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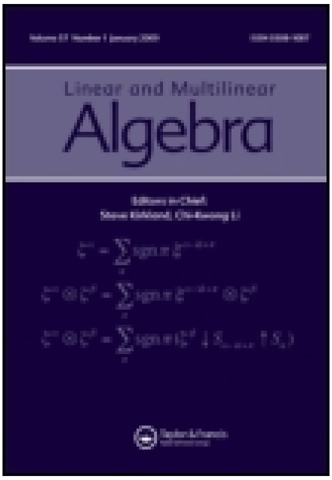

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Invertibility and Dedekind finiteness in structural matrix rings

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Invertibility and Dedekind finiteness in structural matrix rings

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An example in Szigeti and van Wyk [J. Szigeti and L. van Wyk, *Subrings which are closed with respect to taking the inverse*, J. Algebra 318 (2007), pp. 1068–1076] suggests that Dedekind finiteness may play a crucial role in a characterization of the structural subrings $M_n(\theta, R)$ of the full $n \times n$ matrix ring $M_n(R)$ over a ring R, which are closed with respect to taking inverses. It turns out that $M_n(\theta, R)$ is closed with respect to taking inverses in $M_n(R)$ if all the equivalence classes with respect to $\theta \cap \theta^{-1}$, except possibly one, are of a size less than or equal to p (say) and $M_p(R)$ is Dedekind finite. Another purpose of this article is to show that $M_n(\theta, R)$ is Dedekind finite if and only if $M_m(R)$ is Dedekind finite, where m is the maximum size of the equivalence classes (with respect to $\theta \cap \theta^{-1}$). This provides a positive result for the inheritance of Dedekind finiteness by a matrix ring (albeit not a full matrix ring) from a smaller (full) matrix ring.

Keywords: Dedekind finite ring; invertibility; structural matrix ring

AMS Subject Classifications: 15A09; 15A30; 16S50; 16U60

1. Introduction

A ring R is called Dedekind finite (or von Neumann finite, or weakly 1-finite, or affine finite, or directly finite, or inverse symmetric) if whenever $ab = 1$ $(a, b \in R)$, then $ba = 1$. If the full $m \times m$ matrix ring $M_m(R)$ is Dedekind finite, then R is called weakly m-finite in [2]. It is well known that left Noetherian rings and polynomial identity rings (PI-rings) are Dedekind finite. We also note that a reversible ring R ($xy = 0$ implies that $yx = 0$ for all $x, y \in R$) is Dedekind finite [1]. Reduced rings are examples of reversible rings.

The Dedekind finiteness of $M_m(R)$ implies that the Dedekind finiteness of $M_n(R)$ for any integer n with $n \leq m$. Full matrix rings often inherit properties of the base ring R, for example, being a left Noetherian ring or a PI-ring. On the other hand, $M_n(R)$ does not inherit Dedekind finiteness from $M_m(R)$ if $n > m$. The simplest case of $m = 1$ and $n = 2$ was considered in [2,9].

In this article, we study the structural subring $M_n(\theta, R)$ of $M_n(R)$ determined by a quasi-order θ on the set $\{1, 2, \ldots, n\}$. The class of structural matrix rings has been

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studied extensively (see, e.g. [3,4,11]). Based on results by Kaplansky–Amitsur and Rasmyslov–Kemer–Braun, it was shown in [10] that if the base ring R of $M_n(\theta, R)$ is a PI-ring, then $M_n(\theta, R)$ is closed with respect to taking inverses in $M_n(R)$. An example of an invertible 2×2 upper triangular matrix A with entries from any non-Dedekind finite ring R was also given such that A^{-1} is not upper triangular.

