Quaestiones Mathematicae

ISSN: 1607-3606 (Print) 1727-933X (Online) Journal homepage: http://www.tandfonline.com/loi/tqma20

GOOD IDEALS IN MATRIX RINGS OVER COMMUTATIVE PIR's

N. J. Groenewald & L. van Wyk

Quaestiones Mathematicae
22(1999), 391-403

GOOD IDEALS IN MATRIX RINGS OVER COMMUTATIVE PIR's

N.J. GROENEWALD AND L. VAN WYK

ABSTRACT. An ideal A of a ring R is called a good ideal if the coset product r_1r_2+A of any two cosets r_1+A and r_2+A of A in the factor ring R/A equals their set product $(r_1+A)\circ(r_2+A) := \{(r_1+a_1)(r_2+a_2)\colon\ a_1,a_2\in A\}$. Good ideals were introduced in [3] to give a characterization of regular right duo rings. We characterize the good ideals of blocked triangular matrix rings over commutative principal ideal rings and show that the condition $A\circ A=A$ is sufficient for A to be a good ideal in this class of matrix rings, none of which are right duo. It is not known whether good ideals in a base ring carries over to good ideals in complete matrix rings over the base ring. Our characterization shows that this phenomenon occurs indeed for complete matrix rings of certain sizes if the base ring is a blocked triangular matrix ring over a commutative principal ideal ring.

1991 *Mathematics Subject Classification:* Primary 16S50, 16D25.

1. Introduction. Every ring herein is associative with identity, and subrings inherit the identity. Ideal means two-sided ideal, and all direct sums are finite. We assume throughout the sequel that PIR means commutative principal ideal ring.

Good ideals were introduced in [3] to give an internal characterization of regular rings in the class of right duo rings. The definition of a good ideal (see Abstract above) implies that the condition

$$A\circ A(=\{a_1a_2\colon\ a_1,a_2\in A\})=A$$

is necessary for an ideal A to be a good ideal. It is not known whether this condition is in general sufficient too. In [9] the good ideals in the class of blocked triangular matrix rings over simple rings were described, and in [9, Corollary 2.3] it was shown that in the rings in this class, none of which are right duo, the mentioned condition is indeed sufficient.

In this paper we generalize the characterization of the good ideals in blocked triangular matrix rings over simple rings in [9] to blocked triangular matrix rings over the much larger class of PIR's. (Complete matrix rings are special cases of blocked triangular matrix rings.) In [9, Corollary 2.4] it was shown that the good

Key words and phrases: Matrix ring, blocked triangular matrix ring, good ideal, commutative principal ideal ring.

ideals of a blocked triangular matrix ring $\mathbb{M}(B, R)$ over a simple ring R are precisely the ideals $\mathcal{A}$ of $\mathbb{M}(B, R)$ having the property that whenever arbitrary elements of R are allowed in a non-diagonal block of the matrices in $\mathcal{A}$, say the (i, j)-th block, then arbitrary elements of R are also allowed in at least one of the (i, j)-th block's two parental blocks, viz. the (i, i)-th and the (j, j)-th blocks.

Our generalization in Section 2, in particular Theorem 2.5, shows that a necessary condition for an ideal $\mathcal{A}$ of a blocked triangular matrix ring $\mathbb{M}(B, R)$ over a PIR R to be good is that the sum of the ideals of R allowed in the two parental blocks of a non-diagonal block of the matrices in $\mathcal{A}$ equals the ideal of R allowed in that non-diagonal block of the matrices in $\mathcal{A}$. A consequence of our arguments (see Corollary 2.6) is that in this larger class of rings the condition $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$ is still sufficient for an ideal $\mathcal{A}$ to be a good ideal.

It is not known whether good ideals in a base ring carry over to good ideals in complete matrix rings over the base ring. In Section 3 we show that this happens indeed if $n|m$ in passing from an $n \times n$ blocked triangular matrix ring $\mathbb{M}(B, R)$ over a PIR R to the $m \times m$ complete matrix ring $\mathbb{M}_m\left(\mathbb{M}(B, R)\right)$ over $\mathbb{M}(B, R)$. We thus mean by this that the $n \times n$ blocked triangular matrix ring $\mathbb{M}(B, R)$ acts as the base ring.

Examples are provided in both Section 2 and Section 3.

For the ease of the reader we provide the pertinent definitions and summarize the description of the ideals of a blocked triangular matrix ring in [8].

We call a subring $\mathcal{R}$ of a complete matrix ring $\mathbb{M}_n(R)$ (also denoted by $\mathbb{M}_{n\times n}(R)$) a blocked triangular matrix ring over R if it is of the form

$$\begin{bmatrix} \mathbb{M}_{n_1\times n_1}(R_{1,1}) & \mathbb{M}_{n_1\times n_2}(R_{1,2}) & \cdots & \mathbb{M}_{n_1\times n_t}(R_{1,t}) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbb{M}_{n_{t-1}\times n_t}(R_{t-1,t}) \\ 0 & \cdots & 0 & \mathbb{M}_{n_t\times n_t}(R_{t,t}) \end{bmatrix}, \tag{1}$$

