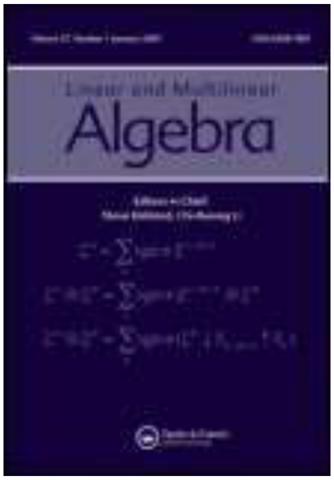

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Determinants for $n \times n$ matrices and the symmetric Newton formula in the 3×3 case

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One of the aims of this paper is to provide a short survey on the $\mathbb{Z}_2$-graded, the symmetric and the left (right) generalizations of the classical determinant theory for square matrices with entries in an arbitrary (possibly non-commutative) ring. This will put us in a position to give a motivation for our main results. We use the preadjoint matrix to exhibit a general trace expression for the symmetric determinant. The symmetric version of the classical Newton trace formula is also presented in the 3×3 case.

Keywords: $\mathbb{Z}_2$-graded; symmetric; left (right) determinants and characteristic polynomials; Cayley–Hamilton identities; the symmetric 3×3 Newton trace formula

AMS Subject Classifications: 15A15; 15A24; 15B33; 16S50

1. Introduction

The universal notion of a determinant has a long history. Determinants of matrices with entries in non-commutative rings have been considered by many mathematicians, among them Cayley, Study, Ore, Dieudonné and others. An excellent recent survey dealing with almost all existing determinants is [1], where the so-called Gelfand–Retakh quasideterminants are used as a main organizing tool.

It seems that the symmetric determinant does not fit into the general framework presented in [1]. One of the aims of this paper is to provide a short introduction to the symmetric and the corresponding left and right versions of the classical determinant theory. The natural symmetrization of the determinant formula and of the adjoint matrix lead to extremely useful concepts. It turns out that these constructions can serve as a starting point of a new symmetric determinant theory for square matrices over an arbitrary ring. The most important feature of this theory is that it can be used to solve systems of left (or right) linear equations and to exhibit left and right Cayley–Hamilton identities for matrices over a Lie nilpotent ring (see [2–5]).

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The algebra of $n \times n$ matrices over an exterior (Grassmann) algebra E is our basic example. The Lie nilpotent property of E plays a central role in the various applications of the symmetric and the corresponding left and right determinants.

The so-called (and not widely known) $\mathbb{Z}_2$-graded determinant (in [6]) is also designed to 'attack' matrices over an exterior algebra. Its construction heavily depends on the natural $\mathbb{Z}_2$-grading $E = E_0 \oplus E_1$. The $\mathbb{Z}_2$-graded determinant (and adjoint) can be used to give an explicit inverse formula and to exhibit left and right Cayley–Hamilton identities for an $n \times n$ matrix over E.

The 'misterious' superdeterminant of a supermatrix due to Kantor and Trishin (in [7]) is another concept which is closely related to the $\mathbb{Z}_2$-grading of the exterior algebra. The treatment in [7] leads to the solution of certain special systems of left (or right) linear equations and to an invariant Cayley–Hamilton identity for supermatrices. Unfortunately, the lack of complete understanding prevents us from dealing with the KT-superdeterminant.

Using the fact that E is a local ring, the Dieudonné determinant is a well-defined $\cup_{n=1}^{\infty}\mathrm{GL}_n(E) \longrightarrow E_{\mathrm{ab}}^{\times}$ map satisfying certain natural rules. Here $E_{\mathrm{ab}}^{\times}$ denotes the Abelianized multiplicative group of units in E (see [8]). The Dieudonné determinant is an important tool in algebraic K-theory, but we cannot use it to solve systems of linear equations (over a local ring) and to derive Cayley–Hamilton identities.

In the rest of this introductory section, we try to explain why we restrict our attention to the matrix algebras $\mathrm{M}_n(E)$ and $\mathrm{M}_{n,t}(E)$.