This example suggests that Dedekind finiteness may play a crucial role in a characterization of structural subrings of $M_n(R)$, which are closed with respect to taking inverses. Indeed, we prove that $M_n(\theta, R)$ is closed with respect to taking inverses in $M_n(R)$ if all the equivalence classes with respect to $\theta \cap \theta^{-1}$, except possibly one, are of a size less than or equal to p (say) and $M_p(R)$ is Dedekind finite. As a consequence, we obtain the main results in [10] including the above-mentioned sufficient one. Another purpose of this article is to show that $M_n(\theta, R)$ is Dedekind finite if and only if $M_m(R)$ is Dedekind finite, where m is the maximum size of the equivalence classes (with respect to $\theta \cap \theta^{-1}$). This provides a positive result for the inheritance of Dedekind finiteness by a matrix ring (albeit not a full matrix ring) from a smaller (full) matrix ring.

It should be noted that every structural matrix ring is an intersection of conjugates of a complete blocked triangular matrix ring [5]. Hence in order to study the closure of a structural matrix ring with respect to taking inverses, it suffices to consider complete blocked triangular matrix rings. However, in our development matrices are considered as functions instead of rectangular arrays, which allows us to obtain direct and rather condensed proofs.

2. Invertibility and Dedekind finiteness

For finite sets U and V a matrix of size $U \times V$ over a unitary ring R is a function $A: U \times V \to R$, and for the non-empty subsets $U' \subseteq U$ and $V' \subseteq V$ the $U' \times V'$ submatrix of A is simply the restricted function $A \restriction U' \times V'$. The addition of $U \times V$ matrices is coordinatewise, and the product of a matrix A of size $U \times V$ and a matrix B of size $V \times W$ is the following $U \times W$ matrix:

$$(AB)(u, w) = \sum_{v \in V} A(u, v)B(v, w),$$

where $(u, w) \in U \times W$ and V is finite. If $\mathcal{V} = \{T_k \mid 1 \le k \le d\}$ is a partition of V, then for the subsets $U' \subseteq U$ and $W' \subseteq W$ we have

$$AB \restriction U' \times W' = \sum_{T \in \mathcal{V}} (A \restriction U' \times T)(B \restriction T \times W').$$

Let $M_{U \times V}(R)$ denote the set of all $U \times V$ matrices over the ring R. The zero matrix of size $U \times V$ is denoted by $0_{U \times V}$. If $U = V$, then $I_{V \times V}$ stands for the identity matrix in $M_{V \times V}(R)$. If $\theta \subseteq V \times V$ is a quasi-order (i.e. reflexive and transitive relation on V), then

$$M_{V \times V}(\theta, R) = \{A \in M_{V \times V}(R) \mid A(i,j) = 0 \quad \text{for all } (i,j) \in (V \times V) \setminus \theta\}$$

is the structural subring of $M_{V \times V}(R)$ determined by θ. Now $\theta \cap \theta^{-1}$ is an equivalence relation on V and θ induces a partial order relation $\le_\theta$ on the set $V/\theta \cap \theta^{-1}$ of equivalence classes.

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LEMMA 2.1 *If $A \in M_{V\times V}(\theta, R)$, $B \in M_{V\times V}(R)$, $BA = I_{V\times V}$ and T^* is a minimal element of the finite poset $(V/\theta \cap \theta^{-1}, \leq_\theta)$, then $A \restriction U \times T^* = 0_{U\times T^*}$,*

$$(B\restriction T^* \times T^*)(A\restriction T^* \times T^*) = I_{T^*\times T^*} \quad \textit{and} \quad (B\restriction U \times T^*)(A\restriction T^* \times T^*) = 0_{U\times T^*},$$

where $U = V \setminus T^*$.

Proof The minimality of T^* and $U \cap T^* = \emptyset$ ensures that $(U \times T^*) \cap \theta = \emptyset$. Now $A \restriction U \times T^* = 0_{U\times T^*}$ is a consequence of $A \in M_{V\times V}(\theta, R)$. Since $\{T^*, U\}$ is a partition of V, we have

$$\begin{aligned} I_{T^*\times T^*} &= I_{V\times V}\restriction T^* \times T^* = BA\restriction T^* \times T^* \\ &= (B\restriction T^* \times T^*)(A\restriction T^* \times T^*) + (B\restriction T^* \times U)(A\restriction U \times T^*) \\ &= (B\restriction T^* \times T^*)(A\restriction T^* \times T^*) \end{aligned}$$

and

$$\begin{aligned} 0_{U\times T^*} &= I_{V\times V}\restriction U \times T^* = BA\restriction U \times T^* \\ &= (B\restriction U \times T^*)(A\restriction T^* \times T^*) + (B\restriction U \times U)(A\restriction U \times T^*) \\ &= (B\restriction U \times T^*)(A\restriction T^* \times T^*). \end{aligned}$$

■

We note that the following dual of Lemma 2.1 also holds.