where for every $i \leq j$, $R_{i,j} = \{0\}$ or R, and $n_1 + \cdots + n_t = n$. (If $t = 1$, i.e. if $n_1 = n$, then $\mathcal{R}$ is an $n \times n$ complete matrix ring.) Since $\mathcal{R}$ is closed under multiplication, it follows that if $R_{i,k} = R = R_{k,j}$ for some k such that $i \leq k \leq j$, then $R_{i,j} = R$. Also, since $\mathcal{R}$ inherits the identity of $\mathbb{M}_n(R)$, we have that $R_{i,i} = R$ for every i. For $X \in \mathcal{R}$ we call the submatrix of X corresponding to the position of $\mathbb{M}_{n_i\times n_j}(R_{i,j})$ in (1) the (i, j)-th block of X and denote it by $X_{i,j}$. If $i = j$ (respectively $i \neq j$), then we call $X_{i,j}$ a diagonal (respectively non-diagonal) block of X, and for a non-diagonal block $X_{i,j}$ we call $X_{i,i}$ and $X_{j,j}$ the parental blocks of $X_{i,j}$.

Let B denote the $n \times n$ Boolean matrix in $\mathcal{R}$ with 1 in every position of every block $B_{i,j}$ for which $R_{i,j} = R$. Then we call B a blocked triangular Boolean matrix.

(See, for example, [5] for basic notions on Boolean matrices.) If $R_{i,j} = R$ in (1) for all i and j such that $i \leq j$, then $\mathcal{R}$ and (the corresponding Boolean matrix) B are called a complete blocked triangular matrix ring (see, for example, [6]) and a complete blocked triangular Boolean matrix respectively.

Henceforth we write $\mathbb{M}(B, R)$ instead of $\mathcal{R}$ in order to stress the specific shape or form of the matrices (i.e. positions where non-zero elements of the base ring R are allowed) in the particular blocked triangular matrix ring as a subring of $\mathbb{M}_n(R)$.

Let $i \leq j$. If $R_{i,j} = R$, then we set

$$\Lambda_{i,j} := \{l\colon\ i \leq l \leq j \text{ and } R_{i,l} = R = R_{l,j}\}.$$

(Note that $\Lambda_{i,i} = \{i\}$ for every i.) Consider any set-inclusion preserving function f from the set of $\Lambda_{i,j}$'s to the set of ideals of R, and let $\mathcal{A}_f$ be the set comprising all the matrices in $\mathbb{M}(B, R)$ with elements from $f(\Lambda_{i,j})$ in their (i,j)-th blocks. Then by [8, Lemma 1.1 and Proposition 1.2] the $\mathcal{A}_f$'s are precisely the ideals of $\mathbb{M}(B, R)$.

2. Good ideals in blocked triangular matrix rings over commutative PIR's. We rely heavily on the well-known result (see [4], [10]) that a PIR is a direct sum of PID's and special PIR's. A *special* PIR is a PIR S which has a unique prime ideal (θ) and this ideal is nilpotent. (Therefore a special PIR is a local ring.) If k is the nilpotency index of (θ) (which is the nilpotency index of θ), then every non-zero element in S can be written in the form $x\theta^l$, where x is invertible in S, $0 \leq l < k$, l is unique and x is unique modulo θ^{k-l}. Furthermore, every ideal of S is of the form (θ^j), $0 \leq j \leq k$.

We mention that special PIR's are chain rings. These include the rings $\mathbb{Z}_{p^n}$ of integers modulo p^n (p a prime), the Galois fields $GF(p^n)$ and the Galois rings $GR(p^n, r) = \mathbb{Z}_{p^n}[x]/(g(x))$ of characteristic p^n and rank r, where $g(x)$ is monic of degree r and irreducible modulo the prime p. (See [1] and [7] for further examples of finite chain rings.)

If $0 < l < k$, with k as above, then the uniqueness of l in any given ideal (θ^l) ensures that $\theta^l \notin (\theta^l) \circ (\theta^l)$, and so $(\theta^l) \circ (\theta^l) \subsetneq (\theta^l)$, implying that a special PIR has no non-trivial good ideals. The same is true for every PID S. In fact, if (a) is a good ideal of S (for some $a \in S$), then $a \in (a) \circ (a) = a^2 S$, forcing a to be 0 or invertible. Similar arguments, using $\theta^l E_{1,1} \in \mathbb{M}_n((\theta^l))$ and $aE_{1,1} \in \mathbb{M}_n((a))$ respectively, where $E_{i,j}$ denotes the (i,j)-th matrix unit, i.e. the matrix with 1 in position (i,j) and zeros elsewhere, show that

LEMMA 2.1. *The two trivial ideals $\{0\}$ and $\mathbb{M}_n(S)$ are the sole good ideals of the complete matrix ring $\mathbb{M}_n(S)$ over a PID S or a special PIR S for every $n \geq 1$.*

Henceforth R will be a PIR, and we write R as the direct sum

$$R = R_1 \oplus R_2 \oplus \cdots \oplus R_N$$

of PID's and special PIR's, for some N. The ideals of R, being the direct sum of rings with identity, are of the form $A_1 \oplus A_2 \oplus \cdots \oplus A_N$, with A_q an ideal of R_q, $q = 1, \ldots, N$. Invoking the definition of a good ideal, one verifies directly that $A_1 \oplus A_2 \oplus \cdots \oplus A_N$ is a good ideal of R if and only if A_q is a good ideal of R_q for every q. Since we may assume, without loss of generality, that

$$\mathbb{M}_n(R) = \mathbb{M}_n(R_1) \oplus \mathbb{M}_n(R_2) \oplus \cdots \oplus \mathbb{M}_n(R_N),$$

it follows that if we combine the foregoing results with Lemma 2.1, then we obtain the following characterization of the good ideals of a PIR R and of the good ideals of a complete matrix ring $\mathbb{M}_n(R)$ over R:

PROPOSITION 2.2. *Let $R = R_1 \oplus R_2 \oplus \cdots \oplus R_N$ be a PIR, with every R_q a PID or a special PIR, and let $n \geq 1$. Then $\mathbb{M}_n(A_1 \oplus A_2 \oplus \cdots \oplus A_N)$, with every A_q an ideal of R_q, is a good ideal of $\mathbb{M}_n(R)$ if and only if $A_q = \{0\}$ or R_q, $q = 1, \ldots, N$.*

COROLLARY 2.3. *There is a 1-1 correspondence between the good ideals of a PIR R and the good ideals of $\mathbb{M}_n(R)$ for every $n \geq 1$ via $A \leftrightarrow \mathbb{M}_n(A)$.*

We next concentrate on a blocked triangular matrix ring $\mathbb{M}(B, S)$ over a PID S or a special PIR S. Consider the setting in (1), with S the base ring. For $l = 1, \ldots, t$, $\mathbb{M}_{n_l}(S)$ is a homomorphic image of $\mathbb{M}(B, S)$, and so since the homomorphic image of a good ideal is certainly a good ideal, it follows that if $\mathcal{A}_f$ is a good ideal of $\mathbb{M}(B, S)$, with f a set-inclusion preserving function from the set of $\Lambda_{i,j}$'s to the set of ideals of S, then $\mathbb{M}_{n_l}(f(\Lambda_{l,l}))$ is a good ideal of $\mathbb{M}_{n_l}(S)$. We conclude from Proposition 2.2, with $N = 1$, that $f(\Lambda_{l,l}) = \{0\}$ or S.

The above arguments imply that the only ideals of S allowed in the diagonal blocks of the matrices in a good ideal $\mathcal{A}_f$ of a blocked triangular matrix ring $\mathbb{M}(B, S)$ over a PID S or a special PIR S are the two trivial ideals of S. We use this result in the proof of the next theorem, in which we characterize the good ideals of a blocked triangular matrix ring over a PID or a special PIR.

THEOREM 2.4. *An ideal $\mathcal{A}_f$ of a blocked triangular matrix ring $\mathbb{M}(B, S)$ over a PID S or a special PIR S is good if and only if the following two conditions hold for all i and j such that $1 \leq i \leq j \leq t$ and $S_{i,j} = S$:*

(i) $f(\Lambda_{i,j}) = \{0\}$ *or* S.

(ii) $f(\Lambda_{i,i}) = S$ *or* $f(\Lambda_{j,j}) = S$ *whenever* $f(\Lambda_{i,j}) = S$.

Proof. Let $\mathcal{A}_f$ be a good ideal of $\mathbb{M}(B, S)$, and let $1 \leq i < j \leq t$, with $S_{i,j} = S$. (The paragraph preceding this theorem caters, as far as (i) is concerned, for the case $i = j$.) We first show that if $f(\Lambda_{i,j}) \neq \{0\}$, then $f(\Lambda_{i,j}) = S$. Hence, assume that $f(\Lambda_{i,j}) \neq \{0\}$, and suppose now that $f(\Lambda_{i,i}) = \{0\} = f(\Lambda_{j,j})$. The fact that f is set-inclusion preserving ensures that we may assume that $\Lambda_{i,j}$ is minimal (with respect to set-inclusion) in the set

$$\{\Lambda_{i,j} \colon\ i < j,\ S_{i,j} = S \ \text{ and } \ f(\Lambda_{i,j}) \neq \{0\}\}. \tag{2}$$

Let $X, Y \in \mathcal{A}_f$. Since

$$(XY)_{i,j} = \sum_{\substack{k \\ i \leq k \leq j \\ S_{i,k} = S = S_{k,j}}} X_{i,k} Y_{k,j}, \tag{3}$$

and since $\Lambda_{i,k}, \Lambda_{k,j} \subsetneq \Lambda_{i,j}$ for every k in this sum for which $k \neq i$ and $k \neq j$, it follows from the minimality of $\Lambda_{i,j}$ in (2) that $\Lambda_{i,k} = \{0\} = \Lambda_{k,j}$, and hence $X_{i,k} = 0 = Y_{k,j}$, for such k's. Hence we conclude from (3) that

$$(XY)_{i,j} = X_{i,i}Y_{i,j} + X_{i,j}Y_{j,j}. \tag{4}$$

Furthermore, the supposition that $f(\Lambda_{i,i}) = \{0\} = f(\Lambda_{j,j})$ implies that $X_{i,i} = 0 = Y_{j,j}$, and thus (4) implies that $(XY)_{i,j} = 0$. Therefore the (i,j)-th block of every matrix in $\mathcal{A}_f \circ \mathcal{A}_f$ is 0. However, the assumption that $f(\Lambda_{i,j}) \neq \{0\}$ says that the (i,j)-th block of some matrices in $\mathcal{A}_f \circ \mathcal{A}_f$ is non-zero. This contradicts the fact that $\mathcal{A}_f \circ \mathcal{A}_f = \mathcal{A}_f$. We conclude that if $f(\Lambda_{i,j}) \neq \{0\}$, then $f(\Lambda_{i,i}) = S$ or $f(\Lambda_{j,j}) = S$, which in turn implies that $f(\Lambda_{i,j}) = S$, because $\Lambda_{i,i}, \Lambda_{j,j} \subseteq \Lambda_{i,j}$ and f is set-inclusion preserving. This proves (i). The proof of (ii) is now similar to the proof of [9, Theorem 2.2].

