JORDAN DERIVATIONS AND ANTIDERIVATIONS
OF GENERALIZED MATRIX ALGEBRAS

YANBO LI, LEON VAN WYK AND FENG WEI

(*Communicated by P. Šemrl*)

Abstract. Let $\mathscr{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ be a generalized matrix algebra defined by the Morita context $(A, B, {}_A M_B, {}_B N_A, \Phi_{MN}, \Psi_{NM})$. In this article we mainly study the question of whether there exist the so-called "proper" Jordan derivations for the generalized matrix algebra $\mathscr{G}$. It is shown that if one of the bilinear pairings Φ_{MN} and Ψ_{NM} is nondegenerate, then every antiderivation of $\mathscr{G}$ is zero. Furthermore, if the bilinear pairings Φ_{MN} and Ψ_{NM} are both zero, then every Jordan derivation of $\mathscr{G}$ is the sum of a derivation and an antiderivation. Several constructive examples and counterexamples are presented.

1. Introduction

Let us begin with the definition of generalized matrix algebras given by a Morita context. Let $\mathscr{R}$ be a commutative ring with identity. A Morita context consists of two $\mathscr{R}$-algebras A and B, two bimodules ${}_A M_B$ and ${}_B N_A$, and two bimodule homomorphisms called the pairings $\Phi_{MN} : M \underset{B}{\otimes} N \longrightarrow A$ and $\Psi_{NM} : N \underset{A}{\otimes} M \longrightarrow B$ satisfying the following commutative diagrams:

$$
\begin{array}{ccc}
M \underset{B}{\otimes} N \underset{A}{\otimes} M & \xrightarrow{\Phi_{MN} \otimes I_M} & A \underset{A}{\otimes} M \\
{\scriptstyle I_M \otimes \Psi_{NM}} \downarrow & & \downarrow {\scriptstyle \cong} \\
M \underset{B}{\otimes} B & \xrightarrow{\cong} & M
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
N \underset{A}{\otimes} M \underset{B}{\otimes} N & \xrightarrow{\Psi_{NM} \otimes I_N} & B \underset{B}{\otimes} N \\
{\scriptstyle I_N \otimes \Phi_{MN}} \downarrow & & \downarrow {\scriptstyle \cong} \\
N \underset{A}{\otimes} A & \xrightarrow{\cong} & N .
\end{array}
$$

Let us write this Morita context as $(A, B, {}_A M_B, {}_B N_A, \Phi_{MN}, \Psi_{NM})$. If $(A, B, {}_A M_B, {}_B N_A, \Phi_{MN}, \Psi_{NM})$ is a Morita context, then the set

$$
\begin{bmatrix} A & M \\ N & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \middle| a \in A, m \in M, n \in N, b \in B \right\}
$$

form an $\mathscr{R}$-algebra under matrix-like addition and matrix-like multiplication, where We assume that at least one of the two bimodules M and N is distinct from zero. Such an $\mathscr{R}$-algebra is called a *generalized matrix algebra* of order 2 and is usually denoted by $\mathscr{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$. This kind of algebra was first introduced by Morita in [20], where the author investigated Morita duality theory of modules and its applications to Artinian algebras.

Let $\mathscr{R}$ be a commutative ring with identity, A be a unital algebra over $\mathscr{R}$ and $\mathscr{Z}(A)$ be the center of A. Recall that an $\mathscr{R}$-linear mapping Θ_d from A into itself is called a *derivation* if $\Theta_d(ab) = \Theta_d(a)b + a\Theta_d(b)$ for all $a, b \in A$. Further, an $\mathscr{R}$-linear mapping Θ_{Jord} from A into itself is called a *Jordan derivation* if $\Theta_{\mathrm{Jord}}(a^2) = \Theta_{\mathrm{Jord}}(a)a + a\Theta_{\mathrm{Jord}}(a)$ for all $a \in A$. Every derivation is obviously a Jordan derivation. The inverse statement is not true in general. Those Jordan derivations which are not derivations are said to be *proper*. An $\mathscr{R}$-linear mapping Θ_{antid} from A into itself is called an *antiderivation* if $\Theta_{\mathrm{antid}}(ab) = \Theta_{\mathrm{antid}}(b)a + b\Theta_{\mathrm{antid}}(a)$ for all $a, b \in A$.

In 1957 Herstein [10] proved that every Jordan derivation from a prime ring of characteristic not 2 into itself is a derivation. This result has been generalized to different rings and algebras in various directions (see e.g. [1, 3, 4, 6, 9, 11, 13, 14, 17, 21, 24] and references therein). Zhang and Yu [24] showed that every Jordan derivation on a triangular algebra is a derivation. Xiao and Wei [21] extended this result to the higher case and obtained that any Jordan higher derivation on a triangular algebra is a higher derivation. Johnson [12] considered a more challenging question for which Banach algebras A there are no proper Jordan derivations from A into an arbitrary Banach A-bimodule M. It turned out that this is true for some important classes of algebras (in particular, for the algebra of all $n \times n$ complex matrices). Motivated by Johnson's work, Benkovic investigated the structure of Jordan derivations from the upper triangular matrix algebra $\mathscr{T}_n(\mathscr{R})$ into its bimodule and proved that every Jordan derivation from $\mathscr{T}_n(\mathscr{R})$ into its bimodule is the sum of a derivation and an antiderivation. Recently, Li, Xiao and Wei [15, 16, 22] jointly studied linear mappings of generalized matrix algebras, such as derivations, Lie derivations, commuting mappings and semi-centralizing mappings. Our main purpose is to develop the theory of linear mappings of triangular algebras to the case of generalized matrix algebras, which has a much broader background. People pay much less attention to linear mappings of generalized matrix algebras, to the best of our knowledge there are fewer articles dealing with linear mappings of generalized matrix algebras except for [2, 15, 16, 22].

The problem that we address in this article is to study whether there exist proper Jordan derivations for generalized matrix algebras. The outline of this article is as follows. The second section presents two basic examples of generalized matrix algebras which we will revisit later. In the third section we describe the general form of Jordan derivations and antiderivations on generalized matrix algebras. We observe that any antiderivation on a class of generalized matrix algebra is zero (see Proposition 3.10). Furthermore, it is shown that every Jordan derivation on another class of generalized matrix algebras is the sum of a derivation and an antiderivation (see Theorem 3.11).

