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ISOMORPHISM OF GENERALIZED TRIANGULAR MATRIX-RINGS AND RECOVERY OF TILES

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We prove an isomorphism theorem for generalized triangular matrix-rings, over rings having only the idempotents 0 and 1, in particular, over indecomposable commutative rings or over local rings (not necessarily commutative). As a consequence, we obtain a recovery result for the tile in a tiled matrix-ring.

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Matrix-rings play a fundamental role in mathematics and its applications. A difficult question is to decide whether a given ring is isomorphic to a matrix-ring or one of its variants. Several "hidden matrix-rings" have been shown in the literature (see [5]). These rings did not appear as being matrix-rings at the first sight, nevertheless they proved out to be isomorphic to matrix-rings. Another type of problem concerned to matrices is to decide whether two rings of matrices are isomorphic or not. For instance, it is known that for commutative rings R and S, the matrix-rings $M_2(R)$ and $M_2(S)$ are isomorphic if and only if the rings R and S are isomorphic, for the simple reason that R is isomorphic to the center of $M_2(R)$. However, if R and S are not commutative, this is not true anymore. Examples have been given in [7], also in [6] for simple Noetherian integral domains R,S, or in [2] for prime Noetherian R,S. A different but related problem is the recovery of the tile in a triangular matrix-ring. More precisely, if R is a ring and I, J are two-sided ideals of R such that the rings $\left(\begin{smallmatrix} R & I \\ 0 & R \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} R & J \\ 0 & R \end{smallmatrix}\right)$ are isomorphic, what can we say about I and J? Are they isomorphic as R-bimodules? If we do not impose any condition to the ring, then there is no hope to recover the tile. For instance, in [3] a ring R was constructed such that

$$\begin{pmatrix} R & R \\ 0 & R \end{pmatrix} \simeq \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}. \tag{1}$$

It was proved in [1] that if R satisfies a certain finiteness condition (in particular in the case where R is a left Noetherian), the above isomorphism cannot hold. For the situation where the tile is not necessarily 0 or the whole ring R, the situation behaves worse. Even when the ring is finite, the tile cannot be recovered. It was proved in [4] that if $R = \left(\begin{smallmatrix} A & 0 & A \\ 0 & A & A \\ 0 & 0 & A \end{smallmatrix}\right)$, A is a ring, and

$$I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A \\ 0 & 0 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

then the rings $\left(\begin{smallmatrix} R & I \\ 0 & R \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} R & J \\ 0 & R \end{smallmatrix}\right)$ are isomorphic, while I and J are not isomorphic as R-bimodules.

The aim of this paper is to obtain a recovery result for the tile in the case where the underlying ring R has only trivial idempotents, that is, R has only two idempotents, 0 and 1. Relevant examples of such rings are for instance: indecomposable commutative rings and local rings (not necessarily commutative). In fact we can investigate the isomorphism among more general matrix-type rings. Recall that if R and S are two rings, and M is an R,S-bimodule (this means left R and right S), we can define the generalized triangular matrix-ring $\left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$, with multiplication induced by the bimodule actions and the usual rule for matrix multiplication. With this notation we can prove the following theorem.

Theorem 1. *Let R and S be rings having only trivial idempotents, and let M,N be two R,S-bimodules. Then a map $\phi : \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right) \to \left(\begin{smallmatrix} R & N \\ 0 & S \end{smallmatrix}\right)$ is a ring isomorphism if and only if there exist $a \in N$, $f \in \operatorname{Aut}(R)$, $g \in \operatorname{Aut}(S)$, and an isomorphism $v : M \to N$ of additive groups satisfying $v(rx) = f(r)v(x)$ and $v(xs) = v(x)g(s)$ for any $x \in M$, $r \in R$, $s \in S$, such that*

$$\phi \begin{pmatrix} r & x \\ 0 & s \end{pmatrix} = \begin{pmatrix} f(r) & f(r)a - ag(s) + v(x) \\ 0 & g(s) \end{pmatrix}, \tag{3}$$

for any $r \in R$, $x \in M$, and $s \in S$.

In particular, we obtain a recovery result for the tile. This is not exactly an isomorphism, but an isomorphism relative to some automorphisms of the ring. We recall that if $f,g \in \operatorname{Aut}(R)$, and X,Y are two R,R-bimodules, then an additive map $v : X \to Y$ is called an f,g-morphism if $v(rxr') = f(r)v(x)g(r')$, for any $r,r' \in R$, $x \in X$.

