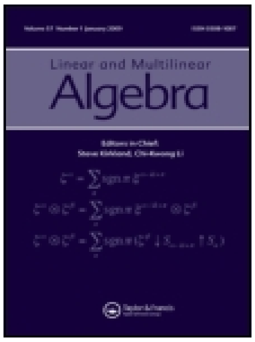

Linear and Multilinear Algebra

ISSN: 0308-1087 (Print) 1563-5139 (Online) Journal homepage: http://www.tandfonline.com/loi/glma20

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Linear and Multilinear Algebra, 2016
Vol. 64, No. 2, 290–296, http://dx.doi.org/10.1080/03081087.2015.1034643

Jordan isomorphisms of 2-torsionfree triangular rings

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Communicated by W.-F. Ke

(*Received 21 January 2015; accepted 17 March 2015*)

We construct a class of Jordan isomorphisms from a triangular ring $\mathcal{T}$, and we show that if $\mathcal{T}$ is 2-torsionfree, any Jordan isomorphism from $\mathcal{T}$ to another ring is of this form, up to a ring isomorphism. As an application, we show that for triangular rings in a large class, any Jordan isomorphism to another ring is a direct sum of a ring isomorphism and a ring anti-isomorphism. In particular, this applies to complete upper block triangular matrix rings and indecomposable triangular rings.

Keywords: triangular algebra; Jordan isomorphism; complete upper block triangular matrix ring

AMS Subject Classifications: 16S50; 16W20; 16W10; 16D20

1. Introduction and preliminaries

Let $\mathcal{T}$ and $\mathcal{U}$ be rings. An additive isomorphism $\varphi : \mathcal{T} \to \mathcal{U}$ is called a Jordan isomorphism if $\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$ for any $x, y \in \mathcal{T}$. Ring isomorphisms and ring anti-isomorphisms are examples of Jordan isomorphisms. It was proved that they are the only such examples in certain special cases: when $\mathcal{T}, \mathcal{U}$ are prime rings of characteristic not 2 (see [1,2]), when $\mathcal{T} = \mathcal{U}$ is a ring of upper triangular matrices over a field with more than 2 elements (see [3]), or more generally over a 2-torsionfree commutative ring having only trivial idempotents (see [4]). If $\mathcal{T}$ is an upper triangular matrix ring over a 2-torsionfree ring, then any Jordan isomorphism from $\mathcal{T}$ to another ring is a direct sum of an isomorphism and an anti-isomorphism (see [5]). Jordan homomorphisms from upper triangular matrix rings onto upper triangular matrix rings were investigated in [6] for base rings having only trivial idempotents.

Our interest is in the case where $\mathcal{T} = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is a triangular ring, i.e. R and S are rings with identity, M is a left R, right S-bimodule and the addition and multiplication obey the usual rules for matrices. It was proved in [7] that any Jordan isomorphism from $\mathcal{T}$ to another ring is either a ring isomorphism or a ring anti-isomorphism, provided that M is an indecomposable bimodule, faithful as a left R-module and as a right S-module,

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and $\mathcal{T}$ is 2-torsionfree (such a $\mathcal{T}$ is called an indecomposable triangular ring). We will refine the method of [7], which itself used some techniques of [4], and describe in Theorem 2.2 Jordan isomorphisms from $\mathcal{T}$ to another ring assuming only that $\mathcal{T}$ is 2-torsionfree. In fact, we construct a class of Jordan isomorphisms from $\mathcal{T}$ to Morita rings associated to Morita contexts with zero Morita maps, and show that up to a ring isomorphism, any Jordan isomorphism from $\mathcal{T}$ lies in this class. As an application, we give in Theorem 3.1, a quite large class of rings for which any Jordan isomorphism to another ring is a direct sum of a ring isomorphism and a ring anti-isomorphism. Immediate consequences of this are the above-mentioned result of [7] and the fact that a Jordan isomorphism from a complete upper block triangular matrix ring $\mathcal{T}$ over a 2-torsionfree ring Γ is a direct sum of a ring isomorphism and a ring anti-isomorphism; this was proved in [5] for upper triangular matrix rings. If Γ has only trivial central idempotents, it follows that any Jordan isomorphism from $\mathcal{T}$ is either a ring isomorphism or a ring anti-isomorphism; for upper triangular matrix rings this recovers results of [4,7].