2. Examples of generalized matrix algebras

We have presented many examples of generalized matrix algebras in [16], such as standard generalized matrix algebras and quasi-hereditary algebras, generalized matrix algebras of order n, inflated algebras, upper and lower triangular matrix algebras, block upper and lower triangular matrix algebras, nest algebras. For later discussion convenience, we have to give another two new generalized matrix algebras.

2.1. Generalized matrix algebras from smash product algebras

Let H be a finite dimensional Hopf algebra over a field $\mathbb{K}$ with comultiplication $\Delta : H \longrightarrow H \otimes H$, counit $\varepsilon : H \longrightarrow \mathbb{K}$ and antipode $S : H \longrightarrow H$. Clearly, S is bijective. Moreover, the space of left integrals $\int_H l = \{x \in H | hx = \varepsilon(h)x, \forall h \in H\}$ is one-dimensional. We substitute the "sigma notation" for Δ in the present article. Now assume that A is an H-module algebra, that is, A is a $\mathbb{K}$-algebra which is a left H-module, such that

(1) $h \cdot (ab) = \sum_{(h)} (h_1 \cdot a)(h_2 \cdot b)$ and

(2) $h \cdot 1_A = \varepsilon(h)1_A$.

for all $h \in H, a, b \in A$. Then the *smash product algebra* $A\#H$ is defined as follows, for any $a, b \in A, h, k \in H$:

(1) as a $\mathbb{K}$-space, $A\#H = A \otimes H$. We write $a\#h$ for the element $a \otimes h$

(2) multiplication is given by $(a\#h)(b\#k) = \sum_{(h)} a(h_1 \cdot b)\#h_2 k$.

The *invariants subalgebra* of H on A is the set $A^H = \{x \in A | h \cdot x = \varepsilon(h)x, \forall h \in H\}$. A is a left $A\#H$-module in the standard way, that is

$$a\#h \rightarrow b = a(h \cdot b)$$

for all $a, b \in A$ and $h \in H$. For a given $t \in \int l$, then $th \in \int l$ for all $h \in H$. Since $\int l$ is one-dimensional, there exists $\alpha \in H^*$ such that $th = \alpha(h)t$ for all $h \in H$. It is easy to see that α is multiplicative, and it is a group-like element of H^*. Hence

$$h^\alpha = \alpha \rightarrow h = \sum_{(h)} \alpha(h_2)h_1, \quad \forall h \in H$$

defines an automorphism on H. Thus A is a right $A\#H$-module via

$$a \leftarrow b\#h = S^{-1}h^\alpha \cdot (ab), \quad \forall a \in A, b\#h \in A\#H.$$

The close relationship between $A\#H$ and A^H enables us to formalize the following generalized matrix algebra. Now A is a left (or right) A^H-module simply by left (or

right) multiplication. Simultaneously, A is also a left (or right) $A\#H$-module. Thus $M =_{A^H} A_{A\#H}$ and $N =_{A\#H} A_{A^H}$, together with the mappings

$$\Psi_{NM} : A \otimes_{A^H} A \longrightarrow A\#H \text{ defined by} \quad \Psi_{NM}(a,b) = (a\#t)(b\#1)$$
$$\Phi_{MN} : A \otimes_{A\#H} A^H \longrightarrow A^H \text{ defined by} \quad \Phi_{NM}(a,b) = t \cdot (ab)$$

give rise to a new generalized matrix algebra

$$\mathscr{G}_{\text{SPA}} = \begin{bmatrix} A^H & M \\ N & A\#H \end{bmatrix}.$$

We refer the reader to [19] about the basic properties of $\mathscr{G}_{\text{SPA}}$.

2.2. Generalized matrix algebras from group algebras

Let A be an associative algebra over a field $\mathbb{K}$ and G be a finite group of automorphisms acting on A. The *fixed ring* A^G of the action G on A is the set $\{a \in A | a^g = a, \forall g \in G\}$. The *skew group algebra* $A * G$ is the set of all formal sums $\sum_{g \in G} a_g g$, $a_g \in A$. The addition operation is componentwise and the multiplication operation is defined distributively by the formula

$$ag \cdot bh = ab^{g^{-1}} gh$$

for all $a, b \in A$ and $g, h \in G$. Clearly, A is a left and right A^G-module. A can also be viewed as a left or right $A * G$-module as follows: for any $x = \sum_{g \in G} a_g g \in A * G$ and $a \in A$, we define $x \cdot a = \sum_{g \in G} a_g a^{g^{-1}}$ and $a \cdot x = \sum_{g \in G} (a a_g)^g$. Then we obtain a generalized matrix algebra

$$\mathscr{G}_{\text{GA}} = \begin{bmatrix} A^G & M \\ N & A * G \end{bmatrix},$$

where $M =_{A^G} A_{A*G}$ and $N =_{A*G} A_{A^G}$. The bilinear pairings Φ_{MN} and Ψ_{NM} can be established via

$$\Phi_{MN} : A \otimes_{A*G} A \longrightarrow A^G$$
$$(x,y) \longmapsto \sum_{g \in G} (xy)^g$$

and

$$\Psi_{NM} : A \otimes_{A^G} A \longrightarrow A * G$$
$$(x,y) \longmapsto \sum_{g \in G} xy^{g^{-1}} g.$$

3. Jordan derivations of generalized matrix algebras

Let $\mathscr{G}$ be a generalized matrix algebra of order 2 based on the Morita context $(A, B, {}_A M_B, {}_B N_A, \Phi_{MN}, \Psi_{NM})$ and let us denote it by

$$\mathscr{G} := \begin{bmatrix} A & M \\ N & B \end{bmatrix}.$$

Here, at least one of the two bimodules M and N is distinct from zero. The main aim of this section is to show that any Jordan derivation on a class of generalized matrix algebras is the sum of a derivation and an antiderivation. Our motivation originates from the following several results. Benkovic [1] proved that every Jordan derivation from the algebra of all upper triangular matrices into its bimodule is the sum of a derivation and an antiderivation. Ma and Ji [18] extended this result to the case of generalized Jordan derivations and obtained that every generalized Jordan derivation from the algebra of all upper triangular matrices into its bimodule is the sum of a generalized derivation and an antiderivation. Zhang and Yu in [24] showed that every Jordan derivation on a triangular algebra is a derivation. Therefore, it is appropriate to describe and characterize Jordan derivations of $\mathscr{G}$. Note that the forms of derivations and Lie derivations of $\mathscr{G}$ were given in [16].