The Cayley–Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the $n \times n$ matrix algebra $\mathrm{M}_n(K)$ over a field K (see [9–12]). In case of $\mathrm{char}(K) = 0$, Kemer's pioneering work (see [13]) on the T-ideals of associative algebras revealed the importance of the identities satisfied by the full $n \times n$ matrix algebra $\mathrm{M}_n(E)$ and by the algebra of (n, t) supermatrices $\mathrm{M}_{n,t}(E)$, where

$$E = K\left\langle v_1, v_2, \ldots, v_i, \ldots \mid v_i v_j + v_j v_i = 0 \quad \text{for all } 1 \le i \le j\right\rangle$$

is the exterior (Grassmann) algebra generated by the infinite sequence of anticommutative indeterminates $(v_i)_{i\ge 1}$.

Let $K\langle x_1, x_2, \ldots, x_i, \ldots\rangle$ denote the polynomial K-algebra generated by the infinite sequence $x_1, x_2, \ldots, x_i, \ldots$ of non-commuting indeterminates. The prime T-ideals of this (free associative K-)algebra are exactly the T-ideals of the identities satisfied by $\mathrm{M}_n(K)$ for $n \ge 1$. The T-prime T-ideals are the prime T-ideals plus the T-ideals of the identities of $\mathrm{M}_n(E)$ for $n \ge 1$ and of $\mathrm{M}_{n,t}(E)$ for $n - 1 \ge t \ge 1$. Another remarkable result is that for a sufficiently large $n \ge 1$, any T-ideal contains the T-ideal of the identities satisfied by $\mathrm{M}_n(E)$.

Accordingly, the importance of matrices (and supermatrices) over certain non-commutative rings is an evidence in the theory of PI-rings, nevertheless this fact has been obvious for a long time in other branches of algebra (e.g. in the structure theory of semisimple rings). Thus, the algebras $\mathrm{M}_n(E)$ and $\mathrm{M}_{n,t}(E)$ served as the main motivation for the development of the symmetric and the $\mathbb{Z}_2$-graded determinants.

In Section 2, we deal with the study determinant of a quaternionic matrix (see [14,15]) and we show why a similar embedding approach does not work for $\mathrm{M}_n(E)$. The main result in Section 2 is based on an embedding of the two generated exterior algebra $E^{(2)}$ into a 2×2 matrix algebra over a commutative ring and gives a Cayley–Hamilton identity of degree

$2n$ in $\mathrm{M}_n(E^{(2)})$. Section 3 is devoted to a simplified version of the $\mathbb{Z}_2$-graded determinant. In Section 4, we present a short introduction to the symmetric theory of determinants, we collect and explain some known results, and point out some similarities and differences between the (traditional) commutative base ring case and the general case.

We do not intend to give a detailed study of the relationships between the symmetric and any of the already existing determinant notions. Nevertheless, a thorough comparison between the symmetric and the $\mathbb{Z}_2$-graded determinant would be essential. For a 2×2 matrix over E, the second right (left) determinant is the double of the right (left) $\mathbb{Z}_2$-graded determinant. A similar comparison in the 3×3 case would probably require computer calculations.

The treatment in Section 4 puts us in a position to give a motivation for our new results in Sections 5 and 6. First we prove that $\mathrm{sdet}(A) = \mathrm{tr}(AA^*) = \mathrm{tr}(A^*A)$, where $\mathrm{sdet}(A)$ is the symmetric determinant, $\mathrm{tr}(A)$ is the sum of the diagonal entries and A^* is the so called preadjoint matrix of the $n \times n$ matrix $A \in \mathrm{M}_n(R)$. Then we present the following symmetric version of the Newton trace formula for a 3×3 matrix $A \in \mathrm{M}_3(R)$:

$$\mathrm{sdet}(A) = \mathrm{tr}^3(A) - \mathrm{tr}(A) \cdot \mathrm{tr}(A^2) - \mathrm{tr}(A \cdot \mathrm{tr}(A) \cdot A) - \mathrm{tr}(A^2) \cdot \mathrm{tr}(A) + \mathrm{tr}(A^3) + \mathrm{tr}\left((A^\top)^3\right),$$

where $A^\top$ denotes the transpose of A. The symmetric characteristic polynomial of this A and the corresponding general Cayley–Hamilton identity are also presented by traces.