LEMMA 2.2 *If $A \in M_{V\times V}(\theta, R)$, $B \in M_{V\times V}(R)$, $AB = I_{V\times V}$ and T^{**} is a maximal element of the finite poset $(V/\theta \cap \theta^{-1}, \leq_\theta)$, then $A \restriction T^{**} \times W = 0_{T^{**}\times W}$,*

$$(A\restriction T^{**} \times T^{**})(B\restriction T^{**} \times T^{**}) = I_{T^{**}\times T^{**}}$$

and

$$(A\restriction T^{**} \times T^{**})(B\restriction T^{**} \times W) = 0_{T^{**}\times W},$$

where $W = V \setminus T^{**}$.

THEOREM 2.3 *Let θ be a quasi-order on V such that $V/\theta \cap \theta^{-1} = \{T_1, T_2, \ldots, T_d\}$ has d elements and*

$$|T_1| \leq |T_2| \leq \cdots \leq |T_{d-1}| = p \leq |T_d|.$$

If $M_p(R)$ is Dedekind finite, $A \in M_{V\times V}(\theta, R)$, $B \in M_{V\times V}(R)$ and $AB = BA = I_{V\times V}$, then $B \in M_{V\times V}(\theta, R)$, and so $M_{V\times V}(\theta, R)$ is closed with respect to taking inverses in $M_{V\times V}(R)$.

Proof We prove that $B \in M_{V\times V}(\theta, R)$ by using induction on the number of elements of $V/\theta \cap \theta^{-1}$.

If $|V/\theta \cap \theta^{-1}| = 1$, then $V/\theta \cap \theta^{-1} = \{V\}$, $\theta = V \times V$ and $B \in M_{V\times V}(R) = M_{V\times V}(\theta, R)$ obviously holds.

Assume that our claim holds for all matrices of size $U \times U$ and for all quasi-orders $\vartheta \subseteq U \times U$ with the properties $|U/\vartheta \cap \vartheta^{-1}| = d$ and $|T| \leq p$ for all $T \in U/\vartheta \cap \vartheta^{-1}$ except possibly one. Now let

$$V/\theta \cap \theta^{-1} = \{T_k \mid 1 \leq k \leq d+1\}$$

be a $d+1 \geq 2$ element set and fix a pair $(i,j) \in (V \times V) \setminus \theta$. We have $i \in T_s$ and $j \in T_r$ for some unique indices $1 \leq r, s \leq d+1$, $r \neq s$. Take a maximal element T^{**} and a minimal element T^* in the finite poset $(V/\theta \cap \theta^{-1}, \leq_\theta)$ such that $T_s \leq_\theta T^{**}$ and $T^* \leq_\theta T_r$. Since $T^{**} = T^*$ would imply that $T_s \leq_\theta T_r$, which is in contradiction with $(i,j) \notin \theta$, we deduce that $T^{**} \neq T^*$. Thus either $|T^{**}| \leq p$ or $|T^*| \leq p$.

Case 1 If $|T^*| \leq p$, then $i \in T^*$ would imply that $(i,j) \in \theta$, a contradiction. It follows that $i \in U$, where $U = V \setminus T^*$.