PROPOSITION 3.1. [16, Proposition 4.2] *An additive mapping* Θ_d *from* $\mathscr{G}$ *into itself is a derivation if and only if it has the form*

$$\Theta_d\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) - mn_0 - m_0n & am_0 - m_0b + \tau_2(m) \\ n_0a - bn_0 + v_3(n) & n_0m + nm_0 + \mu_4(b) \end{bmatrix}, \quad (\bigstar 1)$$

$$\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathscr{G},$$

where $m_0 \in M, n_0 \in N$ *and*

$$\delta_1 : A \longrightarrow A, \quad \tau_2 : M \longrightarrow M, \quad v_3 : N \longrightarrow N, \quad \mu_4 : B \longrightarrow B$$

are all $\mathscr{R}$-*linear mappings satisfying the following conditions:*

(1) δ_1 *is a derivation of* A *with* $\delta_1(mn) = \tau_2(m)n + mv_3(n)$;

(2) μ_4 *is a derivation of* B *with* $\mu_4(nm) = n\tau_2(m) + v_3(n)m$;

(3) $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ *and* $\tau_2(mb) = \tau_2(m)b + m\mu_4(b)$;

(4) $v_3(na) = v_3(n)a + n\delta_1(a)$ *and* $v_3(bn) = bv_3(n) + \mu_4(b)n$.

PROPOSITION 3.2. *An additive mapping* Θ_{Jord} *from* $\mathscr{G}$ *into itself is a Jordan derivation if and only if it is of the form*

$$\Theta_{\text{Jord}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) - mn_0 - m_0n + \delta_4(b) & am_0 - m_0b + \tau_2(m) + \tau_3(n) \\ n_0a - bn_0 + v_2(m) + v_3(n) & \mu_1(a) + n_0m + nm_0 + \mu_4(b) \end{bmatrix}, \quad (\bigstar 2)$$

$$\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathscr{G},$$

where $m_0 \in M, n_0 \in N$ *and*

$$\delta_1 : A \longrightarrow A, \quad \delta_4 : B \longrightarrow A, \quad \tau_2 : M \longrightarrow M, \quad \tau_3 : N \longrightarrow M,$$
$$v_2 : M \longrightarrow N, \quad v_3 : N \longrightarrow N \quad \mu_1 : A \longrightarrow B \quad \mu_4 : B \longrightarrow B$$

are all $\mathscr{R}$-*linear mappings satisfying the following conditions:*

(1) δ_1 is a Jordan derivation on A and $\delta_1(mn) = -\delta_4(nm) + \tau_2(m)n + mv_3(n)$;

(2) μ_4 is a Jordan derivation on B and $\mu_4(nm) = -\mu_1(mn) + n\tau_2(m) + v_3(n)m$;

(3) $\delta_4(b^2) = 2\delta_4(b) = 0$ for all $b \in B$ and $\mu_1(a^2) = 2\mu_1(a) = 0$ for all $a \in A$;

(4) $\tau_2(am) = a\tau_2(m) + \delta_1(a)m + m\mu_1(a)$ and $\tau_2(mb) = \tau_2(m)b + m\mu_4(b) + \delta_4(b)m$;

(5) $v_3(bn) = bv_3(n) + \mu_4(b)n + n\delta_4(b)$ and $v_3(na) = v_3(n)a + n\delta_1(a) + \mu_1(a)n$;

(6) $\tau_3(na) = a\tau_3(n)$, $\tau_3(bn) = \tau_3(n)b$, $n\tau_3(n) = 0$, $\tau_3(n)n = 0$;

(7) $v_2(am) = v_2(m)a$, $v_2(mb) = bv_2(m)$, $mv_2(m) = 0$, $v_2(m)m = 0$.

Proof. Suppose that the Jordan derivation Θ_{Jd} is of the form

$$\Theta_{\mathrm{Jord}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) + \delta_2(m) + \delta_3(n) + \delta_4(b) & \tau_1(a) + \tau_2(m) + \tau_3(n) + \tau_4(b) \\ v_1(a) + v_2(m) + v_3(n) + v_4(b) & \mu_1(a) + \mu_2(m) + \mu_3(n) + \mu_4(b) \end{bmatrix},$$

for all $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathscr{G}$, where $\delta_1, \delta_2, \delta_3, \delta_4$ are $\mathscr{R}$-linear mappings from A, M, N, B to A, respectively; $\tau_1, \tau_2, \tau_3, \tau_4$ are $\mathscr{R}$-linear mappings from A, M, N, B to M, respectively; v_1, v_2, v_3, v_4 are $\mathscr{R}$-linear mappings from A, M, N, B to N, respectively; $\mu_1, \mu_2, \mu_3, \mu_4$ are $\mathscr{R}$-linear mappings from A, M, N, B to B, respectively.

For any $G \in \mathscr{G}$, we will intensively employ the Jordan derivation equation

$$\Theta_{\mathrm{Jord}}(G^2) = G\Theta_{\mathrm{Jord}}(G) + \Theta_{\mathrm{Jord}}(G)G. \tag{3.1}$$

Taking $G = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ into (3.1) we have

$$\Theta_{\mathrm{Jord}}(G^2) = \begin{bmatrix} \delta_1(a^2) & \tau_1(a^2) \\ v_1(a^2) & \mu_1(a^2) \end{bmatrix} \tag{3.2}$$

and

$$G\Theta_{\mathrm{Jord}}(G) + \Theta_{\mathrm{Jord}}(G)G = \begin{bmatrix} a\delta_1(a) + \delta_1(a)a & a\tau_1(a) \\ v_1(a)a & 0 \end{bmatrix}. \tag{3.3}$$

By (3.2) and (3.3) we know that δ_1 is a Jordan derivation of A,

$$\tau_1(a^2) = a\tau_1(a), \qquad v_1(a^2) = v_1(a)a \tag{3.4}$$

and

$$\mu_1(a^2) = 0. \tag{3.5}$$

for all $a \in A$. Similarly, putting $G = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ in (3.1) gives

$$\Theta_{\mathrm{Jord}}(G^2) = \begin{bmatrix} \delta_4(b^2) & \tau_4(b^2) \\ v_4(b^2) & \mu_4(b^2) \end{bmatrix} \tag{3.6}$$

and

$$G\Theta_{\mathrm{Jord}}(G) + \Theta_{\mathrm{Jord}}(G)G = \begin{bmatrix} 0 & \tau_4(b)b \\ bv_4(b) & b\mu_4(b) + \mu_4(b)b \end{bmatrix}. \tag{3.7}$$

Combining (3.6) with (3.7) yields that μ_4 is a Jordan derivation of B,

$$\tau_4(b^2) = \tau_4(b)b, \qquad v_4(b^2) = bv_4(b) \tag{3.8}$$

and

$$\delta_4(b^2) = 0. \tag{3.9}$$

for all $b \in B$.