All rings will be with identity and all modules will be unital. If $\varphi : \mathcal{T} \to \mathcal{U}$ is a Jordan isomorphism between 2-torsionfree rings, then $\varphi(xyx) = \varphi(x)\varphi(y)\varphi(x)$ for any $x, y \in \mathcal{T}$, $\varphi(1_{\mathcal{T}}) = 1_{\mathcal{U}}$, and φ maps idempotents to idempotents, see [4,7].

2. The main result

A Morita context with zero Morita maps is just a quadruple (A, B, N_1, N_2), where A and B are rings, N_1 is a left A, right B-bimodule; N_2 is a left B, right A-bimodule. The Morita ring associated with such a Morita context is $\begin{pmatrix} A & N_1 \\ N_2 & B \end{pmatrix}$, with component-wise addition and multiplication defined by

$$\begin{pmatrix} a & n_1 \\ n_2 & b \end{pmatrix}\begin{pmatrix} a' & n'_1 \\ n'_2 & b' \end{pmatrix} = \begin{pmatrix} aa' & an'_1 + n_1 b' \\ n_2 a' + bn'_2 & bb' \end{pmatrix}$$

The following result gives a class of Jordan isomorphisms from a triangular matrix ring.

Proposition 2.1 *Let $\mathcal{T} = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be a triangular ring such that $M = M_1 \oplus M_2$ as bimodules. Also let (A, B, N_1, N_2) be a Morita context with zero Morita maps. Assume that:*

- *$\rho : R \to A$, $\sigma : S \to B$ are Jordan isomorphisms,*
- *$\psi_1 : M_1 \to N_1$, $\psi_2 : M_2 \to N_2$ are isomorphisms of additive groups such that $\psi_1(rm) = \rho(r)\psi_1(m)$, $\psi_1(ms) = \psi_1(m)\sigma(s)$ for any $m \in M_1, r \in R, s \in S$, $\psi_2(rm) = \psi_2(m)\rho(r)$, $\psi_2(ms) = \sigma(s)\psi_2(m)$ for any $m \in M_2, r \in R, s \in S$, (shortly, ψ_1 is an additive (ρ, σ)-isomorphism and ψ_2 is an additive (ρ, σ)-anti-isomorphism).*

  Then, the map

$$\Phi(\rho, \sigma, \psi_1, \psi_2) : \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \to \begin{pmatrix} A & N_1 \\ N_2 & B \end{pmatrix}$$

  defined by

$$\Phi(\rho, \sigma, \psi_1, \psi_2)\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} \rho(r) & \psi_1(m_1) \\ \psi_2(m_2) & \sigma(s) \end{pmatrix},$$

  where $m = m_1 + m_2$, $m_1 \in M_1, m_2 \in M_2$, is a Jordan isomorphism.

Moreover,

(1) $\Phi(\rho, \sigma, \psi_1, \psi_2)$ *is a ring isomorphism if and only if so are* ρ *and* σ*, and* $M_2 = 0$ *(or equivalently* $N_2 = 0$*).*
(2) $\Phi(\rho, \sigma, \psi_1, \psi_2)$ *is a ring anti-isomorphism if and only if so are* ρ *and* σ*, and* $M_1 = 0$ *(or equivalently* $N_1 = 0$*).*

Proof Denote $\Phi = \Phi(\rho, \sigma, \psi_1, \psi_2)$, which is clearly a bijective additive map. It is a straightforward computation to check that it is a Jordan isomorphism.