Corollary 2 (recovery of the tile)**.** *Let R be a ring having only trivial idempotents, and I,J be ideals of R. Then the matrix-rings $\left(\begin{smallmatrix} R & I \\ 0 & R \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} R & J \\ 0 & R \end{smallmatrix}\right)$ are isomorphic if and only if I and J are f,g-isomorphic as the R,R-bimodules for some $f,g \in \operatorname{Aut}(R)$.*

A complete recovery of the tile (up to isomorphism) is obtained in some special cases when the ring has only the trivial automorphism.

Corollary 3. *Let R be a ring having only trivial idempotents such that, the only automorphism of R is the identity. If I,J are ideals of R, then the matrix-rings $\left(\begin{smallmatrix} R & I \\ 0 & R \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} R & J \\ 0 & R \end{smallmatrix}\right)$ are isomorphic if and only if I and J are isomorphic as the R,R-bimodules.*

Proof of Theorem 1. An element $\left(\begin{smallmatrix} r & x \\ 0 & s \end{smallmatrix}\right) \in \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ is idempotent if and only if $r^2 = r$, $s^2 = s$, and $rx + xs = x$. Since the only idempotents of R and S are 0 and 1, we have that any of r and s is either 0 or 1. If $r = 0$ and $s = 0$, we find $x = 0$. If $r = 1$ and $s = 1$, we find again $x = 0$. If $r = 1$ and $s = 0$, then x can be anything in M, and the same in the case where $r = 0$ and $s = 1$. Thus, apart from 0 and the identity element, the idempotents of $\left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ are the elements of the form

$$\begin{aligned} e_x &= \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \quad x \in M, \\ f_x &= \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}, \quad x \in M. \end{aligned} \tag{4}$$

It is easy to see that the following relations hold:

$$e_x e_y = e_y, \qquad f_x f_y = f_x, \qquad e_x f_y = \begin{pmatrix} 0 & x+y \\ 0 & 0 \end{pmatrix}, \qquad f_x e_y = 0, \tag{5}$$

for any $x, y \in M$. We denote by e'_z, f'_z, $z \in N$, the similar idempotents of $\left(\begin{smallmatrix} R & N \\ 0 & S \end{smallmatrix}\right)$. Let $\phi : \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right) \to \left(\begin{smallmatrix} R & N \\ 0 & S \end{smallmatrix}\right)$ be a ring isomorphism. Then $\phi(e_0)$ must be a nontrivial idempotent of $\left(\begin{smallmatrix} R & N \\ 0 & S \end{smallmatrix}\right)$. We distinguish two cases.

Case 1. We have $\phi(e_0) = e'_a$ for some $a \in N$. Then if for some $x \in M$ we have $\phi(e_x) = f'_b$ for some $b \in N$, we see that

$$e'_a = \phi(e_0) = \phi(e_x e_0) = \phi(e_x)\phi(e_0) = f'_b e'_a = 0, \tag{6}$$

a contradiction. Therefore, $\phi(e_x) = e'_{u(x)}$ for some $u(x) \in N$ for any $x \in M$. Then we have that

$$\phi(f_x) = \phi(I_2 - e_{-x}) = I_2 - e'_{u(-x)} = f'_{-u(-x)}. \tag{7}$$

Thus, for any $x \in M$ we have

$$\phi\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \phi(e_0 f_x) = \phi(e_0)\phi(f_x) = e'_a f'_{-u(-x)} = \begin{pmatrix} 0 & a - u(-x) \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Denote $v : M \to N$, $v(x) = a - u(-x)$. Then clearly v is a morphism of additive groups. Moreover, v is an isomorphism. Indeed, if $\phi^{-1}(e'_z) = f_h$ for some $z \in N, h \in M$, then $\phi(f_h) = e'_z$, a contradiction. Thus $\phi(\{e_x \mid x \in M\}) = \{e'_z \mid z \in N\}$, showing that u is surjective, so then v is also surjective. Obviously, v is injective.

Now

$$\phi\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = \phi\left(e_0\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}\right) = e'_a\phi\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} R & N \\ 0 & 0 \end{pmatrix} \tag{9}$$

thus $\phi\left(\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} f(r) & h(r) \\ 0 & 0 \end{smallmatrix}\right)$ for some additive maps $f : R \to R$, $h : R \to N$. Since ϕ is a ring morphism, we obtain that

$$\begin{aligned} f(r_1r_2) &= f(r_1)f(r_2), \quad f(1) = 1, \\ h(r_1r_2) &= f(r_1)h(r_2), \quad h(1) = a, \end{aligned} \tag{10}$$

for any $r_1, r_2 \in R$. Similarly, one gets $\phi\left(\begin{smallmatrix} 0 & 0 \\ 0 & s \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & p(s) \\ 0 & g(s) \end{smallmatrix}\right)$ for some additive maps $g : S \to S$, $p : S \to N$ satisfying

$$\begin{aligned} g(s_1s_2) &= g(s_1)g(s_2), \quad g(1) = 1, \\ p(s_1s_2) &= p(s_1)g(s_2), \quad p(1) = -a. \end{aligned} \tag{11}$$