Let us choose $G = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ in (3.1). Then

$$\Theta_{\text{Jord}}(G^2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.10}$$

and

$$G\Theta_{\text{Jord}}(G) + \Theta_{\text{Jord}}(G)G = \begin{bmatrix} mv_2(m) & m\mu_2(m) + \delta_2(m)m \\ 0 & v_2(m)m \end{bmatrix}. \tag{3.11}$$

The relations (3.10) and (3.11) jointly imply that

$$mv_2(m) = 0, \qquad v_2(m)m = 0 \tag{3.12}$$

and

$$\delta_2(m)m + m\mu_2(m) = 0 \tag{3.13}$$

for all $m \in M$. Likewise, if we choose $G = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, then

$$\Theta_{\text{Jord}}(G^2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.14}$$

and

$$G\Theta_{\text{Jord}}(G) + \Theta_{\text{Jord}}(G)G = \begin{bmatrix} \tau_3(n)n & 0 \\ n\delta_3(n) + \mu_3(n)n & n\tau_3(n) \end{bmatrix}. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$n\tau_3(n) = 0, \qquad \tau_3(n)n = 0 \tag{3.16}$$

and

$$\mu_3(n)n + n\delta_3(n) = 0 \tag{3.17}$$

for all $n \in N$. Let us consider $G = \begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}$ in (3.1) and set $\tau_1(1) = m_0$ and $v_1(1) = n_0$. Since δ_1 is a Jordan derivation of A, $\delta_1(1) = 0$. Moreover, (3.5) implies that $\mu_1(1) = 0$. Therefore

$$\Theta_{\text{Jord}}(G^2) = \begin{bmatrix} \delta_2(m) & m_0 + \tau_2(m) \\ n_0 + v_2(m) & \mu_2(m) \end{bmatrix}. \tag{3.18}$$

On the other hand, from (3.12) and (3.13) we have that

$$G\Theta_{\text{Jord}}(G) + \Theta_{\text{Jord}}(G)G = \begin{bmatrix} 2\delta_2(m) + mn_0 & m_0 + \tau_2(m) \\ n_0 + v_2(m) & n_0m \end{bmatrix}. \tag{3.19}$$

By (3.18) and (3.19) we arrive at

$$\delta_2(m) = -mn_0 \quad \text{and} \quad \mu_2(m) = n_0 m \tag{3.20}$$

for all $m \in M$. Let us take $G = \begin{bmatrix} 1 & 0 \\ n & 0 \end{bmatrix}$ in (3.1). Applying (3.16) and (3.17) leads to

$$\mu_3(n) = nm_0 \quad \text{and} \quad \delta_3(n) = -m_0 n \tag{3.21}$$

for all $n \in N$. Furthermore, if we choose $G = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$ in (3.1), then it follows from (3.8) and (3.9) that $2\delta_4(b) = 0$,

$$\nu_4(b) = -bn_0 \quad \text{and} \quad \tau_4(b) = -m_0 b \tag{3.22}$$

for all $b \in B$. Taking $G = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ into (3.1) and using (3.4) and (3.5) we obtain $2\mu_1(a) = 0$,

$$\tau_1(a) = am_0 \quad \text{and} \quad \nu_1(a) = n_0 a \tag{3.23}$$

for all $a \in A$. Let us put $G = \begin{bmatrix} a & m \\ 0 & 0 \end{bmatrix}$ in (3.1). Then the relations (3.5), (3.19) and (3.23) imply that

$$\Theta_{\text{Jord}}(G^2) = \begin{bmatrix} \delta_1(a^2) + \delta_2(am) & a^2 m_0 + \tau_2(am) \\ n_0 a^2 + \nu_2(am) & n_0 am \end{bmatrix}. \tag{3.24}$$

On the other hand, by the relations (3.4), (3.12), (3.13), (3.20) and (3.23) we get

$$G\Theta_{\text{Jord}}(G) + \Theta_{\text{Jord}}(G)G$$
$$= \begin{bmatrix} a\delta_1(a) + \delta_1(a)a + amn_0 & a^2 m_0 + a\tau_2(m) + \delta_1(a)m + m\mu_1(a) \\ n_0 a^2 + \nu_2(m)a & n_0 am \end{bmatrix}. \tag{3.25}$$

Combining (3.24) with (3.25) yields $\nu_2(am) = \nu_2(m)a$ and

$$\tau_2(am) = a\tau_2(m) + \delta_1(a)m + m\mu_1(a)$$

for all $a \in A, m \in M$. Similarly, taking $G = \begin{bmatrix} a & 0 \\ n & 0 \end{bmatrix}$ into (3.1) gives $\tau_3(na) = a\tau_3(n)$ and

$$\nu_3(na) = \nu_3(n)a + n\delta_1(a) + \mu_1(a)n$$

for all $n \in N, a \in A$. Let us choose $G = \begin{bmatrix} 0 & m \\ 0 & b \end{bmatrix}$ in (3.1). We will get $\nu_2(mb) = b\nu_2(m)$ and

$$\tau_2(mb) = \tau_2(m)b + m\mu_4(b) + \delta_4(b)m$$

for all $m \in M, b \in B$. Putting $G = \begin{bmatrix} 0 & 0 \\ n & b \end{bmatrix}$ in (3.1) and employing the same computational approach we conclude that $\tau_3(bn) = \tau_3(n)b$ and $\nu_3(bn) = b\nu_3(n) + \mu_4(b)n + n\delta_4(b)$ for all $b \in B, n \in N$. Finally, let us set $G = \begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix}$ in (3.1). We have that $\delta_1(mn) = -\delta_4(nm) + \tau_2(m)n + m\nu_3(n)$ and $\mu_4(nm) = -\mu_1(mn) + n\tau_2(m) + \nu_3(n)m$ for all $m \in M, n \in N$.