For (1), let $x = \begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$, $y = \begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix}$ in $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ with $m = m_1 + m_2$, $m' = m'_1 + m'_2$, $m_1, m'_1 \in M_1$, $m_2, m'_2 \in M_2$. Then a direct computation shows that $\Phi(xy) = \Phi(x)\Phi(y)$ if and only if $\rho(rr') = \rho(r)\rho(r')$, $\sigma(ss') = \sigma(s)\sigma(s')$, and $\psi_2(m'_2)\rho(r) - \sigma(s)\psi_2(m'_2) = \psi_2(m_2)\rho(r') - \sigma(s')\psi_2(m_2)$. The first two relations mean that ρ and σ are ring morphisms, while the third one (for any x, y) means that $N_2 = 0$. Indeed, if r, s and s' are 0, we get $\psi_2(m_2)\rho(r') = 0$, so $N_2 A = 0$ and then, N_2 must be 0. A similar argument proves (2). $\square$

Our main result shows that up to a ring isomorphism, any Jordan isomorphism of a 2-torsionfree triangular ring is of the form constructed in Proposition 2.1, for some decomposition of M as a direct sum of bimodules.

THEOREM 2.2 *Let* $\mathcal{T} = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *be a 2-torsionfree triangular ring and let* $\mathcal{U}$ *be a ring. If* $\varphi : \mathcal{T} \to \mathcal{U}$ *is a Jordan isomorphism, then there exist rings* A*,* B*, bimodules* ${}_A{N_1}_B$*,* ${}_B{N_2}_A$*, a ring isomorphism* $\gamma : \mathcal{U} \to \begin{pmatrix} A & N_1 \\ N_2 & B \end{pmatrix}$ *and a decomposition* $M = M_1 \oplus M_2$ *such* $\gamma\varphi = \Phi(\rho, \sigma, \psi_1, \psi_2)$ *for certain Jordan isomorphisms* $\rho : R \to A$*,* $\sigma : S \to B$ *and additive isomorphisms* $\psi_1 : M_1 \to N_1$*,* $\psi_2 : M_2 \to N_2$ *satisfying the conditions of Proposition 2.1.*

Proof Regard R, S and M as embedded in $\mathcal{T}$, thus $\mathcal{T} = R \oplus M \oplus S$. Let $e = \varphi(1_R)$ and $f = \varphi(1_S)$, which are orthogonal idempotents with $e + f = 1_{\mathcal{U}}$. As in the proof of [7, Theorem 3.1], we have that $\varphi(R) = e\mathcal{U}e$, a ring with identity e, and $\varphi(S) = f\mathcal{U}f$, a ring with identity f. Moreover, for any $r \in R$, $m \in M$, $s \in S$,

$$\varphi(r)\varphi(ms) = \varphi(rm)\varphi(s) \text{ and } \varphi(ms)\varphi(r) = \varphi(s)\varphi(rm).$$

In particular one has

$$\begin{gathered} \varphi(r)\varphi(m) = \varphi(rm)f, \;\; \varphi(m)\varphi(r) = f\varphi(rm) \\ \varphi(m)\varphi(s) = e\varphi(ms), \;\; \varphi(s)\varphi(m) = \varphi(ms)e \\ e\varphi(m) = \varphi(m)f, \;\; \varphi(m)e = f\varphi(m). \end{gathered}$$

We note that $\varphi(R)\varphi(M) \subseteq \varphi(M)$, nevertheless $\varphi(M)$ is not a $\varphi(R)$-module, since in general $e\varphi(m) \neq \varphi(m)$. However, $e\varphi(M) = \varphi(M)f = e\mathcal{U}f$ is a left $\varphi(R)$, right $\varphi(S)$-bimodule, and $f\varphi(M) = \varphi(M)e = f\mathcal{U}e$ is a left $\varphi(S)$, right $\varphi(R)$-bimodule. It is proved in [7, Theorem 3.1] that $M = \varphi^{-1}(e\varphi(M)) \oplus \varphi^{-1}(f\varphi(M))$ as left R, right S-bimodules.

Actually R, S, M are subject to more conditions in [7], but they are not needed for proving this relation. Denote $M_1 = \varphi^{-1}(e\varphi(M))$ and $M_2 = \varphi^{-1}(f\varphi(M))$.