Then $h(r) = h(r1) = f(r)h(1) = f(r)a$ for any $r \in R$, and similarly $p(s) = -ag(s)$ for any $s \in S$. We obtain that

$$\begin{aligned} \phi\begin{pmatrix} r & x \\ 0 & s \end{pmatrix} &= \phi\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} + \phi\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} + \phi\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} \\ &= \begin{pmatrix} f(r) & f(r)a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & v(x) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ag(s) \\ 0 & g(s) \end{pmatrix} \\ &= \begin{pmatrix} f(r) & f(r)a - ag(s) + v(x) \\ 0 & g(s) \end{pmatrix}, \end{aligned} \tag{12}$$

for any $r \in R$, $s \in S$, and $x \in M$. By using the relation

$$\phi\left(\begin{pmatrix} r & x \\ 0 & s \end{pmatrix}\begin{pmatrix} r' & x' \\ 0 & s' \end{pmatrix}\right) = \phi\begin{pmatrix} r & x \\ 0 & s \end{pmatrix}\phi\begin{pmatrix} r' & x' \\ 0 & s' \end{pmatrix}, \tag{13}$$

we obtain, by computing the $(1,2)$-slots in the two sides, that $f(r)v(x') + v(x)g(s') = v(rx') + v(xs')$ for any $r \in R$, $x, x' \in M$, $s' \in S$. For $s' = 0$, we find $v(rx') = f(r)v(x')$, and for $r = 0$, we obtain $v(xs') = v(x)g(s')$.

It remains to show that f and g are bijective. Clearly, $\ker(f) = 0$ since $f(r) = 0$ implies $\phi\left(\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and then r must be 0. Also f is surjective since for any

$b \in R$, there exists $\left(\begin{smallmatrix} r & x \\ 0 & s \end{smallmatrix}\right) \in \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ with $\phi\left(\begin{smallmatrix} r & x \\ 0 & s \end{smallmatrix}\right) = \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right)$, in particular, $f(r) = b$. Thus f is a ring isomorphism, and so is g.

Case 2. We have $\phi(e_0) = f'_a$ for some $a \in N$. Then for any $x \in M$, we have that

$$f'_a = \phi(e_0) = \phi(e_x e_0) = \phi(e_x)\phi(e_0) = \phi(e_x) f'_a. \tag{14}$$

If $\phi(e_x) = e'_z$ for some $x \in M$, $z \in N$, we obtain that

$$f'_a = e'_z f'_a = \begin{pmatrix} 0 & z+a \\ 0 & 0 \end{pmatrix}, \tag{15}$$

a contradiction. Thus, $\phi(e_x) = f'_{u(x)}$ for any $x \in M$, where $u : M \to N$ is a map. Hence $\phi(f_x) = \phi(I_2 - e_{-x}) = I_2 - f'_{u(-x)} = e'_{-u(-x)}$, and then

$$\phi\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \phi(e_0 f_x) = \phi(e_0)\phi(f_x) = f'_{u(0)} e'_{-u(-x)} = 0, \tag{16}$$

a contradiction, for $x \neq 0$. Therefore this case cannot occur.

For the other way around, it is straightforward to check that any map ϕ of the given form is an isomorphism of rings. □

Examples. (1) Let m and n be two nonnegative integers, and let $\mathbb{Z}$ be the ring of integers which has only 0 and 1 as idempotents. Then by Corollary 3 the rings $\left(\begin{smallmatrix} \mathbb{Z} & m\mathbb{Z} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathbb{Z} & n\mathbb{Z} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$ are isomorphic if and only if $m = n$.

(2) Let $\mathbb{Z}[i]$ be the ring of Gauss integers which is a principal ideal domain (PID), in particular, it also has only trivial idempotents. If $x, y \in \mathbb{Z}[i]$, then the rings $\left(\begin{smallmatrix} \mathbb{Z}[i] & x\mathbb{Z}[i] \\ 0 & \mathbb{Z}[i] \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathbb{Z}[i] & y\mathbb{Z}[i] \\ 0 & \mathbb{Z}[i] \end{smallmatrix}\right)$ are isomorphic if and only if either $x = uy$ or $x = u\overline{y}$ for some $u \in \{1, -1, i, -i\}$, where $\overline{y}$ denotes the complex conjugate of y. Indeed, this follows from Corollary 2 and the fact that the only automorphisms of $\mathbb{Z}[i]$ are the identity and the complex conjugation.

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