If Θ_{Jord} has the form ($\bigstar 2$) and satisfies conditions $(1) - (7)$, the assertion that Θ_{Jord} is a Jordan derivation of $\mathscr{G}$ will follow from direct computations. We complete the proof of this proposition.

From now on, we always assume in this section that M is faithful as a left A-module and also as a right B-module, but no any constraint conditions concerning the bimodule N. Then we have the following:

COROLLARY 3.3. *Let $\mathcal{G}$ be a 2-torsion free generalized matrix algebra over the commutative ring $\mathcal{R}$. An additive mapping Θ_{Jord} form $\mathcal{G}$ into itself is a Jordan derivation of $\mathcal{G}$ if and only if it has the form*

$$\Theta_{\mathrm{Jord}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) - mn_0 - m_0n & am_0 - m_0b + \tau_2(m) + \tau_3(n) \\ n_0a - bn_0 + \nu_2(m) + \nu_3(n) & n_0m + nm_0 + \mu_4(b) \end{bmatrix}, \quad (\bigstar 3)$$

$$\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G},$$

where $m_0 \in M, n_0 \in N$ and

$$\delta_1 : A \longrightarrow A, \quad \tau_2 : M \longrightarrow M, \quad \tau_3 : N \longrightarrow M,$$
$$\nu_2 : M \longrightarrow N, \quad \nu_3 : N \longrightarrow N \quad \mu_4 : B \longrightarrow B$$

are all $\mathcal{R}$-linear mappings satisfying conditions

(1) δ_1 *is a derivation on A and* $\delta_1(mn) = \tau_2(m)n + m\nu_3(n)$;

(2) μ_4 *is a derivation on B and* $\mu_4(nm) = n\tau_2(m) + \nu_3(n)m$;

(3) $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ *and* $\tau_2(mb) = \tau_2(m)b + m\mu_4(b)$;

(4) $\nu_3(na) = \nu_3(n)a + n\delta_1(a)$ *and* $\nu_3(bn) = b\nu_3(n) + \mu_4(b)n$;

(5) $\tau_3(na) = a\tau_3(n)$, $\tau_3(bn) = \tau_3(n)b$, $n\tau_3(n) = 0$, $\tau_3(n)n = 0$;

(6) $\nu_2(am) = \nu_2(m)a$, $\nu_2(mb) = b\nu_2(m)$, $m\nu_2(m) = 0$, $\nu_2(m)m = 0$.

Proof. Let Θ_{Jord} be a Jordan derivation of $\mathcal{G}$. Then Θ_{Jord} has the form of $(\bigstar 2)$ and satisfies all additional conditions $(1) - (7)$ of Proposition 3.2. Since $\mathcal{G}$ is a 2-torsion free generalized matrix algebra, $\delta_4 = 0$ and $\mu_1 = 0$ by condition (3) of Proposition 3.2. Condition (3) of Proposition 3.2 vanishes in the present case. Condition (4) of Proposition 3.2 correspondingly becomes

$$\tau_2(am) = a\tau_2(m) + \delta_1(a)m$$

and

$$\tau_2(mb) = \tau_2(m)b + m\mu_4(b).$$

Clearly, we only need to prove that δ_1 is a derivation of A and that μ_4 is a derivation of B. Then for arbitrary elements $a_1, a_2 \in A$, we have

$$\tau_2(a_1a_2m) = a_1a_2\tau_2(m) + \delta_1(a_1a_2)m \tag{3.26}$$

and

$$\begin{aligned} \tau_2(a_1a_2m) &= a_1\tau_2(a_2m) + \delta_1(a_1)a_2m \\ &= a_1a_2\tau_2(m) + a_1\delta_1(a_2)m + \delta_1(a_1)a_2m. \end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27) gives

$$\delta_1(a_1 a_2)m = a_1 \delta_1(a_2)m + \delta_1(a_1)a_2 m. \qquad (3.28)$$

Note that M is faithful as left A-module. Relation (3.28) implies that

$$\delta_1(a_1 a_2) = a_1 \delta_1(a_2) + \delta_1(a_1)a_2$$

for all $a_1, a_2 \in A$. So δ_1 is a derivation of A. Similarly, we can show that μ_4 is a derivation of B.

Conversely, if an additive mapping Θ_{Jord} of $\mathscr{G}$ is of the form $(\bigstar 3)$ and satisfies all additional conditions $(1) - (6)$, then the fact that is a Jordan derivation of $\mathscr{G}$ will follow form direct computations.

In view of Herstein's result and recent intensive works [1, 3, 4, 6, 12, 17, 18, 23, 21, 24], the following question is at hand.

QUESTION 3.4. Is each Jordan derivation on a generalized matrix algebra $\mathscr{G}$ a derivation, or equivalently, do there exist proper Jordan derivations on generalized matrix algebras?

The following counterexample provides an explicit answer to the above question. It is shown that Jordan derivations of generalized matrix algebras need not be derivations. Equivalently, there indeed exist proper Jordan derivations on certain generalized matrix algebras.

EXAMPLE 3.5. Let A be a commutative unital $\mathscr{R}$-algebra and let

$$\mathscr{G} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \middle| a_{ij} \in A \right\}$$

be a generalized matrix algebra of order 2. For arbitrary $X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathscr{G}$, $Y = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathscr{G}$, we define the sum $X + Y$ as usual. The multiplication XY is given by the rule

$$XY = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}. \qquad (\spadesuit)$$

Such kind of generalized matrix algebras are called *trivial generalized matrix algebras*. That is, the bilinear pairings $\Phi_{AA} = \Psi_{AA} = 0$ are both zero. Let us establish an $\mathscr{R}$-linear mapping

$$\Gamma_{\text{Jord}} : \mathscr{G} \longrightarrow \mathscr{G}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & a_{12} + a_{21} \\ a_{12} - a_{21} & 0 \end{bmatrix}, \forall \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathscr{G}.$$

By straightforward computations, we know that Γ_{Jord} is a Jordan derivation of $\mathscr{G}$, but not a derivation.

On the other hand, we can also define two $\mathscr{R}$-linear mappings

$$\Theta_1 : \mathscr{G} \longrightarrow \mathscr{G}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & a_{12} \\ -a_{21} & 0 \end{bmatrix}, \ \forall \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathscr{G}$$

and

$$\Theta_2 : \mathscr{G} \longrightarrow \mathscr{G}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & a_{21} \\ a_{12} & 0 \end{bmatrix}, \ \forall \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathscr{G}.$$

It is easy to see that Θ_1 is a derivation of $\mathscr{G}$ and Θ_2 is an anti-derivation of $\mathscr{G}$. Therefore Γ_{Jord} is the sum of the derivation Θ_1 and the anti-derivation Θ_2.

