, every matrix A in $\mathcal{A}$ can be viewed as

$$A = \left[\begin{array}{c|c} \tilde{A} & M_A \\ \hline 0 & B \end{array}\right],$$

where $\tilde{A}$, B and M_A are $(n_1+1) \times (n_1+1)$, $(n_2+n_3+\cdots+n_q) \times (n_2+n_3+\cdots+n_q)$ and $(n_1+1) \times (n_2+n_3+\cdots+n_q)$ matrices, respectively. Here, as in (17), we have

$$\tilde{A} = \left[\begin{array}{c|c} A_{11} & N_{\tilde{A}} \\ \hline 0 & \overline{\tilde{A}} \end{array}\right],$$

with $A_{11} \in F$, $\overline{\tilde{A}} \in \mathcal{A}_1$ and $B \in \mathcal{A}_2 \oplus \cdots \oplus \mathcal{A}_q$. (Note that $A_{11} = \tilde{A}_{1,1}$ and that the last n_1 rows of M_A comprises only zeroes.)

Assume first that $A_{11} = 1$ for every $A \in \mathcal{A}$ such that $\overline{\tilde{A}} = I_{n_1}$. Then

$$A = \left[\begin{array}{c|c} I_{n_1+1} & 0 \\ \hline 0 & 0 \end{array}\right] \in \mathcal{A}.$$

Therefore, if $\pi_{1,j}(\mathcal{A}) \neq \{0\}$ for some $j > n_1+1$, then $e_{1j} \in \mathcal{A}$, as before. As A is an idempotent in $\mathcal{A}$, and hence central, we get $e_{1j} = Ae_{1j} = e_{1j}A = 0$; a contradiction. We conclude that $\pi_{1,j}(\mathcal{A}) = \{0\}$ for all $j > n_1+1$, i.e.

$$\mathcal{A} = \tilde{\mathcal{A}} \oplus \mathcal{A}_2 \oplus \cdots \oplus \mathcal{A}_q,$$

with $\tilde{\mathcal{A}} := \{\tilde{A} : A \in \mathcal{A}\} \subseteq U^{\star}_{n_1+1}(F)$.

Next, if there is an $A \in \mathcal{A}$ with $A_{11} \neq 1$ such that

$$\tilde{A} = \left[\begin{array}{c|c} A_{11} & N_{\tilde{A}} \\ \hline 0 & I_{n_1} \end{array}\right],$$

then an adaptation of the foregoing arguments readily shows that

$$\mathcal{A} = F \oplus \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_q$$

(with $1 + n_1 + \cdots + n_q = n$). $\square$

Now, we want to show that Theorem 9 is not longer true if we consider subrings of upper triangular matrix algebras over a field F which are not F-subalgebras.

Example 10. Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. Consider the subring

$$\mathcal{A} := \left\{ \begin{bmatrix} a & b & c & 0 & 0 & 0 \\ 0 & a & d & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{a} & e & f \\ 0 & 0 & 0 & 0 & \overline{a} & g \\ 0 & 0 & 0 & 0 & 0 & \overline{a} \end{bmatrix} : a, b, c, d, e, f, g \in \mathbb{C} \right\}$$

of $U_6(\mathbb{C})$, where for any $a \in \mathbb{C}$ by $\overline{a}$ we denote the conjugate of a. Note that $\mathcal{A}$ is not a $\mathbb{C}$-subalgebra of the $\mathbb{C}$-algebra $U_6(\mathbb{C})$, but it is obviously an $\mathbb{R}$-subalgebra of $U_6(\mathbb{C})$. It has structural Jacobson radical, since

$$J(\mathcal{A}) = \left\{ \begin{bmatrix} 0 & \mathbb{C} & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\}.$$

Also $\mathcal{A}$ is Lie nilpotent of index 2, as can be verified directly. However, $\mathcal{A} \notin U_6^{\star}(\mathbb{C})$, and $\mathcal{A}$ is not direct sum of (at least two) subrings as in Theorem 9. In fact, $\mathcal{A}$ is an indecomposable ring, because it can be easily shown that 0 and 1 are the only central idempotents of $\mathcal{A}$ (in fact, the only idempotents of $\mathcal{A}$).

It is noteworthy that even a commutative indecomposable subring of $U_n(F)$ with structural Jacobson radical need not be in $U_n^{\star}(F)$, as the subring

$$\left\{ \begin{bmatrix} a & 0 & b \\ 0 & \overline{a} & 0 \\ 0 & 0 & a \end{bmatrix} : a, b \in \mathbb{C} \right\}$$

of $U_3(\mathbb{C})$ shows.