2. The embedding of $\mathbb{H}$ and the study determinant

There are well-known embeddings of the complex number field $\mathbb{C}$ and of the skew field $\mathbb{H}$ of the real quaternions into matrices:

$$a+bi \longmapsto \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad a+bi+cj+dk = a+bi+(c+di)j \longmapsto \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}.$$

The above definitions provide injective $\mathbb{R}$-algebra homomorphisms $\mu : \mathbb{C} \to \mathrm{M}_2(\mathbb{R})$ and $\nu : \mathbb{H} \to \mathrm{M}_2(\mathbb{C})$. There is a natural $\mu_2 : \mathrm{M}_2(\mathbb{C}) \to \mathrm{M}_2(\mathrm{M}_2(\mathbb{R}))$ extension of μ:

$$\mu_2\left(\begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix}\right) = \begin{bmatrix} \mu(z_{1,1}) & \mu(z_{1,2}) \\ \mu(z_{2,1}) & \mu(z_{2,2}) \end{bmatrix}.$$

Since $\mathrm{M}_2(\mathrm{M}_2(\mathbb{R})) \cong \mathrm{M}_4(\mathbb{R})$, the composition $\vartheta = \mu_2 \circ \nu : \mathbb{H} \longrightarrow \mathrm{M}_4(\mathbb{R})$ is the following map:

$$a+bi+cj+dk \longmapsto \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}.$$

Using the natural extensions

$$\nu_n : \mathrm{M}_n(\mathbb{H}) \longrightarrow \mathrm{M}_n(\mathrm{M}_2(\mathbb{C})) \cong \mathrm{M}_{2n}(\mathbb{C}) \text{ and } \vartheta_n : \mathrm{M}_n(\mathbb{H}) \longrightarrow \mathrm{M}_n(\mathrm{M}_4(\mathbb{R})) \cong \mathrm{M}_{4n}(\mathbb{R}),$$

an $n \times n$ matrix over $\mathbb{H}$ can be viewed as a $2n \times 2n$ matrix over $\mathbb{C}$ or as a $4n \times 4n$ matrix over $\mathbb{R}$. Now we can define the study determinant of a quaternionic matrix $A \in \mathrm{M}_n(\mathbb{H})$ as the ordinary determinant $\mathrm{S}\det(A) = \det_{\mathbb{C}} \nu_n(A)$ in $\mathrm{M}_{2n}(\mathbb{C})$. If we take the absolute value of the complex number $\det_{\mathbb{C}} \nu_n(A)$, then $|\det_{\mathbb{C}} \nu_n(A)|^2 = \det_{\mathbb{R}} \vartheta_n(A)$. The study determinant has some nice properties and it is used frequently in differential geometry and Lie theory. The Cayley–Hamilton identity for $\vartheta_n(A)$ yields the same identity (with real coefficients) of degree $4n$ for A itself.

A similar approach to get a useful determinant notion and a Cayley–Hamilton identity in $\mathrm{M}_n(E)$ is impossible. The reason is that the infinitely generated exterior algebra E cannot be embedded into a full matrix algebra over a commutative ring (E does not satisfy any of the standard identities). On the other hand, the embedding approach gives the following:

Theorem 2.1 *Let $A \in \mathrm{M}_n\left(E^{(2)}\right)$ be an $n \times n$ matrix over the two generated exterior algebra*

$$E^{(2)} = K\left\langle v_1, v_2 \mid v_1 v_2 = -v_2 v_1,\, v_1^2 = v_2^2 = 0\right\rangle.$$

Then A satisfies a Cayley–Hamilton identity of the form

$$A^{2n} + c_{2n-1}A^{2n-1} + \cdots + c_1 A + c_0 I = 0$$

where $c_i \in K$, $0 \le i \le 2n-1$.