Using $BA = I_{V \times V}$, the application of Lemma 2.1 gives $A \restriction U \times T^* = 0_{U \times T^*}$,

$$(B \restriction T^* \times T^*)(A \restriction T^* \times T^*) = I_{T^* \times T^*} \quad \text{and} \quad (B \restriction U \times T^*)(A \restriction T^* \times T^*) = 0_{U \times T^*}.$$

The Dedekind finiteness of $M_{T^* \times T^*}(R)$ is a consequence of $|T^*| \leq p$. It follows that

$$(A \restriction T^* \times T^*)(B \restriction T^* \times T^*) = I_{T^* \times T^*},$$

whence $B \restriction U \times T^* = 0_{U \times T^*}$ follows. Thus, we have

$$\begin{aligned} I_{U \times U} &= I_{V \times V} \restriction U \times U = BA \restriction U \times U \\ &= (B \restriction U \times T^*)(A \restriction T^* \times U) + (B \restriction U \times U)(A \restriction U \times U) = (B \restriction U \times U)(A \restriction U \times U) \end{aligned}$$

and

$$\begin{aligned} I_{U \times U} &= I_{V \times V} \restriction U \times U = AB \restriction U \times U \\ &= (A \restriction U \times U)(B \restriction U \times U) + (A \restriction U \times T^*)(B \restriction T^* \times U) = (A \restriction U \times U)(B \restriction U \times U) \end{aligned}$$

is a consequence of $A \restriction U \times T^* = 0_{U \times T^*}$. Consider the quasi-order $\vartheta = \theta \cap (U \times U)$ on the set U. Since

$$A \restriction U \times U \in M_{U \times U}(\vartheta, R) \quad \text{and} \quad U/\vartheta \cap \vartheta^{-1} = \{T_k \mid 1 \leq k \leq d+1, T_k \neq T^*\},$$

the induction hypothesis ensures that $B \restriction U \times U \in M_{U \times U}(\vartheta, R)$.

If $j \in T^*$, then $B \restriction U \times T^* = 0_{U \times T^*}$ implies that $B(i,j) = 0$. If $j \notin T^*$, then $(i,j) \in U \times U$ and $(i,j) \notin \vartheta$ imply that $B(i,j) = 0$. We have thus proved that $B \in M_{V \times V}(\theta, R)$.

Case 2 If $|T^{**}| \leq p$, then $j \in T^{**}$ would imply that $(i,j) \in \theta$, a contradiction. It follows that $j \in W$, where $W = V \setminus T^{**}$. Using $AB = I_{V \times V}$ and Lemma 2.2, we can proceed in a similar way as in Case 1 to get $B \in M_{V \times V}(\theta, R)$. ■

Note that in the formulation of the above theorem, $p = |T_{d-1}|$ is the size of (one of) the second largest block(s) T_{d-1} in $V/\theta \cap \theta^{-1}$.

THEOREM 2.4 *If $A \in M_{V \times V}(\theta, R)$, $B \in M_{V \times V}(R)$, $BA = I_{V \times V}$ and $M_{T \times T}(R)$ is Dedekind finite for all $T \in V/\theta \cap \theta^{-1}$, then $AB = I_{V \times V}$.*

Proof We prove that $AB = I_{V \times V}$ by using induction on the number of elements of $V/\theta \cap \theta^{-1}$.

If $|V/\theta \cap \theta^{-1}| = 1$, then $V/\theta \cap \theta^{-1} = \{V\}$, $\theta = V \times V$ and the Dedekind finiteness of $M_{V \times V}(R)$ implies that $AB = I_{V \times V}$.

Assume that our claim holds for all matrices of size $U \times U$ and for all quasi-orders $\vartheta \subseteq U \times U$ with $|U/\vartheta \cap \vartheta^{-1}| = d$. Now let

$$V/\theta \cap \theta^{-1} = \{T_k \mid 1 \leq k \leq d+1\}$$

be a $d+1 \geq 2$ element set, and take a minimal element T^* in the finite poset $(V/\theta \cap \theta^{-1}, \leq_\theta)$.