Conversely, suppose (i) and (ii) hold. In this case the proof of [9, Theorem 2.1] suffices. □

Note that (ii) in Theorem 2.4 can be replaced by $f(\Lambda_{i,i}) + f(\Lambda_{j,j}) = f(\Lambda_{i,j})$. We now state the main result of Section 2, in which we characterize the good ideals of a blocked triangular matrix ring over a PIR.

THEOREM 2.5. *Let $R = R_1 \oplus R_2 \oplus \cdots \oplus R_N$ be a PIR, with every R_q a PID or a special PIR. An ideal $\mathcal{A}_f$ of a blocked triangular matrix ring $\mathbb{M}(B,R)$ over R is good if and only if the following two conditions hold for all i and j such that $1 \leq i \leq j \leq t$ and $R_{i,j} = R$:*

(i) *$f(\Lambda_{i,i}) = A_{i,1} \oplus A_{i,2} \oplus \cdots \oplus A_{i,N}$, with $A_{i,q} = \{0\}$ or R_q, $q = 1, 2, \ldots, N$.*

(ii) *$f(\Lambda_{i,i}) + f(\Lambda_{j,j}) = f(\Lambda_{i,j})$.*

Proof. Since we may assume, without loss of generality, that

$$\mathbb{M}(B,R) = \mathbb{M}(B,R_1) \oplus \mathbb{M}(B,R_2) \oplus \cdots \oplus \mathbb{M}(B,R_N),$$

there are ideals $\mathcal{A}_{f_q}$ of $\mathbb{M}(B,R_q)$, corresponding to set-inclusion preserving functions f_q from the set of $\Lambda_{i,j}$'s to the set of ideals of R_q, $q = 1, 2, \ldots, N$, such that $\mathcal{A}_f = \mathcal{A}_{f_1} \oplus \mathcal{A}_{f_2} \oplus \cdots \oplus \mathcal{A}_{f_N}$. Hence, if $R_{i,j} = R$, then

$$f(\Lambda_{i,j}) = f_1(\Lambda_{i,j}) \oplus f_2(\Lambda_{i,j}) \oplus \cdots \oplus f_N(\Lambda_{i,j}), \tag{5}$$

and so

$$f(\Lambda_{i,i}) + f(\Lambda_{j,j}) = f(\Lambda_{i,j}) \quad \Leftrightarrow \quad f_q(\Lambda_{i,i}) + f_q(\Lambda_{j,j}) = f_q(\Lambda_{i,j}),\ q = 1, \ldots, N. \tag{6}$$

Since we are dealing with the direct sum $\mathbb{M}(B,R) = \mathbb{M}(B,R_1) \oplus \mathbb{M}(B,R_2) \oplus \cdots \oplus \mathbb{M}(B,R_N)$, it follows that $\mathcal{A}_f$ is a good ideal of $\mathbb{M}(B,R)$ if and only if $\mathcal{A}_{f_q}$ is a good ideal of $\mathbb{M}(B,R_q)$ for $q = 1, \ldots, N$. Therefore the desired result now follows directly from (5), (6), Theorem 2.4 and the remark following Theorem 2.4. □

COROLLARY 2.6. *The condition $\mathcal{A}_f \circ \mathcal{A}_f = \mathcal{A}_f$ is sufficient for an ideal $\mathcal{A}_f$ of a blocked triangular matrix ring $\mathbb{M}(B, R)$ over a PIR R to be a good ideal.*

Proof. Suppose $\mathcal{A}_f$ is not a good ideal of $\mathbb{M}(B, R)$. Then (i) or (ii) in Theorem 2.5 does not hold. Assume first that (i) does not hold, with $A_{i',q'}$ an ideal of $R_{q'}$ such that $\{0\} \subsetneqq A_{i',q'} \subsetneqq R_{q'}$ for some i', $1 \le i' \le t$, and some q', $1 \le q' \le N$. The arguments preceding Lemma 2.1 show that $\mathbb{M}_{n_{i'}}(A_{i',q'}) \circ \mathbb{M}_{n_{i'}}(A_{i',q'}) \subsetneqq \mathbb{M}_{n_{i'}}(A_{i',q'})$ in $\mathbb{M}_{n_{i'}}(R_{q'})$, and so $\mathbb{M}_{n_{i'}}\left(f(\Lambda_{i',i'})\right) \circ \mathbb{M}_{n_{i'}}\left(f(\Lambda_{i',i'})\right) \subsetneqq \mathbb{M}_{n_{i'}}\left(f(\Lambda_{i',i'})\right)$ in $\mathbb{M}_{n_{i'}}(R)$, which in turn implies that $\mathcal{A}_f \circ \mathcal{A}_f \subsetneqq \mathcal{A}_f$ in $\mathbb{M}(B, R)$.