Take $A = e\mathcal{U}e$, $B = f\mathcal{U}f$, $N_1 = e\mathcal{U}f$, $N_2 = f\mathcal{U}e$. Consider the Morita context (A, B, N_1, N_2) with zero Morita maps, and the associated Morita ring $\begin{pmatrix} A & N_1 \\ N_2 & B \end{pmatrix}$.

We have $\mathcal{U} = e\mathcal{U}e \oplus e\mathcal{U}f \oplus f\mathcal{U}e \oplus f\mathcal{U}f = \varphi(R) \oplus e\varphi(M) \oplus f\varphi(M) \oplus \varphi(S)$. We show that $(e\varphi(M))(f\varphi(M)) = (f\varphi(M))(e\varphi(M)) = 0$. Indeed, let $y \in e\varphi(M)$ and $y' \in f\varphi(M)$. Then, there are $m, m' \in M$ with $y = \varphi(m)$ and $y' = \varphi(m')$. Since $mm' = m'm = 0$, one has $yy' + y'y = \varphi(m)\varphi(m') + \varphi(m')\varphi(m) = \varphi(mm' + m'm) = 0$. But $yy' \in e\mathcal{U}e$ and $y'y \in f\mathcal{U}f$, so we must have $yy' = y'y = 0$. This shows that there is a ring isomorphism $\gamma : \mathcal{U} \to \begin{pmatrix} A & N_1 \\ N_2 & B \end{pmatrix}$, defined by $\gamma(u) = \begin{pmatrix} eue & euf \\ fue & fuf \end{pmatrix}$ for any $u \in \mathcal{U}$.

Since φ is a Jordan isomorphism, it induces by restriction and corestriction Jordan isomorphisms $\rho : R \to A$ and $\sigma : S \to B$, and also additive isomorphisms $\psi_1 : M_1 \to N_1$ and $\psi_2 : M_2 \to N_2$. It is clear that $\gamma\varphi \begin{pmatrix} r & m_1 + m_2 \\ 0 & s \end{pmatrix} = \begin{pmatrix} \rho(r) & \psi_1(m_1) \\ \psi_2(m_2) & \sigma(s) \end{pmatrix}$ for any $r \in R, s \in S, m_1 \in M_1, m_2 \in M_2$. It remains to show that ψ_1 is a (ρ, σ)-isomorphism and ψ_2 is a (ρ, σ)-anti-isomorphism.

Let $r \in R$ and $m \in M_1$. We have $\psi_1(rm) = \varphi(rm) \in N_1 = e\varphi(M)$, so $\varphi(rm) = e\varphi(n) = \varphi(n)f$ for some $n \in M$. Then, $\varphi(rm)f = \varphi(n)f^2 = \varphi(n)f = \varphi(rm)$, so $\varphi(rm) = \varphi(rm)f = \varphi(r)\varphi(m) = \rho(r)\psi_1(m)$.
If $s \in S$ and $m \in M_1$ then, $\psi_1(ms) = \varphi(ms) = e\varphi(n)$ for some $n \in M$. Then, $\varphi(ms) = e\varphi(n) = e^2\varphi(n) = e\varphi(ms) = \varphi(m)\varphi(s) = \psi_1(m)\sigma(s)$.

Similarly, one can prove that $\psi_2(rm) = \psi_2(m)\rho(r)$ and $\psi_2(ms) = \sigma(s)\psi_2(m)$ for any $r \in R, m \in M_2$ and $s \in S$. □

The following result shows in some sense what is the obstruction for ρ and σ in Theorem 2.2 to being ring morphisms or ring anti-morphisms.