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The application of Lemma 2.1 gives $A\restriction U\times T^*=0_{U\times T^*}$,

$$(B\restriction T^*\times T^*)(A\restriction T^*\times T^*)=I_{T^*\times T^*}\quad\text{and}\quad(B\restriction U\times T^*)(A\restriction T^*\times T^*)=0_{U\times T^*},$$

where $U=V\setminus T^*$. The Dedekind finiteness of $M_{T^*\times T^*}(R)$ implies that

$$(A\restriction T^*\times T^*)(B\restriction T^*\times T^*)=I_{T^*\times T^*},$$

whence $B\restriction U\times T^*=0_{U\times T^*}$ and

$$\begin{aligned}I_{U\times U}&=I_{V\times V}\restriction U\times U=BA\restriction U\times U\\&=(B\restriction U\times T^*)(A\restriction T^*\times U)+(B\restriction U\times U)(A\restriction U\times U)=(B\restriction U\times U)(A\restriction U\times U)\end{aligned}$$

follow. Consider the quasi-order $\vartheta=\theta\cap(U\times U)$ on the set U. In view of

$$A\restriction U\times U\in M_{U\times U}(\vartheta,R)\quad\text{and}\quad U/\vartheta\cap\vartheta^{-1}=\{T_k\mid 1\le k\le d+1,T_k\ne T^*\},$$

the induction hypothesis ensures that

$$(A\restriction U\times U)(B\restriction U\times U)=I_{U\times U}.$$

Now

$$\begin{aligned}0_{T^*\times U}&=I_{V\times V}\restriction T^*\times U=BA\restriction T^*\times U\\&=(B\restriction T^*\times T^*)(A\restriction T^*\times U)+(B\restriction T^*\times U)(A\restriction U\times U),\end{aligned}$$

whence

$$\begin{aligned}0_{T^*\times U}&=(A\restriction T^*\times T^*)\cdot 0_{T^*\times U}\cdot(B\restriction U\times U)\\&=(A\restriction T^*\times T^*)(B\restriction T^*\times T^*)(A\restriction T^*\times U)(B\restriction U\times U)\\&\quad+(A\restriction T^*\times T^*)(B\restriction T^*\times U)(A\restriction U\times U)(B\restriction U\times U)\\&=(A\restriction T^*\times U)(B\restriction U\times U)+(A\restriction T^*\times T^*)(B\restriction T^*\times U)=AB\restriction T^*\times U\end{aligned}$$

can be derived. Using $A\restriction U\times T^*=B\restriction U\times T^*=0_{U\times T^*}$, we obtain that

$$\begin{aligned}AB\restriction U\times T^*&=(A\restriction U\times T^*)(B\restriction T^*\times T^*)+(A\restriction U\times U)(B\restriction U\times T^*)=0_{U\times T^*},\\AB\restriction T^*\times T^*&=(A\restriction T^*\times T^*)(B\restriction T^*\times T^*)+(A\restriction T^*\times U)(B\restriction U\times T^*)\\&=(A\restriction T^*\times T^*)(B\restriction T^*\times T^*)=I_{T^*\times T^*}\end{aligned}$$

and

$$\begin{aligned}AB\restriction U\times U&=(A\restriction U\times T^*)(B\restriction T^*\times U)+(A\restriction U\times U)(B\restriction U\times U)\\&=(A\restriction U\times U)(B\restriction U\times U)=I_{U\times U}.\end{aligned}$$

Obviously,

$$AB\restriction T^*\times U=0_{T^*\times U},\quad AB\restriction U\times T^*=0_{U\times T^*}$$

and

$$AB\restriction T^*\times T^*=I_{T^*\times T^*},\quad AB\restriction U\times U=I_{U\times U}$$

imply that $AB=I_{V\times V}$. ∎

The following dual of Theorem 2.4 can be proved analogously.

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THEOREM 2.5 *If* $A \in M_{V\times V}(\theta, R)$, $B \in M_{V\times V}(R)$, $AB = I_{V\times V}$ *and* $M_{T\times T}(R)$ *is Dedekind finite for all* $T \in V/\theta \cap \theta^{-1}$, *then* $BA = I_{V\times V}$.

COROLLARY 2.6 *Let* θ *be a quasi-order on* V *such that* $V/\theta \cap \theta^{-1} = \{T_1, T_2, \ldots, T_d\}$ *has* d *elements and*

$$|T_1| \leq |T_2| \leq \cdots \leq |T_{d-1}| \leq |T_d| = m.$$

If $M_m(R)$ *is Dedekind finite,* $A \in M_{V\times V}(\theta, R)$, $B \in M_{V\times V}(R)$ *and* $BA = I_{V\times V}$, *then* $B \in M_{V\times V}(\theta, R)$, *and so* $M_{V\times V}(\theta, R)$ *is closed with respect to taking (left) inverses in* $M_{V\times V}(R)$.