Next, assume that (i) holds (for $i = 1, \dots, t$), but that (ii) does not hold, with i' and j' such that $f(\Lambda_{i',i'}) + f(\Lambda_{j',j'}) \subsetneqq f(\Lambda_{i',j'})$. Then by (6) there is some q' for which

$$f_{q'}(\Lambda_{i',i'}) + f_{q'}(\Lambda_{j',j'}) \subsetneqq f_{q'}(\Lambda_{i',j'}). \tag{7}$$

Since (i) holds, it follows, considering $\mathbb{M}(B, R_{q'})$, that $f_{q'}(\Lambda_{i,i}) = \{0\}$ or $R_{q'}$ for every i. If (i) in Theorem 2.4 does not hold for $S = R_{q'}$, then $\{0\} \subsetneqq f_{q'}(\Lambda_{i'',j''}) \subsetneqq R_{q'}$ for some i'' and j'' such that $i'' \neq j''$. But then we can impose a minimality condition on $\Lambda_{i'',j''}$ as in the proof of Theorem 2.4 and show that $\mathcal{A}_{f_{q'}} \circ \mathcal{A}_{f_{q'}} \subsetneqq \mathcal{A}_{f_{q'}}$ in $\mathbb{M}(B, R_{q'})$. This implies then that $\mathcal{A}_f \circ \mathcal{A}_f \subsetneqq \mathcal{A}_f$ in $\mathbb{M}(B, R)$.

Hence assume now that (i) in Theorem 2.4 holds for $S = R_{q'}$. The remark following the proof of Theorem 2.4, together with (7), implies then that

$$f_{q'}(\Lambda_{i',i'}) = \{0\} = f_{q'}(\Lambda_{j',j'}) \quad \text{and} \quad f_{q'}(\Lambda_{i',j'}) = R_q. \tag{8}$$

In the proof of Theorem 2.4 we mentioned that, assuming that (i) in Theorem 2.4 holds for $S = R_{q'}$, then the proof of [9, Theorem 2.2] shows that (8) implies that $\mathcal{A}_{f_{q'}} \circ \mathcal{A}_{f_{q'}} \subsetneqq \mathcal{A}_{f_{q'}}$ in $\mathbb{M}(B, R_{q'})$. As before we conclude that $\mathcal{A}_f \circ \mathcal{A}_f \subsetneqq \mathcal{A}_f$ in $\mathbb{M}(B, R)$.

We have thus proved that if $\mathcal{A}_f$ is not a good ideal of $\mathbb{M}(B, R)$, then $\mathcal{A}_f \circ \mathcal{A}_f \subsetneqq \mathcal{A}_f$, which completes the proof. □

EXAMPLE 2.7. Let p and q be prime numbers, with $p \neq q$, and let $n_1, n_2 \geq 1$. Consider the $(n_1 + n_2) \times (n_1 + n_2)$ blocked triangular matrix ring

$$\mathbb{M}(B, \mathbb{Z}_{p^2q}) = \begin{bmatrix} \mathbb{M}_{n_1}(\mathbb{Z}_{p^2q}) & \mathbb{M}_{n_1 \times n_2}(\mathbb{Z}_{p^2q}) \\ 0 & \mathbb{M}_{n_2}(\mathbb{Z}_{p^2q}) \end{bmatrix}, \quad \text{with} \quad B := \begin{bmatrix} 1_{n_1 \times n_1} & 1_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & 1_{n_2 \times n_2} \end{bmatrix},$$

over the PIR $\mathbb{Z}_{p^2q}$. Here $1_{n_i \times n_j}$ (respectively $0_{n_i \times n_j}$) denotes the $n_i \times n_j$ Boolean matrix with 1 (respectively 0) everywhere.

By Theorem 2.5 the good ideals of $\mathbb{M}(B, \mathbb{Z}_{p^2q})$ are

$$\begin{bmatrix} \mathbb{M}_{n_1}((q)) & \mathbb{M}_{n_1 \times n_2}(\mathbb{Z}_{p^2q}) \\ 0 & \mathbb{M}_{n_2}((p^2)) \end{bmatrix}, \begin{bmatrix} \mathbb{M}_{n_1}((p^2)) & \mathbb{M}_{n_1 \times n_2}(\mathbb{Z}_{p^2q}) \\ 0 & \mathbb{M}_{n_2}((q)) \end{bmatrix},$$

$$\begin{bmatrix} \mathbb{M}_{n_1}((0)) & \mathbb{M}_{n_1 \times n_2}(A) \\ 0 & \mathbb{M}_{n_2}(A) \end{bmatrix}, \begin{bmatrix} \mathbb{M}_{n_1}(A) & \mathbb{M}_{n_1 \times n_2}(A) \\ 0 & \mathbb{M}_{n_2}((0)) \end{bmatrix}, \begin{bmatrix} \mathbb{M}_{n_1}(A) & \mathbb{M}_{n_1 \times n_2}(A) \\ 0 & \mathbb{M}_{n_2}(A) \end{bmatrix},$$

$$\begin{bmatrix} \mathbb{M}_{n_1}(A) & \mathbb{M}_{n_1 \times n_2}(\mathbb{Z}_{p^2q}) \\ 0 & \mathbb{M}_{n_2}(\mathbb{Z}_{p^2q}) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbb{M}_{n_1}(\mathbb{Z}_{p^2q}) & \mathbb{M}_{n_1 \times n_2}(\mathbb{Z}_{p^2q}) \\ 0 & \mathbb{M}_{n_2}(A) \end{bmatrix},$$

for $A = (0), (q), (p^2), \mathbb{Z}_{p^2q}$.