6. The maximum dimension of an F-subalgebra of $M_n(F)$ satisfying the identity $[x_1, y_1]\,[x_2, y_2] \cdots [x_q, y_q] = 0$

As mentioned earlier, one of the main motivations for this paper is Domokos's paper [5], which deals with the identity in (2). The main result in [5] is the following:

Theorem 11. *Let F be a field, and $\mathcal{A}$ a finite dimensional* D_q *F-algebra. If M is a finitely generated faithful module over $\mathcal{A}$, then*

$$\dim_F(M) \geq \sqrt{\frac{\dim_F(\mathcal{A}) - q}{\frac{1}{2} - \frac{1}{4q}}}. \tag{26}$$

In the proof of the above theorem in [5] Domokos shows, firstly, that for the considered F-algebra $\mathcal{A}$, the following, also very interesting for us, inequality holds for some positive integers $n_1, \dots, n_q$ such that $n_1 + \dots + n_q = \dim_F(M)$:

$$\dim_F(\mathcal{A}) \leq \frac{1}{2}\dim_F(M)^2 + q - \sum_{i=1}^{q}\left(\frac{n_i^2}{2} - \left\lfloor \frac{n_i^2}{4} \right\rfloor\right). \tag{27}$$

As presented at the end of [5], the inequality in (27) is sharp, and so, since (26) can be written as

$$\dim_F(\mathcal{A}) \leq q + (\frac{1}{2} - \frac{1}{4q})\dim_F(M)^2, \tag{28}$$

we have

$$\frac{1}{2}\dim_F(M)^2 + q - \sum_{i=1}^{q}\left(\frac{(n_i^*)^2}{2} - \left\lfloor \frac{(n_i^*)^2}{4} \right\rfloor\right) \leq q + (\frac{1}{2} - \frac{1}{4q})\dim_F(M)^2 \tag{29}$$

for any q-tuple $(n_1^*, \dots, n_q^*)$ (with $n_1^* + \dots + n_q^* = \dim_F(M)$) for which the inequality in (27) is sharp. In Section 7 we will explore when the two upper bounds in (29) for a D_q F-subalgebra of $M_n(F)$ are equal.

For any $n_1, \dots, n_q$ such that $n_1 + \dots + n_q = n$, an F-subalgebra $\mathcal{A}$ of $M_n(F)$ satisfying (2) is constructed in [5], with

$$\dim_F(A) = q + \sum_{i=1}^{q}\left\lfloor \frac{n_i^2}{4} \right\rfloor + \sum_{1 \leq i < j \leq q} n_i n_j. \tag{30}$$

Taking $M = F^n$, the right hand side in (27) takes the form

$$\frac{1}{2}(n_1 + \dots + n_q)^2 + q - \sum_{i=1}^{q}\left(\frac{n_i^2}{2} - \left\lfloor \frac{n_i^2}{4} \right\rfloor\right), \tag{31}$$

which clearly equals the expression in (30), and so the mentioned sharpness follows.

In [5] the considered $n_1, \dots, n_q$ are mentioned as existing numbers which guarantee that $\frac{1}{2}\dim_F(M)^2 + q - \sum_{i=1}^{q}\left(\frac{n_i^2}{2} - \left\lfloor \frac{n_i^2}{4} \right\rfloor\right)$ is as large as possible, and obviously such a sequence $n_1, \dots, n_q$ with this property exists. However, unfortunately [5] does not reveal how to directly and precisely find these n_i's. We now proceed to show how to do it by working with $M = F^n$.

In order to find the n_i's, $i = 1, 2, \dots, q$, with $n_1 + \dots + n_q = n$, such that the dimension of the algebra $\mathcal{A}$ is a maximum, it follows from (31) that we need to maximize

$$f(n_1, \dots, n_q) := \frac{1}{2}(n^2 - n_1^2 - \dots - n_q^2) + q + \sum_{i=1}^{q}\left\lfloor \frac{n_i^2}{4} \right\rfloor. \tag{32}$$