As a matter of fact, there exist some generalized matrix algebras whose multiplication satisfies the rule ($\spadesuit$). Let $\mathscr{R}'$ be an associative ring with identity and $\mathscr{Z}(\mathscr{R}')$ be its center. Let us consider the usual 2×2 matrix ring $\begin{bmatrix} \mathscr{R}' & \mathscr{R}' \\ \mathscr{R}' & \mathscr{R}' \end{bmatrix}$. It will become a generalized matrix algebra under the usual addition and the following multiplication rule

$$\begin{bmatrix} a & c \\ d & b \end{bmatrix} \begin{bmatrix} e & g \\ h & f \end{bmatrix} = \begin{bmatrix} ae + sch & ag + cf \\ de + bh & sdg + bf \end{bmatrix},$$

where $s \in \mathscr{Z}(\mathscr{R}')$. A trivial generalized matrix algebra arises in the case of $s = 0$. The usual 2×2 matrix ring is produced when $s = 1$.

In view of Example 3.5 and our main motivation, we now begin to describe the forms of anti-derivations on the generalized matrix algebra $\mathscr{G}$. As we will see below, Example 3.5 can be lifted and extracted to a more general conclusion.

PROPOSITION 3.6. *An additive mapping Θ_{antid} from $\mathscr{G}$ into itself is an antiderivation if and only if it has the form*

$$\Theta_{\mathrm{antid}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} 0 & am_0 - m_0 b + \tau_3(n) \\ n_0 a - b n_0 + v_2(m) & 0 \end{bmatrix}, \quad (\bigstar 4)$$

$$\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathscr{G},$$

where $m_0 \in M, n_0 \in N$ and

$$\tau_3 : N \longrightarrow M, \quad v_2 : M \longrightarrow N$$

are $\mathscr{R}$-linear mappings satisfying the following conditions:

(1) $[a, a']m_0 = 0$, $m_0[b, b'] = 0$, $n_0[a, a'] = 0$, $[b, b']n_0 = 0$ *for all $a' \in A, b' \in B$;*

(2) $m_0 n = 0$, $n m_0 = 0$, $m n_0 = 0$, $n_0 m = 0$;

(3) $\tau_3(na) = a\tau_3(n)$, $\tau_3(bn) = \tau_3(n)b$, $n\tau_3(n') = 0$, $\tau_3(n)n' = 0$ *for all $n' \in N$;*

(4) $v_2(am) = v_2(m)a$, $v_2(mb) = bv_2(m)$, $mv_2(m') = 0$, $v_2(m)m' = 0$ *for all* $m' \in M$.

Proof. Suppose that the Jordan derivation Θ_{antid} is of the form

$$\Theta_{\text{antid}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) + \delta_2(m) + \delta_3(n) + \delta_4(b) & \tau_1(a) + \tau_2(m) + \tau_3(n) + \tau_4(b) \\ v_1(a) + v_2(m) + v_3(n) + v_4(b) & \mu_1(a) + \mu_2(m) + \mu_3(n) + \mu_4(b) \end{bmatrix},$$

for all $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathscr{G}$, where $\delta_1, \delta_2, \delta_3, \delta_4$ are $\mathscr{R}$-linear mappings from A, M, N, B to A, respectively; $\tau_1, \tau_2, \tau_3, \tau_4$ are $\mathscr{R}$-linear mappings from A, M, N, B to M, respectively; v_1, v_2, v_3, v_4 are $\mathscr{R}$-linear mappings from A, M, N, B to N, respectively; $\mu_1, \mu_2, \mu_3, \mu_4$ are $\mathscr{R}$-linear mappings from A, M, N, B to B, respectively.

For any $G_1, G_2 \in \mathscr{G}$, we will intensively employ the antiderivation equation

$$\Theta_{\text{antid}}(G_1 G_2) = \Theta_{\text{antid}}(G_2)G_1 + G_2\Theta_{\text{antid}}(G_1). \tag{3.29}$$

Taking $G_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix}$ into (3.29) yields

$$\Theta_{\text{antid}}(G_1 G_2) = \begin{bmatrix} \delta_1(aa') & \tau_1(aa') \\ v_1(aa') & \mu_1(aa') \end{bmatrix} \tag{3.30}$$

and

$$\Theta_{\text{antid}}(G_2)G_1 + G_2\Theta_{\text{antid}}(G_1) = \begin{bmatrix} \delta_1(a')a + a'\delta_1(a) & a'\tau_1(a) \\ v_1(a')a & 0 \end{bmatrix}. \tag{3.31}$$

It follows from (3.30) with (3.31) that δ_1 is an antiderivation of A, $\mu_1 = 0$ and

$$v_1(aa') = v_1(a')a \tag{3.32}$$

for all $a, a' \in A$. Let us set $a' = 1$ in (3.32) and denote $v_1(1)$ by n_0. Then $v_1(a) = n_0 a$. Furthermore, (3.32) implies that $n_0 a a' = n_0 a' a$ for all $a, a' \in A$, that is, $n_0[a, a'] = 0$ for all $a, a' \in A$. If we denote $\tau_1(1)$ by m_0, then we obtain $\tau_1(a) = am_0$ and $[a, a']m_0 = 0$ for all $a, a' \in A$.

Let us choose $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ 0 & b' \end{bmatrix}$ in (3.29). By the same computational approach we conclude that μ_4 is an antiderivation of B, $\delta_4 = 0$ and

$$\tau_4(b) = \tau_4(1)b, \quad v_4(b) = bv_4(1), \quad \tau_4(1)[b, b'] = 0, \quad [b, b']v_4(1) = 0 \tag{3.33}$$

for all $b, b' \in B$. We claim that $\tau_4(1) = -m_0$. In fact, this can be obtained by taking $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in (3.29). Likewise, we assert that $v_4(1) = -n_0$. Thus the relation (3.33) becomes

$$\tau_4(b) = -m_0 b, \quad v_4(b) = -bn_0, \quad m_0[b, b'] = 0, \quad [b, b']n_0 = 0$$

for all $b, b' \in B$.