Proof The assignments

$$1 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad v_1 \longmapsto \begin{bmatrix} x & x \\ 0 & -x \end{bmatrix}, \quad v_2 \longmapsto \begin{bmatrix} y & 0 \\ -2y & -y \end{bmatrix}$$

define a K-embedding $\varepsilon : E^{(2)} \longrightarrow \mathrm{M}_2(K[x, y]/(x^2, y^2))$, where $(x^2, y^2) \lhd K[x, y]$ is the ideal generated by the monomials x^2 and y^2. Now consider the induced K-embedding

$$\varepsilon_n : \mathrm{M}_n(E^{(2)}) \longrightarrow \mathrm{M}_n\left(\mathrm{M}_2\left(K[x, y]/(x^2, y^2)\right)\right) \cong \mathrm{M}_{2n}\left(K[x, y]/(x^2, y^2)\right).$$

The trace of any 2×2 block

$$\begin{aligned} &\varepsilon(b_0 + b_1 v_1 + b_2 v_2 + b_3 v_1 v_2) \\ &\quad = \begin{bmatrix} b_0 + b_1 x + b_2 y - b_3 xy + (x^2, y^2) & b_1 x - b_3 xy + (x^2, y^2) \\ -2b_2 y + 2b_3 xy + (x^2, y^2) & b_0 - b_1 x - b_2 y + b_3 xy + (x^2, y^2) \end{bmatrix} \end{aligned}$$

in $\varepsilon_n(A)$ is of the form $2b_0 + (x^2, y^2)$. Since the trace of $\varepsilon_n(A)$ is the sum of the traces of the diagonal 2×2 blocks, we have $\mathrm{tr}(\varepsilon_n(A)) = 2b + (x^2, y^2)$ for some $b \in K$. The coefficients of the characteristic polynomial of $\varepsilon_n(A)$ are rational polynomial expressions (with zero constant terms) of the traces $\mathrm{tr}((\varepsilon_n(A))^k) = \mathrm{tr}(\varepsilon_n(A^k))$, $k \ge 1$ (Newton formulae). Thus, the Cayley–Hamilton identity for $\varepsilon_n(A)$ is of the form

$$\begin{aligned} &(\varepsilon_n(A))^{2n} + \left(c_{2n-1} + \left(x^2, y^2\right)\right)(\varepsilon_n(A))^{2n-1} + \cdots + \left(c_1 + \left(x^2, y^2\right)\right)\varepsilon_n(A) \\ &\quad + \left(c_0 + \left(x^2, y^2\right)\right) I = 0 \end{aligned}$$

with $c_i \in K$, $0 \le i \le 2n-1$. It follows that

$$\begin{aligned} &\varepsilon_n\left(A^{2n} + c_{2n-1}A^{2n-1} + \cdots + c_1 A + c_0 I\right) \\ &\quad = \varepsilon_n\left(A^{2n}\right) + c_{2n-1}\varepsilon_n\left(A^{2n-1}\right) + \cdots + c_1\varepsilon_n(A) + c_0 I = 0 \end{aligned}$$

holds in $M_{2n}(K[x,y]/(x^2,y^2))$, and thus the injectivity of ε_n gives the desired identity. $\square$

3. The $\mathbb{Z}_2$-graded determinant

A $\mathbb{Z}_2$-grading of an (associative) ring R is a pair (R_0, R_1), where R_0 and R_1 are additive subgroups of R such that $R = R_0 \oplus R_1$ and $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \{0, 1\}$ and $i + j$ is taken modulo 2. The relation $R_0 R_0 \subseteq R_0$ ensures that R_0 is a subring of R. It is easy to see that the existence of $1 \in R$ implies that $1 \in R_0$.