We claim that the maximum can be obtained by choosing the n_i's in such a way that

$$|n_i - n_j| \leq 1 \tag{33}$$

for all $1 \leq i, j \leq q$. Indeed, suppose without loss of generality that

$$n_2 - n_1 > 1, \tag{34}$$

and replace n_1 and n_2 in (32) by $n_1 + 1$ and $n_2 - 1$, respectively. Then, for $T := f(n_1 + 1, n_2 - 1, n_3, \ldots, n_q)$, we have

$$\begin{aligned} T = \frac{1}{2}\Big(n^2 - (n_1+1)^2 - (n_2-1)^2 - n_3^2 - \cdots - n_q^2\Big) + \\ + q + \left\lfloor \frac{(n_1+1)^2}{4} \right\rfloor + \left\lfloor \frac{(n_2-1)^2}{4} \right\rfloor + \sum_{i=3}^{q} \left\lfloor \frac{n_i^2}{4} \right\rfloor. \end{aligned}$$

With $S := f(n_1, n_2, \ldots, n_q)$ one obtains

$$\begin{aligned} T - S = n_2 - n_1 - 1 + \left\lfloor \frac{(n_1+1)^2}{4} \right\rfloor - \left\lfloor \frac{(n_1)^2}{4} \right\rfloor + \\ + \left\lfloor \frac{(n_2-1)^2}{4} \right\rfloor - \left\lfloor \frac{(n_2)^2}{4} \right\rfloor. \end{aligned}$$

We consider the following three possible cases:

1. $\underline{n_1 = 2k_1 + 1,\ n_2 = 2k_2 + 1}$: By (34), $k_2 - k_1 \geq 1$, since k_2 and k_1 are integers. Now,

$$\begin{aligned} T - S &= 2k_2 - 2k_1 - 1 + \left\lfloor \frac{4k_1^2 + 8k_1 + 4}{4} \right\rfloor - \left\lfloor \frac{4k_1^2 + 4k_1 + 1}{4} \right\rfloor \\ &\quad + \left\lfloor \frac{4k_2^2}{4} \right\rfloor - \left\lfloor \frac{4k_2^2 + 4k_2 + 1}{4} \right\rfloor \\ &= 2k_2 - 2k_1 - 1 + k_1^2 + 2k_1 + 1 - k_1^2 - k_1 + k_2^2 - k_2^2 - k_2 \\ &= k_2 - k_1 \geq 1. \end{aligned}$$

2. $\underline{n_1 \text{ is odd and } n_2 \text{ is even, or vice versa}}$: Similar calculations as in case (1) show that $T - S \geq 1$.

The last case produces a slightly different outcome.

3. $\underline{n_1 = 2k_1,\ n_2 = 2k_2}$: By (34) we have $k_2 - k_1 \geq 1$. Again, routine calculations yield

$$T - S = k_2 - k_1 - 1 \geq 0.$$

Consequently, if $n_2 - n_1 = 2$ (i.e. if $k_2 - k_1 = 1$), then $T = S$. We have thus proved (33) and the following result.

Lemma 12. *If n_1 and n_2 are positive integers such that $n_2 - n_1 > 1$, then*

$$n_2 - n_1 - 1 + \left\lfloor \frac{(n_1+1)^2}{4} \right\rfloor - \left\lfloor \frac{(n_1)^2}{4} \right\rfloor + \left\lfloor \frac{(n_2-1)^2}{4} \right\rfloor - \left\lfloor \frac{(n_2)^2}{4} \right\rfloor \geq 0,$$

and equality holds if and only if n_1 and n_2 are both even and $n_2 - n_1 = 2$.

We are now in a position to strengthen the condition in (33).

Lemma 13. *If $n_1, \ldots, n_q$ are positive integers with $n_1 + \cdots + n_q = n$ and $1 \leq q \leq n$, and if $|n_i - n_j| \leq 1$ for all i and j, then $q - r$ of the n_i's are equal to $\left\lfloor \frac{n}{q} \right\rfloor$, and the other r n_i's are equal to $\left\lfloor \frac{n}{q} \right\rfloor + 1$, where r is the (unique) non-negative integer obtained in (10).*

Proof. If $n_i < \left\lfloor \frac{n}{q} \right\rfloor$ (respectively, $n_i > \left\lfloor \frac{n}{q} \right\rfloor$) for all i, then $\sum_{i=1}^{q} n_i < q \left\lfloor \frac{n}{q} \right\rfloor < q(\frac{n}{q}) = n$ (respectively, $\sum_{i=1}^{q} n_i \geq q \left(\left\lfloor \frac{n}{q} \right\rfloor + 1 \right) > q(\frac{n}{q}) = n$), and so we conclude that

$$n_{i^\star} = \left\lfloor \frac{n}{q} \right\rfloor \tag{35}$$

for some $i^\star$.