Putting $G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ in (3.29) and using the fact $\mu_1 = 0$ gives

$$\Theta_{\text{antid}}(G_1 G_2) = \begin{bmatrix} \delta_2(m) & \tau_2(m) \\ v_2(m) & \mu_2(m) \end{bmatrix} \tag{3.34}$$

and

$$\Theta_{\text{antid}}(G_2)G_1 + G_2\Theta_{\text{antid}}(G_1) = \begin{bmatrix} \delta_2(m) + mn_0 & 0 \\ v_2(m) & 0 \end{bmatrix}. \tag{3.35}$$

Combining (3.34) with (3.35) leads to

$$mn_0 = 0, \quad \tau_2 = 0, \quad \mu_2 = 0$$

for all $m \in M$. Interchanging G_1 and G_2 we will get

$$\delta_2 = 0, \quad n_0m = 0$$

for all $m \in M$.

If we take $G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ into (3.29), then

$$\Theta_{\text{antid}}(G_1G_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.36}$$

and

$$\Theta_{\text{antid}}(G_2)G_1 + G_2\Theta_{\text{antid}}(G_1) = \begin{bmatrix} \delta_3(n) & 0 \\ v_3(n) & 0 \end{bmatrix}. \tag{3.37}$$

will follow from the fact $\delta_1(1) = 0$. By (3.36) and (3.37) we obtain that

$$\delta_3 = 0, \quad v_3 = 0. \tag{3.38}$$

Interchanging G_1 and G_2 again yields

$$\mu_3 = 0, \quad m_0n = 0 \tag{3.39}$$

for all $n \in N$. In order to get $nm_0 = 0$, we only need to put $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ in (3.29).

Taking $G_1 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ into (3.29) and applying (3.38) and (3.39) we arrive at

$$\Theta_{\text{antid}}(G_1G_2) = \begin{bmatrix} 0 & \tau_3(na) \\ 0 & 0 \end{bmatrix}. \tag{3.40}$$

The fact $\mu_1 = 0$ and (3.39) imply that

$$\Theta_{\text{antid}}(G_2)G_1 + G_2\Theta_{\text{antid}}(G_1) = \begin{bmatrix} 0 & a\tau_3(n) \\ 0 & 0 \end{bmatrix}. \tag{3.41}$$

The relations (3.40) and (3.41) jointly show that $\tau_3(na) = a\tau_3(n)$ for all $a \in A, n \in N$. Likewise, if we choose $G_1 = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ in (3.29), then $\tau_3(bn) = \tau_3(n)b$ for all $b \in B, n \in N$. The equalities $v_2(am) = v_2(m)a$ and $v_2(mb) = bv_2(m)$ can be obtained by analogous discussions and the details are omitted here.

Let us consider $G_1 = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ n' & 0 \end{bmatrix}$ in (3.29). Then we get $n\tau_3(n') = 0$ and $\tau_3(n)n' = 0$ for all $n, n' \in N$. Putting $G_1 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix}$ in (3.29) yields $mv_2(m') = 0$ and $v_2(m)m' = 0$ for all $m, m' \in M$.

Taking $G_1 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ and $G_2 = \begin{bmatrix} a & m \\ 0 & 0 \end{bmatrix}$ into (3.29), then we get $\delta_1(a)m = 0$ for all $a \in A, m \in M$. Putting $G_1 = \begin{bmatrix} 0 & m \\ 0 & b \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ in (3.29) gives $m\mu_4(b) = 0$ for all $b \in B, m \in M$. It follows from the faithfulness of M that $\delta_1 = 0$ and $\mu_4 = 0$.

Conversely, suppose that Θ_{antid} is of the form ($\bigstar 4$) and satisfies conditions $(1) - (4)$. Then the fact that Θ_{antid} is a antiderivation of $\mathscr{G}$ will follow by direct computations.

Let us next observe the antiderivations of a class of generalized matrix algebras.

DEFINITION 3.7. Let $\mathscr{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ be a generalized matrix algebra originating from the Morita context $(A, B, {}_A M_B, {}_B N_A, \Phi_{MN}, \Psi_{NM})$. The bilinear form $\Phi_{MN} : M \otimes_B N \longrightarrow A$ (resp. $\Psi_{NM} : N \otimes_A M \longrightarrow B$) is called *nondegenerate* if for any $0 \neq m \in M$ and $0 \neq n \in N$, $\Phi_{MN}(m, N) \neq 0$ and $\Phi_{MN}(M, n) \neq 0$ (resp. $\Psi_{NM}(n, M) \neq 0$ and $\Psi_{NM}(N, m) \neq 0$).

EXAMPLE 3.8. Let H be a finite dimensional Hopf algebra over a filed $\mathbb{K}$ and A be an H-module algebra. Let A^H be the invariant subalgebra of H on A, and $A\#H$ be the smash product algebra of A and H. We now consider the generalized matrix algebra

$$\mathscr{G}_{\text{SPA}} = \begin{bmatrix} A^H & M \\ N & A\#H \end{bmatrix}$$

defined in Example 2.1, where $M = {}_{A^H} A_{A\#H}$ and $N = {}_{A\#H} A_{A^H}$. Suppose that M is a faithful right (or left) $A\#H$-module. By [7, Proposition 2.13] we know that the bilinear form Φ_{MN} will be nondegenerate. In this case, we easily check that there is indeed no nonzero antiderivations on $\mathscr{G}_{\text{SPA}}$.

EXAMPLE 3.9. Let $\mathbb{K}$ be a field and A be an associative algebra over $\mathbb{K}$. Let G be a group and $A * G$ be the skew group algebra over $\mathbb{K}$. Suppose that A^G is the fixed ring of the action G on A. We now revisit the generalized matrix algebra

$$\mathscr{G}_{\text{GA}} = \begin{bmatrix} A^G & M \\ N & A * G \end{bmatrix}$$

in Example 2.2, where $M = {}_{A^G} A_{A*G}$ and $N = {}_{A*G} A_{A^G}$. For an arbitrary element $n \in N$, we define

$$n^\perp = \{ m \in M | \Psi_{NM}(n, m) = 0 \}.$$

Similarly, for an arbitrary element $m \in M$, we define

$$m^\perp = \{ n \in N | \Psi_{NM}(n, m) = 0 \}.$$

Then $n^\perp$ is a G-invariant right ideal of A contained in $r_A(n)$, where $r_A(n)$ is the right annihilator of n in A. Indeed, let $m \in n^\perp$ and $g \in G$, then $\Psi_{NM}(n, m^g) = \Psi_{NM}(n, m \cdot g) = \Psi_{NM}(n, m)g = 0$. Hence $n^\perp$ is G-invariant, the rest is obvious. Similarly, we can show that $m^\perp$ is a G-invariant left ideal of A contained in $l_A(m)$, where $l_A(m)$ is the left annihilator of m in A.