A $\mathbb{Z}_2$-grading (R_0, R_1) of the ring R is called central if $R_0 \subseteq Z(R)$ (here $\mathrm{Z}(R)$ denotes the centre of R). The condition $R_0 \subseteq Z(R)$ implies the Lie nilpotence (of index 2) of R. The general notion of the $\mathbb{Z}_2$-graded determinant (in [6]) is defined for an $n \times n$ matrix over an arbitrary ring R with a central $\mathbb{Z}_2$-grading (R_0, R_1).

In order to present a more natural and understandable treatment of the $\mathbb{Z}_2$-graded determinant, we restrict ourselves to the case of the well-known central $\mathbb{Z}_2$-grading $E = E_0 \oplus E_1$ of the (infinitely generated) exterior algebra. If we add one more (anticommutative) generator w to the infinite sequence $(v_i)_{i\geq 1}$, we obtain an extended

$$E_w = K\left\langle w, v_1, v_2, \ldots, v_i, \ldots \mid w^2 = wv_j + v_j w = v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i \leq j\right\rangle$$

exterior algebra. An $n \times n$ matrix $A \in M_n(E)$ can be uniquely written as $A = A_0 + A_1$ with $A_0 \in M_n(E_0)$ and $A_1 \in M_n(E_1)$. The companion matrix of A is defined as $A_0 + A_1 w \in M_n(E_w(0))$, where the notations $E_w(0) = (E_w)_0$ and $E_w(1) = (E_w)_1$ are used for the even and the odd part of the (central) $\mathbb{Z}_2$-grading $E_w = (E_w)_0 \oplus (E_w)_1$ of E_w.

Since $E_w(0)$ is commutative, the ordinary determinant and the ordinary adjoint of $A_0 + A_1 w$ are defined and can be written as

$$\det(A_0 + A_1 w) = d_0 + d_1 w \in E_w(0) \text{ and } \operatorname{adj}(A_0 + A_1 w) = B_0 + B_1 w \in M_n(E_w(0)),$$

where $d_0 \in E_0, d_1 \in E_1, B_0 \in M_n(E_0), B_1 \in M_n(E_1)$ and each of these objects is uniquely determined by A. Clearly, $d_0 = \det(A_0)$, $B_0 = \operatorname{adj}(A_0)$ and the elements $d_1, b^{(1)}_{i,j} \in E_1$ are also polynomial expressions of the entries $a^{(0)}_{i,j}$ and $a^{(1)}_{i,j}$ $\Big($note that $A_0 = \left[a^{(0)}_{i,j}\right]$, $A_1 = \left[a^{(1)}_{i,j}\right]$ and $B_1 = \left[b^{(1)}_{i,j}\right]\Big)$.

THEOREM 3.1 *The elements of the product matrices*

$$A(B_0 + B_1) = (A_0 + A_1)(B_0 + B_1) \text{ and } (B_0 + B_1)A = (B_0 + B_1)(A_0 + A_1)$$

are contained in the subring $E_0[d_1]$ of E generated by d_1 and the elements of E_0, namely:

$$A(B_0 + B_1), (B_0 + B_1)A \in M_n(E_0[d_1]).$$

The containment $E_0 \subseteq \mathrm{Z}(E)$ implies that the subring $E_0[d_1] \subseteq E$ is commutative (the elements of $E_0[d_1]$ are polynomials of d_1 with coefficients in E_0). As a consequence of Theorem 3.1 the determinant and the adjoint of the matrices $A(B_0 + B_1), (B_0 + B_1)A \in M_n(E_0[d_1])$ are defined. We call

$$\operatorname{rgdet}(A) = \det(A(B_0 + B_1)) \text{ the right } \mathbb{Z}_2\text{-graded determinant}$$

and

$$\mathrm{rgadj}(A) = (B_0 + B_1)\mathrm{adj}(A(B_0 + B_1)) \text{ the right } \mathbb{Z}_2\text{-graded adjoint}$$