EXAMPLE 2.8. As before, let B be an $n \times n$ blocked triangular matrix, but not the $n \times n$ universal matrix, i.e. $B \neq 1_{n \times n}$, and let R be a PIR. Then Theorem 2.5 implies that the blocked triangular matrix ring $\mathbb{M}(B, R)$ has non-trivial good ideals. However, it can happen that other types of matrix rings (over PIR's) having blocked triangular form, but which are not blocked triangular matrix rings as defined in (1), have no non-trivial good ideals. For example, consider the matrix ring

$$\mathcal{R} := \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} : a, b, c, d \in S \right\},$$

with S a PID or a special PIR. Then the methods invoked so far can be easily adapted to show that $\mathcal{R}$ has no non-trivial good ideals.

In [3, Example 3] the ideal

$$\mathcal{A} := \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} \quad \text{of} \quad \begin{bmatrix} F & F \\ 0 & F \end{bmatrix},$$

F a field, was used to show that an ideal (in a ring which is not right duo) can be a good ideal, without being generated as a right ideal by an idempotent. In spite of this, note that $\mathcal{A}$ is generated as a left ideal by the idempotent $E_{2,2}$. In [9, Example 2.5] an example was given of a good ideal $\mathcal{A}$ in a blocked triangular matrix ring which is neither generated as a right ideal by an idempotent nor generated as a left ideal by an idempotent.

We show now that this phenomenon can occur even in complete blocked triangular matrix rings.

EXAMPLE 2.9. Let $p = 2$, $q = 3$, $n_1 = 1$, $n_2 = 1$ in Example 2.7. Then

$$\mathcal{A} := \begin{bmatrix} (4) & \mathbb{Z}_{12} \\ 0 & (3) \end{bmatrix} \quad \text{is a good ideal of} \quad \mathcal{R} := \begin{bmatrix} \mathbb{Z}_{12} & \mathbb{Z}_{12} \\ 0 & \mathbb{Z}_{12} \end{bmatrix}.$$

The only idempotents in $\mathcal{R}$ which have non-zero entries in position (1,1) and (2,2), which are the only idempotents standing a chance of generating $\mathcal{A}$ as a left or a right ideal, are of the form

$$X_a := \begin{bmatrix} 4 & a \\ 0 & 9 \end{bmatrix},$$

$a \in \mathbb{Z}_{12}$. However,

$$\begin{bmatrix} 0 & 9 \\ 0 & 0 \end{bmatrix} \notin X_a\mathcal{R}, \qquad \text{and} \qquad \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \notin \mathcal{R}X_a.$$

3. Complete matrix rings over blocked triangular matrix rings. We do not know in general whether, for an arbitrary base ring T, $\mathbb{M}_n(A)$ is a good ideal of the complete matrix ring $\mathbb{M}_n(T)$ given that A is a good ideal of the base ring T. Proposition 3.1 provides a partial answer if the base ring is right duo.

PROPOSITION 3.1. *Let T be a right duo ring, and let A be a finitely generated ideal of T. If A is a good ideal in T, then $\mathbb{M}_n(A)$ is a good ideal in $\mathbb{M}_n(T)$ for every $n \geq 1$.*

Proof. By [3, Lemma 1], $A = eT$ for some idempotent e, and by [2, Theorem 1.3], e is central. Then $\mathbb{M}_n(T)$ is the direct sum of its ideals $\mathbb{M}_n(eT)$ and $\mathbb{M}_n\big((1-e)T\big)$. Furthermore, $\mathbb{M}_n(eT) \circ \mathbb{M}_n(eT) = \mathbb{M}_n(eT)$, since for every $Y \in \mathbb{M}_n(eT)$ we have that $Y = (eI_n)Y \in \mathbb{M}_n(eT) \circ \mathbb{M}_n(eT)$. (Here I_n denotes the identity in $\mathbb{M}_n(T)$.) The idea in the last part of the proof of (1) $\Rightarrow$ (2) in [3, Lemma 1] now gets us home. □

The main purpose of this section (see Corollary 3.4) is to show that if one takes any member of the class of blocked triangular matrix rings over a PIR as the base ring, then the mentioned phenomenon occurs too for matrices of certain sizes. To be precise, we show that there is a 1-1 correspondence between the good ideals of an $n \times n$ blocked triangular matrix ring $\mathbb{M}(B, R)$ over a PIR R and the complete matrix ring $\mathbb{M}_m\big(\mathbb{M}(B, R)\big)$ over $\mathbb{M}(B, R)$ for every m such that $n|m$ via $\mathcal{A}_f \leftrightarrow \mathbb{M}_m(\mathcal{A}_f)$. Note again that the base ring $\mathbb{M}(B, R)$ in this case is not right duo.

In order to prove Corollary 3.4 we show that if $n|m$, then the complete matrix ring $\mathbb{M}_m\big(\mathbb{M}(B, T)\big)$ over the $n \times n$ blocked triangular matrix ring $\mathbb{M}(B, T)$, T an arbitrary ring, is isomorphic to an $mn \times mn$ blocked triangular matrix ring $\mathbb{M}(\bar{B}, T)$, where $\bar{B}$, although in general larger than B, has the same shape as B in the sense of (1).