In particular, if A is a semiprime $\mathbb{K}$-algebra, then $r_A(n) \neq A$ and $l_A(m) \neq A$. This shows that the bilinear form Ψ_{NM} is nondegenerate. Furthermore, if we assume that the module N is faithful as a left $A * G$-module, then the bilinear form Φ_{MN} will be also nondegenerate. Indeed, let $\Phi_{MN}(m, N) = 0$ for some $m \in M$. Then, $0 = N \cdot \Phi_{MN}(m, N) = \Psi_{NM}(N, m) \cdot N$. By faithfulness and nondegeneracy of Ψ_{NM} we deduce that $m = 0$. If one of the bilinear pairings Φ_{MN} and Ψ_{NM} is nondegenerate, then there is no nonzero antiderivations on $\mathscr{G}_{GA}$, which is similar to Example 3.8.

In order to ensure the semiprimeness of the $\mathbb{K}$-algebra A and the nondegeneracy of the bilinear forms Φ_{MN} and Ψ_{NM}, A may be one of the following algebras:

(1) the quantized enveloping algebra $U_q(\mathfrak{sl}_2(\mathbb{K}))$ over the field $\mathbb{K}$,

(2) the quantum $n \times n$ matrix algebra $\mathscr{O}_q(M_n(\mathbb{K}))$ over the field $\mathbb{K}$,

(3) the quantum affine n-space $\mathscr{O}_q(\mathbb{K}^n)$ over the field $\mathbb{K}$,

(4) the double affine Hecke algebra $\widetilde{H}$ over the field $\mathbb{K}$.

(5) the Iwasawa algebra Ω_G over the finite field $\mathbb{F}_p$.

In view of Proposition 3.6, Example 3.8 and Example 3.9 we immediately have

PROPOSITION 3.10. *Let $\mathscr{G}$ be a generalized matrix algebra over the commutative ring $\mathscr{R}$ and Θ_{antid} be an $\mathscr{R}$-linear mapping from $\mathscr{G}$ into itself. If one of the bilinear forms $\Phi_{MN} : M \underset{B}{\otimes} N \longrightarrow A$ and $\Psi_{NM} : N \underset{A}{\otimes} M \longrightarrow B$ is nondegenerate, then Θ_{antid} is an antiderivation of $\mathscr{G}$ if and only if $\Theta_{\text{antid}} = 0$.*

We will end this section by investigating properties of Jordan derivations of generalized matrix algebras with zero bilinear pairings. Such kind of generalized matrix algebras draw our attention, which is due to Haghany's work and Example 3.5. Haghany in [8] studied hopficity and co-hopficity for generalized matrix algebras with zero bilinear parings. As you see in Example 3.5, those generalized matrix algebras exactly have zero bilinear pairings.

THEOREM 3.11. *Let $\mathscr{G}$ be a 2-torsion free generalized matrix algebra over the commutative ring $\mathscr{R}$. If the bilinear pairings Φ_{MN} and Ψ_{NM} are both zero, then every Jordan derivation of $\mathscr{G}$ can be expressed as the sum of a derivation and an antiderivation.*

Proof. Let Θ_{Jord} be a Jordan derivation of $\mathscr{G}$. By Corollary 3.3 we know that Θ_{Jord} is of the form

$$\Theta_{\text{Jord}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) - mn_0 - m_0 n & am_0 - m_0 b + \tau_2(m) + \tau_3(n) \\ n_0 a - bn_0 + \nu_2(m) + \nu_3(n) & n_0 m + nm_0 + \mu_4(b) \end{bmatrix}, \quad (\bigstar 3)$$

$$\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathscr{G}.$$

It follows from Proposition 3.1 and Proposition 3.6 that there exist a derivation Θ'_d and an antiderivation Θ'_{antid} such that

$$
\begin{aligned}
\Theta_{\text{Jord}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) &= \begin{bmatrix} \delta_1(a) - mn_0 - m_0 n & am_0 - m_0 b + \tau_2(m) \\ n_0 a - bn_0 + \nu_3(n) & n_0 m + nm_0 + \mu_4(b) \end{bmatrix} + \begin{bmatrix} 0 & \tau_3(n) \\ \nu_2(m) & 0 \end{bmatrix} \\
&= \Theta'_d\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) + \Theta'_{antid}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right)
\end{aligned}
$$

for all $\begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathscr{G}$. This shows that Θ_{Jord} can be expressed the sum of a derivation Θ'_d and an antiderivation Θ'_{antid}, which is the desired result.

As direct consequences of Theorem 3.11 we have

COROLLARY 3.12. *The Jordan derivation Γ_{Jord} constructed in Example 3.5 can be expressed as the sum of a derivation Θ_1 and an antiderivation Θ_2.*

COROLLARY 3.13. *Let $\mathscr{R}$ be a 2-torsion free commutative ring with identity, A, B be two unital $\mathscr{R}$-algebras and M be a faithful A-B-bimodule. Suppose that $\mathscr{T}_{\mathscr{R}}$ is the triangular algebra consisting of A, B and M. Then each Jordan derivation of $\mathscr{T}_{\mathscr{R}}$ is the sum of a derivation and an antiderivation.*

Clearly, $\mathscr{T}_{\mathscr{R}}$ is a generalized matrix algebra with zero pairings. In view of Theorem 3.11, every Jordan derivation of $\mathscr{T}_{\mathscr{R}}$ can be written as the sum of a derivation and an antiderivation.

Acknowledgements

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Yanbo Li
Department of Information and Computing Sciences
Northeastern University at Qinhuangdao
Qinhuangdao, 066004, P. R. China
e-mail: liyanbo707@gmail.com

Leon van Wyk
Department of Mathematics
Stellenbosch University
Private Bag X1, Matieland 7602
Stellenbosch, South Africa
e-mail: LvW@sun.ac.za

Feng Wei
School of Mathematics
Beijing Institute of Technology
Beijing, 100081, P. R. China
e-mail: daoshuo@hotmail.com

Operators and Matrices
www.ele-math.com
oam@ele-math.com