(with respect to $E = E_0 \oplus E_1$) of the matrix $A \in \mathrm{M}_n(E)$. Since

$$A(B_0 + B_1)\mathrm{adj}(A(B_0 + B_1)) = \det(A(B_0 + B_1))I$$

in $\mathrm{M}_n(E_0[d_1])$, we immediately obtain (in $\mathrm{M}_n(E)$) that:

$$A\mathrm{rgadj}(A) = \mathrm{rgdet}(A)I.$$

Proposition 3.2

(i) *If* $T \in \mathrm{GL}_n(E_0)$ *is an invertible matrix and* $A \in \mathrm{M}_n(E)$, *then* $\mathrm{rgdet}(TAT^{-1}) = \mathrm{rgdet}(A)$ *and* $\mathrm{rgadj}(TAT^{-1}) = T(\mathrm{rgadj}(A))T^{-1}$.
(ii) *If* $A \in \mathrm{M}_n(E_0)$, *then* $\mathrm{rgdet}(A) = (\det(A))^n$ *and* $\mathrm{rgadj}(A) = (\det(A))^{n-1}\mathrm{adj}(A)$.

The polynomial ring $E[t]$ inherits a natural (and central) $\mathbb{Z}_2$-grading $E[t] = E_0[t] \oplus E_1[t]$ from $E = E_0 \oplus E_1$. We define the right $\mathbb{Z}_2$-graded characteristic polynomial of a matrix $A \in \mathrm{M}_n(E)$ as the right $\mathbb{Z}_2$-graded determinant (with respect to $E[t] = E_0[t] \oplus E_1[t]$) of the matrix $tI - A \in \mathrm{M}_n(E[t])$, where I is the identity matrix in $\mathrm{M}_n(E)$:

$$\chi_A(t) = \mathrm{rgdet}(tI - A) = \lambda_0 + \lambda_1 t + \cdots + \lambda_k t^k \in E[t],\ \lambda_0, \lambda_1, \ldots, \lambda_k \in E \text{ and } \lambda_k \neq 0.$$

Since $\mathrm{GL}_n(E_0) \subseteq \mathrm{GL}_n(E_0[t])$, an immediate consequence of Proposition 3.2 is that $\chi_{TAT^{-1}}(t) = \chi_A(t)$ for any invertible matrix $T \in \mathrm{GL}_n(E_0)$.

Proposition 3.3 *If* $\chi_A(t) = \lambda_0 + \lambda_1 t + \cdots + \lambda_k t^k$ *is the right* $\mathbb{Z}_2$*-graded characteristic polynomial of the* $n \times n$ *matrix* $A \in \mathrm{M}_n(E)$, *then* $k = n^2$ *and* $\lambda_{n^2} = 1$, $\lambda_0 = \mathrm{rgdet}(-A)$.

Theorem 3.4 *If* $\chi_A(t) \in E[t]$ *is the right* $\mathbb{Z}_2$*-graded characteristic polynomial of an* $n \times n$ *matrix* $A \in \mathrm{M}_n(E)$ *and* $h(t) \in E[t]$ *is arbitrary, then the left substitution of* A *into the product polynomial* $\chi_A(t)h(t) = \mu_0 + \mu_1 t + \cdots + \mu_m t^m$ *is zero:* $I\mu_0 + A\mu_1 + \cdots + A^m\mu_m = 0$.

4. The symmetric and the right (left) determinants

Let S_n denote the symmetric group of all permutations of the set $\{1, 2, \ldots, n\}$. For an $n \times n$ matrix $A = [a_{i,j}]$ over an arbitrary (possibly non-commutative) ring or algebra R with 1, the element

$$\begin{aligned}\mathrm{sdet}(A) &= \sum_{\tau,\rho \in \mathrm{S}_n} \mathrm{sgn}(\rho) a_{\tau(1),\rho(\tau(1))} \ldots a_{\tau(t),\rho(\tau(t))} \ldots a_{\tau(n),\rho(\tau(n))} \\ &= \sum_{\alpha,\beta \in \mathrm{S}_n} \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) a_{\alpha(1),\beta(1)} \ldots a_{\alpha(t),\beta(t)} \ldots a_{\alpha(n),\beta(n)}\end{aligned}$$

of R can be obviously considered as the symmetric determinant of A.