Proposition 2.3 *With notation as in Theorem 2.2 and its proof, we have that for any $r, r' \in R$, $s, s' \in S$ the following hold:*

$$\rho(rr') - \rho(r)\rho(r') \in \varphi(ann_R(M_1))$$
$$\rho(r'r) - \rho(r)\rho(r') \in \varphi(ann_R(M_2))$$
$$\sigma(ss') - \sigma(s)\sigma(s') \in \varphi(ann_S(M_1))$$
$$\sigma(s's) - \sigma(s)\sigma(s') \in \varphi(ann_S(M_2))$$

Proof Let $r, r' \in R$. Then for any $m \in M$,

$$\varphi(r)\varphi(r')\varphi(m) = \varphi(r)\varphi(r'm)f = \varphi(rr'm)f = \varphi(rr')\varphi(m),$$

so $(\varphi(r)\varphi(r') - \varphi(rr'))\varphi(m) = 0$. Since $\varphi(R)$ is a subring of $\mathcal{U}$, we have that $\varphi(r)\varphi(r') - \varphi(rr') = \varphi(r_0)$ for some $r_0 \in R$. Then $e\varphi(r_0 m) = \varphi(r_0 m)f = \varphi(r_0)\varphi(m) = 0$, so $r_0 m = \varphi^{-1}(\varphi(r_0 m)) = \varphi^{-1}(f\varphi(r_0 m)) \in M_2$. This shows that $r_0 M \subseteq M_2$. As $r_0 M_1 \subseteq M_1$, we must have $r_0 M_1 = 0$, so $r_0 \in ann_R(M_1)$. We conclude that $\rho(rr') - \rho(r)\rho(r') \in \varphi(ann_R(M_1))$.

For the second relation, we see that

$$\varphi(m)\varphi(r)\varphi(r') = f\varphi(rm)\varphi(r') = f\varphi(r'rm) = \varphi(m)\varphi(r'r)$$

so $\varphi(m)(\varphi(r)\varphi(r') - \varphi(r'r)) = 0$. As for the first relation, if we write $\varphi(r)\varphi(r') - \varphi(r'r) = \varphi(r_0)$, we have that $f\varphi(r_0 m) = \varphi(m)\varphi(r_0) = 0$. Hence, $r_0 m \in M_1$ for any m, implying $r_0 \in ann_R(M_2)$ and the second relation. The other two relations can be proved similarly. □

3. Applications

We use Theorem 2.2 and Proposition 2.3 for proving the following result, giving information about Jordan isomorphisms of a large class of rings.

THEOREM 3.1 *Let C be a commutative ring and R, S be C-algebras. Let M be a left R, right S-bimodule, faithful on each side, such that any direct summand of M as a bimodule is of the form cM for some idempotent c in C. Let $\mathcal{T} = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, which is assumed to be 2-torsionfree. Then for any Jordan isomorphism $\varphi : \mathcal{T} \to \mathcal{U}$, where $\mathcal{U}$ is a ring, there exists an idempotent $c \in C$ such that $\varphi_{|c\mathcal{T}}$ is a ring isomorphism and $\varphi_{|(1-c)\mathcal{T}}$ is a ring anti-isomorphism. In particular, φ is the direct sum of a ring isomorphism and a ring anti-isomorphism.*

Proof By Theorem 2.2, and keeping the notation in its statement and its proof, there is a decomposition $M = M_1 \oplus M_2$ as bimodules such that $\gamma\varphi = \Phi(\rho, \sigma, \psi_1, \psi_2)$. By our hypothesis, there is an idempotent $c \in C$ such that $M_1 = cM$ and $M_2 = (1-c)M$. The mapping $z \mapsto (cz, (1-c)z)$ defines a ring isomorphism $\mathcal{T} \simeq c\mathcal{T} \times (1-c)\mathcal{T}$.

We have that $\varphi(c1_R)$ and $\varphi((1-c)1_R)$ are central orthogonal idempotents in $\varphi(R)$, and their sum is e, the identity of $\varphi(R)$. Since $\varphi(cr) = \varphi(crc) = \varphi(c1_R)\varphi(r)\varphi(c1_R) = \varphi(c1_R)\varphi(r)$, we see that $\varphi(cR) = \varphi(c1_R)\varphi(R)$ is a subring of $\varphi(R)$, with identity $\varphi(c1_R)$. Similarly, $\varphi((1-c)1_R) = \varphi((1-c)1_R)\varphi(R)$ is a subring of $\varphi(R)$, with identity $\varphi((1-c)1_R)$.