Henceforth $B = [b_{k,k'}]$ will be an $n \times n$ blocked triangular Boolean matrix, as in Section 1. Let $n|m$, with $nl = m$ for some $l \geq 1$, and consider the $mn \times mn$ Boolean matrix B^* with $b_{k,k'}$ in positions $(k + pn, k' + p'n)$, $p, p' = 0, 1, \dots, m-1$. Next, let $\bar{B}$ be the $mn \times mn$ blocked triangular Boolean matrix resulting from B by replacing 1 (respectively 0) by $1_{m\times m}$ (respectively $0_{m\times m}$), with $1_{m\times m}$ and $0_{m\times m}$ as in Example 2.7. Note that, as far as (1) is concerned, B and $\bar{B}$ have the same shape, and that they possibly differ only in that the (k, k')-block of B is an

$n_k \times n_{k'}$ matrix, whereas the (k,k')-th block of $\bar{B}$ is an $mn_k \times mn_{k'}$ matrix. (See Example 3.2.)

We will now construct a permutation $\sigma \in S_{mn}$ such that $\sigma(B^*) = \bar{B}$, where $\sigma(B^*)$ is the matrix with the (u,v)-th entry of B^* in position $\big(\sigma(u),\sigma(v)\big)$, $1 \leq u,v \leq mn$.

Let $1 \leq k \leq mn$. There are i and j, with $1 \leq i,j \leq n$, such that

$$m(i-1) < k \leq mi \tag{9}$$

and

$$k \equiv j \mod n. \tag{10}$$

(We should rather write i_k and j_k, but for the sake of notation we stick to i and j.) We claim that

$$1 \leq k + (m-1)(j-i) \leq mn. \tag{11}$$

Indeed, since $k \leq mi$, it follows that

$$k + (m-1)(j-i) \leq mj - j + i = (m-1)j + i \leq (m-1)n + n = mn,$$

and since $k \geq m(i-1)+1$, we conclude that

$$k + (m-1)(j-i) \geq -m + 1 + mj - j + i = (m-1)(j-1) + i \geq 0 + 1 = 1.$$

By (11) we may consider the map $\sigma : \mathbb{N}_{mn} \to \mathbb{N}_{mn}$, with $\mathbb{N}_{mn} := \{1, \ldots, mn\}$, defined by

$$\sigma(k) = k + (m-1)(j-i). \tag{12}$$

We claim next that conditions (9) and (10), which lead to the definition of σ in (12), imply that

$$\sigma\big(k + (m-1)(j-i)\big) = k. \tag{13}$$

In order to verify (13), we show, given (9) and (10), that

$$m(j-1) < k + (m-1)(j-i) \leq mj \tag{14}$$

and

$$k + (m-1)(j-i) \equiv i \mod n, \tag{15}$$

which by (12) will imply that $\sigma\big(k+(m-1)(j-i)\big) = k+(m-1)(j-i)+(m-1)(i-j)$, and thus will establish (13). Since $nl = m$, it follows from (9) that

$$k = m(i-1) + nt + j$$

for some t with $0 \leq t \leq l-1$. Hence

$$k + (m-1)(j-i) = mj + nt + i - m \leq mj + n(l-1) + n - m = mj$$

and

$$k + (m-1)(j-i) = mj - m + nt + i \geq m(j-1) + 0 + 1,$$

which proves (14). Also, $(m-1)(j-i) = (nl-1)(j-i) \equiv -(j-i) \mod n$, and so by (10) we have that $k + (m-1)(j-i) \boxminus j - (j-i) \mod n$, proving (15).

The foregoing arguments show that $\sigma \in S_{mn}$; in fact, they show that σ is a product of 2-cycles.

EXAMPLE 3.2. Let $m, n = 3$, and let

$$B := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\sigma = (2\ 4)(3\ 7)(6\ 8)$,

$$B^* = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 0&0&1&0&0&1&0&0&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 0&0&1&0&0&1&0&0&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 0&0&1&0&0&1&0&0&1 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&1&1&1 \end{bmatrix}.$$

THEOREM 3.3. *If B is an $n \times n$ blocked triangular Boolean matrix and $n|m$, then with σ, B^* and $\bar{B}$ as above:*

(i) $\sigma(B^*) = \bar{B}$;

(ii) $\mathbb{M}_m\left(\mathbb{M}(B,T)\right) \cong \mathbb{M}(\bar{B},T)$ *for every ring T.*

Proof. (i) It suffices to show that

$$\sigma(k) = (k-1)m + 1 \tag{16}$$

and

$$0 \leq \sigma(k+pn) - \sigma(k) \leq m-1 \tag{17}$$

for $k = 1, \ldots, n$ and $p = 0, \ldots, m-1$. Loosely speaking this will mean that for any given position (k,k'), $1 \leq k,k' \leq n$, the entries of B^* in the m^2 positions $(k+pn, k'+p'n)$, $p,p' = 0,1,\ldots,m-1$, i.e. the entries of B^* corresponding to the entry of B in positions (k,k'), $1 \leq k,k' \leq n$, are mapped by σ (in no specific order) to the m^2 positions $\left((k-1)m+q, (k'-1)m+q'\right)$, $q,q' = 1,\ldots,m$. Note that the position $\left((k-1)m+1, (k'-1)m+1\right)$ denotes the top left corner of the (k,k')-th block of $\bar{B}$.