Next, if $n_j \leq \left\lfloor \frac{n}{q} \right\rfloor - 1$ for some j, then $n_k \geq \left\lfloor \frac{n}{q} \right\rfloor + 1$ for some k, otherwise again $\sum_{i=1}^{q} < n$. But then $n_k - n_j \geq 2$; a contradiction. Consequently, $n_i \geq \left\lfloor \frac{n}{q} \right\rfloor$ for all i, and therefore $n_i \in \left\{ \left\lfloor \frac{n}{q} \right\rfloor, \left\lfloor \frac{n}{q} \right\rfloor + 1 \right\}$ for all i. Let

$$I := \left\{ i : \; n_i = \left\lfloor \frac{n}{q} \right\rfloor + 1, \; 1 \leq i \leq q \right\}.$$

By (35), $|I| < q$, and so, since

$$n = \sum_{i=1}^{q} n_i = (q - |I|) \left\lfloor \frac{n}{q} \right\rfloor + |I| \left(\left\lfloor \frac{n}{q} \right\rfloor + 1 \right) = q \left\lfloor \frac{n}{q} \right\rfloor + |I|,$$

and $0 \leq |I| < q$, we conclude that $|I| = r$. $\square$

The foregoing results prove:

Theorem 14. *Let $1 \leq q \leq n$, and let $n = q \left\lfloor \frac{n}{q} \right\rfloor + r$, $0 \leq r < q$. Then*

$$\frac{1}{2} \left(n^2 - (q - r) \left\lfloor \frac{n}{q} \right\rfloor^2 - r \left(\left\lfloor \frac{n}{q} \right\rfloor + 1 \right)^2 \right) +$$

$$+q+(q-r)\left\lfloor\frac{\left\lfloor\frac{n}{q}\right\rfloor^2}{4}\right\rfloor+r\left\lfloor\frac{\left(\left\lfloor\frac{n}{q}\right\rfloor+1\right)^2}{4}\right\rfloor$$

is the precise sharp upper bound for the dimension of a D_q *F-subalgebra of* $M_n(F)$*, which can be obtained by choosing* $q-r$ *commutative subalgebras of* $M_{\lfloor\frac{n}{q}\rfloor}(F)$ *of dimension* $\left\lfloor\frac{\lfloor\frac{n}{q}\rfloor^2}{4}\right\rfloor+1$ *and* r *commutative subalgebras of* $M_{\lfloor\frac{n}{q}\rfloor+1}(F)$ *of dimension* $\left\lfloor\frac{(\lfloor\frac{n}{q}\rfloor+1)^2}{4}\right\rfloor+1$ *on the diagonal blocks for the F-algebra presented in (11) in Example 1 (see also [5, page 157]).*

Observe that if $q=1$ (in which case $r=0$), then the above formula yields $1+\left\lfloor\frac{n^2}{4}\right\rfloor$, which, of course, is the commutative case (see [7] or [15]).

Remark 15. The arguments leading up to Lemma 13 show that the numbers $n_1, n_2, \ldots, n_q$ for the F-algebras $M_{n_i}(F)$ on the diagonal blocks of the F-algebra in Theorem 14 can, without loss of generality, be chosen as follows:

$$n_1,\ldots,n_{q-r} := \left\lfloor\frac{n}{q}\right\rfloor, \quad \text{and} \quad n_{q-r+1},\ldots,n_q := \left\lfloor\frac{n}{q}\right\rfloor+1.$$

However, case (3) above (see the three cases preceding Lemma 13) shows that if we view the numbers $n_1,\ldots,n_q$ as a q-tuple $(n_1,\ldots,n_q)$, and if we assume that $n_i \le n_j$ for all i and j such that $i \le j$, then such a q-tuple, for a given n, q and r as in (10), need not be unique. In fact we have the following possibilities:

1. $\underline{\left\lfloor\frac{n}{q}\right\rfloor \text{ is odd}}$: In this case both $\left\lfloor\frac{n}{q}\right\rfloor-1$ and $\left\lfloor\frac{n}{q}\right\rfloor+1$ are even, and $\left(\left\lfloor\frac{n}{q}\right\rfloor+1\right)-\left(\left\lfloor\frac{n}{q}\right\rfloor-1\right)=2$. Hence any pair $(n_i,n_j)=\left(\left\lfloor\frac{n}{q}\right\rfloor,\left\lfloor\frac{n}{q}\right\rfloor\right)$ with $1\le i<j\le q-r$ can be replaced by $\left(\left\lfloor\frac{n}{q}\right\rfloor-1,\left\lfloor\frac{n}{q}\right\rfloor+1\right)$, because case (3) above shows that changing this new pair back to $\left(\left\lfloor\frac{n}{q}\right\rfloor,\left\lfloor\frac{n}{q}\right\rfloor\right)$ does not change the right hand side of (32). Therefore we conclude that there are $\left\lfloor\frac{q-r}{2}\right\rfloor+1$ distinct sets $\{n_1,\ldots,n_{q-r},n_{q-r+1},\ldots,n_q\}$ giving the maximum dimension in Theorem 14. It is easy to see that this set is unique if and only if $r=q-1$.
2. $\underline{\left\lfloor\frac{n}{q}\right\rfloor \text{ is even}}$: Now $\left\lfloor\frac{n}{q}\right\rfloor+1$ is odd. Invoking similar arguments one sees that there are $\left\lfloor\frac{r}{2}\right\rfloor+1$ distinct sets $\{n_1,\ldots,n_{q-r},n_{q-r+1},\ldots,n_q\}$ yielding the mentioned maximum dimension. This set is unique in this case if and only if $r\in\{0,1\}$.

At this point we want to stress that, in contrast to the above remarks, the corresponding set $\{d_1,\ldots,d_{m+1}\}$ in [17, Section 7], giving an example of an L_m F-subalgebra of $M_n(F)$ of index m with maximum dimension, is unique.

7. When do the two upper bounds coincide?

Next we compare the precise sharp upper bound b_1 (say) in Theorem 14 with the integer upper bound $b_2 := q + \lfloor (\frac{1}{2} - \frac{1}{4q})n^2 \rfloor$ (again using $M = F^n$) in (28). With $n = q\lfloor \frac{n}{q} \rfloor + r,\ 0 \leq r < q$, and with $t := \lfloor \frac{n}{q} \rfloor$, we readily have

$$\begin{aligned} b_1 &= \frac{1}{2}\Big((qt+r)^2 - (q-r)t^2 - r(t+1)^2)\Big) \\ &\quad + q + (q-r)\left\lfloor \frac{t^2}{4} \right\rfloor + r\left\lfloor \frac{(t+1)^2}{4} \right\rfloor \\ &= \frac{1}{2}q(q-1)t^2 + (q-1)rt + \frac{1}{2}r(r-1) \\ &\quad + q + (q-r)\left\lfloor \frac{t^2}{4} \right\rfloor + r\left\lfloor \frac{(t+1)^2}{4} \right\rfloor \end{aligned} \tag{36}$$

and

$$b_2 = q + \left\lfloor \frac{2q-1}{4q}(q^2t^2 + 2qtr + r^2) \right\rfloor. \tag{37}$$

We distinguish between when t is even and when t is odd.

(i) $\underline{t \text{ even, say } t = 2k}$: By (36) and (37),

$$\begin{aligned} b_1 &= \frac{1}{2}q(q-1)(4k^2) + 2(q-1)rk + \frac{1}{2}r(r-1) \\ &\quad + q + (q-r)k^2 + r(k^2+k) \\ &= q + 2q^2k^2 - qk^2 + (2q-1)rk + \frac{1}{2}r(r-1), \end{aligned}$$

and

$$\begin{aligned} b_2 &= q + \left\lfloor \frac{2q-1}{4q}(4q^2k^2 + 4qkr + r^2) \right\rfloor \\ &= q + 2q^2k^2 - qk^2 + (2q-1)kr + \left\lfloor \frac{2q-1}{4q}r^2 \right\rfloor. \end{aligned}$$

Hence, $b_1 = b_2$ if and only if

$$\left\lfloor \frac{2q-1}{4q}r^2 \right\rfloor = \frac{1}{2}r(r-1), \tag{38}$$

which, considering that

$$(2q-1)r^2 = \frac{1}{2}r(r-1)(4q) + (2qr - r^2)$$

and that $2qr - r^2 = r(2q - r) \geq 0$ (keeping in mind that $0 \leq r \leq q - 1$), implies that (38) holds if and only if $2qr - r^2 < 4q$, i.e. $r^2 - 2qr + 4q > 0$. Since $0 \leq r < q$ and $x^2 - 2qx + 4q = 0$ for $x = q \pm \sqrt{q(q-4)}$, we conclude that

(α) if $q \in \{1, 2, 3, 4\}$, then (38) holds for all r (i.e. $r = 0, 1, \ldots, q - 1$);
(β) if $q \geq 5$, then (38) holds if and only if $r = 0, 1, \ldots, \left\lfloor q - \sqrt{q(q-4)} \right\rfloor$.