We show that $\varphi_{|cR}$ is a ring morphism. Indeed, by Proposition 2.3, $\varphi(rr') - \varphi(r)\varphi(r') \in \varphi(ann_R(M_1)) = \varphi(ann_R(cM)) = \varphi((1-c)R) = \varphi((1-c)1_R)\varphi(R)$, for any $r, r' \in R$. Now if $r, r' \in cR$, we have $\varphi(rr') - \varphi(r)\varphi(r') \in \varphi(c1_R)\varphi(R)$, as this is a subring. Since $\varphi(c1_R)\varphi(R) \cap \varphi((1-c)1_R)\varphi(R) = 0$, we must have $\varphi(rr') - \varphi(r)\varphi(r') = 0$.

In a similar way, if we use the relation $\varphi(rr') - \varphi(r')\varphi(r) \in \varphi(ann_R(M_2)) = \varphi(cR) = \varphi(c1_R)\varphi(R)$, where $r, r' \in R$, we obtain that for $r, r' \in (1-c)R$,

$$\varphi(rr') - \varphi(r')\varphi(r) \in \varphi(c1_R)\varphi(R) \cap \varphi((1-c)1_R)\varphi(R) = 0,$$

so $\varphi_{|(1-c)R}$ is a ring anti-morphism.

Similarly, one sees that $\varphi_{|cS}$ is a ring morphism and $\varphi_{|(1-c)S}$ is a ring anti-morphism.

We show that $\varphi_{|cM}$ is a $(\varphi_{|cR}, \varphi_{|cS})$-morphism, which in view of Proposition 2.1(1) shows that $\varphi_{|c\mathcal{T}}$ is a ring isomorphism. Indeed, we have $\varphi(cr)\varphi(cm) = \varphi(crcm)f = \varphi(crm)f$. Since $crm = rcm \in M_1$, we have $\varphi(crm) \in \varphi(M_1) = \phi(M)f$, and we get $\varphi(crm)f = \varphi(crm) = \varphi(crcm)$, showing that $\varphi(cr)\varphi(cm) = \varphi(crcm)$. On the other hand, $\varphi(cm)\varphi(cs) = e\varphi(cmcs) = e\varphi(mcs)$. Now $mcs \in M_1$ shows that $\varphi(mcs) \in \varphi(M_1) = e\varphi(M)$, and then $e\varphi(mcs) = \varphi(mcs) = \varphi(cmcs)$. Thus, $\varphi(cm)\varphi(cs) = \varphi(cmcs)$.

In a similar way, one can show that $\varphi_{|(1-c)M}$ is a $(\varphi_{|(1-c)R}, \varphi_{|(1-c)S})$-morphism, and then by Proposition 2.1(2) we see that $\varphi_{|(1-c)\mathcal{T}}$ is a ring anti-isomorphism, and this ends the proof. $\square$

COROLLARY 3.2 [7, Theorem 3.1] *Let* $\mathcal{T} = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *be a 2-torsionfree triangular ring. If M is R-faithful, S-faithful and indecomposable as a bimodule, then every Jordan isomorphism from $\mathcal{T}$ onto another ring is either a ring isomorphism or a ring anti-isomorphism.*

Proof Let $C = \mathbb{Z}$. Since M is indecomposable, a direct summand M_1 of M is of the form cM, with $c = 0$ or $c = 1$. By Theorem 3.1, we obtain that φ is a ring isomorphism for $c = 1$, and a ring anti-isomorphism for $c = 0$. $\square$

We recall that a complete upper block triangular matrix ring over a ring Γ is a ring of the form

$$A = \begin{pmatrix} \mathcal{M}_{d_1}(\Gamma) & \mathcal{M}_{d_1,d_2}(\Gamma) & \cdots & \mathcal{M}_{d_1,d_p}(\Gamma) \\ 0 & \mathcal{M}_{d_2}(\Gamma) & \cdots & \mathcal{M}_{d_2,d_p}(\Gamma) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathcal{M}_{d_p}(\Gamma) \end{pmatrix}$$

for some positive integers $p \geq 2, d_1, d_2, \ldots, d_p$. If all the blocks are of size 1, i.e. $d_1 = \ldots = d_p = 1$, this is just an upper triangular matrix ring.