Let $1 \leq k \leq n$. Then $i = 1$ and $j = k$ in (9) and (10). Hence by (12),

$$\sigma(k) = k + (m-1)(k-1) = (k-1)m + 1, \tag{18}$$

which proves (16). Next, if $0 \leq p \leq m-1$, then, with $ln = m$ as before, it follows that

$$(q-1)l \leq p \leq ql - 1 \tag{19}$$

for some q such that

$$1 \leq q \leq n. \tag{20}$$

Therefore $1 + (q-1)ln \leq k + pn \leq n + (ql-1)n = qln = qm$, and so

$$(q-1)m < k + pn \leq qm. \tag{21}$$

Furthermore,

$$k + pn \equiv k \mod n. \tag{22}$$

We conclude from (9), (10), (12), (21) and (22) that

$$\sigma(k+pn) = k + pn + (m-1)(k-q),$$

and so (18) and (19) imply that

$$\begin{aligned}\sigma(k+pn) - \sigma(k) &= k + pn + (m-1)(k-q) - k - (m-1)(k-1) \\ &= pn + (m-1)(1-q) &(23)\\ &\leq (ql-1)n - mq + m - 1 + q \\ &= m - 1 + q - n. &(24)\end{aligned}$$

By (20) we have that $q - n \leq 0$, and so (24) implies that

$$\sigma(k+pn) - \sigma(k) \leq m - 1. \tag{25}$$

Next, we deduce from (19) and (23) that

$$\sigma(k+pn) - \sigma(k) \geq (q-1)ln - mq + m - 1 + q = -1 + q \geq 0. \tag{26}$$

Thus, (25) and (26) establish (17).

We wish to mention that there are certainly other ways of proving (i), for example, by conjugation by an (appropriately chosen) invertible matrix.

(ii) It is clear that $\mathbb{M}_m(\mathbb{M}(B,T)) \cong \mathbb{M}(B^*,T)$ via ψ (say), and that the map $\varphi : \mathbb{M}(B^*,T) \to \mathbb{M}(\bar{B},T)$, which maps $[r_{k,k'}] \in \mathbb{M}(B^*,T)$ to the matrix with $r_{k,k'}$ in position $(\sigma(k),\sigma(k'))$, is a ring isomorphism too. Thus the composition $\varphi\psi$ completes the proof. $\square$

COROLLARY 3.4. *Let R be a PIR, and let B be an $n \times n$ blocked triangular matrix. Then there is a 1-1 correspondence between the good ideals of the blocked triangular matrix ring $\mathbb{M}(B,R)$ and the complete matrix ring $\mathbb{M}_m(\mathbb{M}(B,R))$ over $\mathbb{M}(B,R)$ for every m such that $n|m$ via $\mathcal{A}_f \leftrightarrow \mathbb{M}_m(\mathcal{A}_f)$.*

Proof. The description of the ideals of a blocked triangular matrix ring in Section 1, together with Theorem 2.5 and the fact that B and $\bar{B}$ have the same shape, shows that there is a 1-1 correspondence between the good ideals of $\mathbb{M}(B,R)$ and the good ideals of $\mathbb{M}(\bar{B},R)$. In order to avoid confusion, we use $\mathcal{A}_{f,B}$ and $\mathcal{A}_{f,\bar{B}}$ (instead of $\mathcal{A}_f$) to distinguish between the ideals of $\mathbb{M}(B,R)$ and the ideals of $\mathbb{M}(\bar{B},R)$ corresponding to a set-inclusion preserving function f from the set of $\Lambda_{i,j}$'s to the set of ideals of R. The desired result then follows from the fact that the proof of Theorem 3.3 shows that $\varphi\psi(\mathbb{M}_m(\mathcal{A}_{f,B})) = \mathcal{A}_{f,\bar{B}}$. □

Note that Corollary 3.4 provides a class of rings which are not right duo, yet they have the property that their good ideals carry over to good ideals in complete matrix rings over them.

REFERENCES

1. CLARK, W.E. AND LIANG, J.J., Enumeration of finite commutative chain rings, *J. Algebra* **27** (1973), 445–453.
2. COURTER, R.C., Finite dimensional right duo algebras are duo, *Proc. Amer. Math. Soc.* **84** (1982), 157–161.
3. HUANG, L. AND XUE, W., An internal characterisation of strongly regular rings, *Bull. Austral. Math. Soc.* **46** (1992), 525–528.
4. HUNGERFORD, T.W., On the structure of principal ideal rings, *Pacific J. Math.* **25** (1968), 543–547.
5. KIM, K.H., *Boolean Matrix Theory and Applications*, Marcel Dekker, New York, 1982.
6. LI, M.S. AND ZELMANOWITZ, J.M., Artinian rings with restricted primeness conditions, *J. Algebra* **124** (1989), 139–148.
7. MCDONALD, B.R., *Finite Rings with Identity*, Marcel Dekker, New York, 1974.
8. VAN WYK, L., Special radicals in structural matrix rings, *Comm. Algebra* **16** (1988), 421–435.
9. ———————, Good ideals in blocked triangular matrix rings over division rings, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **40** (1997), 75–83.
10. ZARISKI, O. AND SAMUEL, P., *Commutative Algebra, Volume I*, Springer-Verlag, New York, 1958.

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF PORT ELIZABETH, P.O. BOX 1600, PORT ELIZABETH 6000, SOUTH AFRICA.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF STELLENBOSCH, PRIVATE BAG XI, MATIELAND 7602, SOUTH AFRICA.

EMAIL: `maanjg@upe.ac.za` `lvw@land.sun.ac.za`

Received September, 1997