(ii) <u>t odd, say $t = 2k + 1$</u>: By (1) and (2),

$$\begin{aligned} b_1 &= \frac{1}{2}q(q-1)(4k^2 + 4k + 1) + (q-1)r(2k+1) + \frac{1}{2}r(r-1) \\ &\quad + q + (q-r)(k^2 + k) + r(k^2 + 2k + 1) \\ &= q(2q-1)(k^2 + k) + \frac{1}{2}q(q-1) + (2q-1)rk + qr + \frac{1}{2}r(r-1) + q \end{aligned}$$

and

$$\begin{aligned} b_2 &= q + \left\lfloor \frac{2q-1}{4q}\Big(q^2(4k^2 + 4k + 1) + 2qr(2k+1) + r^2\Big) \right\rfloor \\ &= q + q(2q-1)(k^2 + k) + (2q-1)rk + \left\lfloor \frac{2q-1}{4q}(q + r^2) \right\rfloor . \end{aligned}$$

Therefore, $b_1 = b_2$ if and only if

$$\begin{aligned} \left\lfloor \frac{2q-1}{4q}(q+r)^2 \right\rfloor &= \frac{1}{2}q(q-1) + qr + \frac{1}{2}r(r-1) \\ &= \frac{1}{2}\Big((q+r)^2 - (q+r)\Big), \end{aligned} \tag{39}$$

which, considering that

$$(2q-1)(q+r)^2 = \frac{1}{2}\Big((q+r)^2 - (q+r)\Big)(4q) + (q^2 - r^2)$$

and that $q^2 - r^2 > 0$ for $r = 0, 1, \ldots, q - 1$, implies that (39) holds if and only if $q^2 - r^2 < 4q$, i.e.

$$r^2 > q^2 - 4q.$$

Noting that $q^2 - 4q < 0$ for $q = 1, 2, 3$, that $q^2 - 4q = 0$ for $q = 4$, that $(q-2)^2 > q^2 - 4q$ and that

$$(q-3)^2 - q^2 - 6q + 9 < q^2 - 4q$$

if $2q > 9$, we conclude that

(α') if $q \in \{1, 2, 3\}$, then (39) holds for all r, i.e., $r = 0, 1, \ldots, q-1$;
(β') if $q = 4$, then (39) holds for all $r > 0$, i.e. $r = 1, 2, \ldots, q-1$;
(γ') if $q \geq 5$, then (39) holds if and only if $r = q-2$ or $q-1$.

We have thus proved

Proposition 16. *The precise sharp upper bound in Theorem 14 for the maximum dimension of a* D_q *F-subalgebra of $M_n(F)$ equals the simplified upper bound* $q + \left\lfloor \left(\frac{1}{2} - \frac{1}{4q}\right) n^2 \right\rfloor$ *for all $n \geq q \geq 1$ if and only if $q \in \{1, 2, 3\}$, and (i) $((\alpha), (\beta))$ and (ii) $((\alpha'), (\beta'), (\gamma'))$ give a complete characterization of the values of n for which the two upper bounds coincide for every $q \geq 4$.*

8. Final remarks and an open problem

It is clear that the Ls_2 F-subalgebra $U_3^\star\big(U_3^\star(F)\big)$ (see Example 3) of $M_9(F)$ has dimension 16. It is even possible for a commutative F-subalgebra of $M_9(F)$ to have dimension greater than 16; in fact, by [7] or [15] (see also [17]), the maximum dimension of a commutative F-subalgebra of $M_9(F)$ is 21. Moreover, by Proposition 16, the maximum dimension of a D_2 F-subalgebra of $M_9(F)$ is 32. Such an F-algebra, which is Ls_2 (and which can be constructed using Theorem 14) leads us to the following problem:

Problem 17. For a field F, does an Ls_2 F-subalgebra of $M_n(F)$ (for some n) with dimension larger than the maximum dimension $2 + \left\lfloor \frac{3n^2}{8} \right\rfloor$ (see, [1], [5] or Proposition 16) of a D_2 F-subalgebra of $M_n(F)$ exist?

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