COROLLARY 3.3 *Let Γ be a 2-torsionfree ring and let $\mathcal{T}$ be a complete upper block triangular matrix ring over Γ. Then for any Jordan isomorphism $\varphi : \mathcal{T} \to \mathcal{U}$, where $\mathcal{U}$ is a ring, there exists a central idempotent $c \in \Gamma$ such that $\varphi_{|c\mathcal{T}}$ is a ring isomorphism and $\varphi_{|(1-c)\mathcal{T}}$ is a ring anti-isomorphism. Thus, φ is the direct sum of a ring isomorphism and a ring anti-isomorphism. In particular, if Γ has only trivial central idempotents, φ is either an isomorphism of rings or an anti-isomorphism of rings.*

Proof We regard $\mathcal{T}$ as a triangular ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R is the first diagonal block of $\mathcal{T}$, say of size $p \times p$, S is the complete upper block triangular matrix ring, say of size $q \times q$, obtained from $\mathcal{T}$ by deleting the first p rows and the first p columns, and $M = \mathcal{M}_{p,q}(\Gamma)$. Let C be the centre of Γ. By direct computation, or by using [8, Propositions 2.1 and 3.1], we see that M is faithful as a left R-module and also as a right S-module, and that the sub-bimodules of M which are direct summands of M are of the form $\mathcal{M}_{p,q}(I)$, where I is a two-sided ideal of Γ. Then, a decomposition of M as a direct sum of sub-bimodules reduces to a decomposition of Γ as a direct sum of ideals. Therefore, a direct summand of the bimodule M is of the form cM for some idempotent $c \in C$. Now we just apply Theorem 3.1. $\square$

In the particular case of upper triangular matrix rings, Corollary 3.3 was proved in [5, Main Theorem 1]. The fact that a Jordan isomorphism from an upper triangular matrix ring over a 2-torsionfree ring Γ having only trivial idempotents is either a ring isomorphism or a ring anti-isomorphism was proved in [4] for commutative Γ, and in [7, Theorem 3.2] in general.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The second author was supported by the UEFISCDI [grant number PN-II-ID-PCE-2011-3-0635], contract no. 253/5.10.2011 of CNCSIS. The third author was supported by the National Research Foundation of South Africa [grant number 81141]. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and therefore, the National Research Foundation does not accept any liability in regard thereto.

References

[1] Herstein IN. Jordan homomorphisms. Trans. Am. Math. Soc. 1956;81:331–351.
[2] Smiley MF. Jordan homomorphisms onto prime rings. Trans. Am. Math. Soc. 1957;84:426–429.
[3] Molnar L, Šemrl P. Some linear preserver problems on upper triangular matrices. Linear Multilinear Algebra. 1998;45:189–206.
[4] Beidar KI, Brešar M, Chebotar MA. Jordan isomorphisms of triangular matrix algebra over a connected commutative ring. Linear Algebra Appl. 2000;312:197–201.
[5] Liu C-K, Tsai W-Y. Jordan isomorphisms of upper triangular matrix rings. Linear Algebra Appl. 2007;426:143–148.
[6] Wang Y, Wang Y. Jordan homomorphisms of upper triangular matrix rings. Linear Algebra Appl. 2013;439:4063–4069.
[7] Wong T-L. Jordan isomorphisms of triangular rings. Proc. Am. Math. Soc. 2005;133:3381–3388.
[8] Dăscălescu S, Preduţ S, van Wyk L. Jordan isomorphisms of generalized structural matrix rings. Linear Multilinear Algebra. 2013;61:369–376.