Proof The Dedekind finiteness of $M_{T\times T}(R)$ (for all $T \in V/\theta \cap \theta^{-1}$) is a consequence of $|T| \leq m$. Thus the application of Theorem 2.4 gives that $AB = I_{V\times V}$. Since $p = |T_{d-1}| \leq m$, the ring $M_p(R)$ is also Dedekind finite and we can use Theorem 2.3 to obtain $B \in M_{V\times V}(\theta, R)$. ■

The (right) dual of Corollary 2.6 also holds.

THEOREM 2.7 *Let* $m \geq 1$ *be an integer and* R *an arbitrary ring. The following conditions are equivalent:*

(1) *The ring* $M_m(R)$ *of* $m \times m$ *matrices over* R *is Dedekind-finite.*
(2) *For every integer* $n \geq 1$ *and quasi-order* θ *on* $V = \{1, \ldots, n\}$ *such that the maximum size of the equivalence classes (with respect to* $\theta \cap \theta^{-1}$*) is* m, *the structural matrix ring* $M_n(\theta, R)$ *is closed with respect to taking inverses in* $M_n(R)$.
(3) *The ring of* (m, m)*-block upper triangular matrices over* R *is closed with respect to taking inverses in* $M_{2m}(R)$.

Proof (1) ⇒ (2): Let $A \in M_n(\theta, R)$ be invertible in $M_n(R)$. If all the equivalence classes of $\theta \cap \theta^{-1}$ are of size at most m, then $A^{-1} \in M_n(\theta, R)$ is an immediate consequence of Corollary 2.6.
(2) ⇒ (3): Obvious.
(3) ⇒ (1): If $XY = I_m$ and $YX \neq I_m$ for some $X, Y \in M_m(R)$, then the $2m \times 2m$ matrices

$$A = \begin{bmatrix} Y & I_m - YX \\ 0 & X \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} X & 0 \\ I_m - YX & Y \end{bmatrix}$$

are inverses of each other. However, A is (m, m)-block upper triangular while B is not [10]. ■

COROLLARY 2.8 *If* R *is a left Noetherian ring or a PI-algebra over a central subring* $C \subseteq Z(R)$, *then the structural matrix ring* $M_n(\theta, R)$ *is closed with respect to taking inverses in* $M_n(R)$.

Proof For each integer $m \geq 1$ the matrix ring $M_m(R)$ inherits the left Noetherian or the PI property of the base ring R. Since a left Noetherian ring or a PI-algebra is always Dedekind finite [7,8,10], we can apply Theorem 2.7. ■

THEOREM 2.9 *Let* θ *be any quasi-order on* $V = \{1, \ldots, n\}$ *such that the maximum size of the equivalence classes (with respect to* $\theta \cap \theta^{-1}$*) is* m. *Then* $M_n(\theta, R)$ *is Dedekind-finite if and only if* $M_m(R)$ *is Dedekind-finite.*

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Proof Any matrix $X \in M_m(R)$ can be viewed as a matrix in $M_n(\theta, R)$ by adding extra 1's on the diagonal and zeros in the non diagonal entries. Thus the 'only if' part is straightforward.

Suppose $M_m(R)$ is Dedekind-finite and let $AB = I_n$ in $M_n(\theta, R)$. Since $|T| \leq m$, the ring $M_{T \times T}(R)$ is Dedekind finite for all $T \in V/\theta \cap \theta^{-1}$. Now Theorem 2.4 (or 2.5) gives that $BA = I_n$. ∎

The following corollary is treated in [6] for the 2×2 upper triangular case.

COROLLARY 2.10 *If R is a Dedekind finite ring, then the ring $UT_n(R)$ of $n \times n$ upper triangular matrices is Dedekind-finite.*

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