The preadjoint matrix $A^* = [a^*_{r,s}]$ of an $n \times n$ matrix $A = [a_{i,j}]$ (over an arbitrary ring or algebra R with 1) is defined as the following natural symmetrization of the classical adjoint:

$$\begin{aligned} a^*_{r,s} &= \sum_{\tau,\rho} \mathrm{sgn}(\rho) a_{\tau(1),\rho(\tau(1))} \ldots a_{\tau(s-1),\rho(\tau(s-1))} a_{\tau(s+1),\rho(\tau(s+1))} \ldots a_{\tau(n),\rho(\tau(n))} \\ &= \sum_{\alpha,\beta} \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) a_{\alpha(1),\beta(1)} \ldots a_{\alpha(s-1),\beta(s-1)} a_{\alpha(s+1),\beta(s+1)} \ldots a_{\alpha(n),\beta(n)}, \end{aligned}$$

where the first sum is taken over all $\tau, \rho \in \mathrm{S}_n$ with $\tau(s) = s$ and $\rho(s) = r$ (the second sum is taken over all $\alpha, \beta \in \mathrm{S}_n$ with $\alpha(s) = s$ and $\beta(s) = r$). We note that the (r, s) entry of A^* is exactly the signed symmetric determinant $(-1)^{r+s}\mathrm{sdet}(A_{s,r})$ of the $(n-1) \times (n-1)$ minor $A_{s,r}$ of A arising from the deletion of the s-th row and the r-th column of A. If R is commutative, then $A^* = (n-1)!\mathrm{adj}(A)$, where $\mathrm{adj}(A)$ denotes the ordinary adjoint of A.

The right adjoint sequence $(P_k)_{k\geq 1}$ of A is defined by the recursion: $P_1 = A^*$ and $P_{k+1} = (AP_1 \cdots P_k)^*$ for $k \geq 1$. Originally, the kth right determinant was defined as the top left entry of the product matrix $AP_1 \cdots P_k$. These definitions were introduced in [2].

The above-mentioned kth right determinant is not invariant with respect to the conjugate action of $\mathrm{GL}_n(\mathrm{Z}(R))$ on $\mathrm{M}_n(R)$. A more appropriate (and invariant) definition for the kth right determinant is the trace of $AP_1 \cdots P_k$ (see [16] and [4]):

$$\mathrm{rdet}_{(k)}(A) = \mathrm{tr}(AP_1 \ldots P_k).$$

The left adjoint sequence $(Q_k)_{k\geq 1}$ can be defined analogously: $Q_1 = A^*$ and $Q_{k+1} = (Q_k \ldots Q_1 A)^*$ for $k \geq 1$. The k-th left determinant of A is

$$\mathrm{ldet}_{(k)}(A) = \mathrm{tr}(Q_k \ldots Q_1 A).$$

Note that $\mathrm{rdet}_{(k+1)}(A) = \mathrm{rdet}_{(k)}(AA^*)$ and $\mathrm{ldet}_{(k+1)}(A) = \mathrm{ldet}_{(k)}(A^*A)$. The basic properties of these determinants are given in the following theorems.

THEOREM 4.1 (see [4,16]) *If $T \in \mathrm{GL}_n(\mathrm{Z}(R))$ is an invertible matrix with entries in the centre $\mathrm{Z}(R)$ of R, then*

$$\begin{aligned} \mathrm{tr}(T^{-1}AT) &= \mathrm{tr}(A), (T^{-1}AT)^* = T^{-1}A^*T, \\ \mathrm{rdet}_{(k)}(T^{-1}AT) &= \mathrm{rdet}_{(k)}(A), \mathrm{ldet}_{(k)}(T^{-1}AT) = \mathrm{ldet}_{(k)}(A). \end{aligned}$$

The next results shed light on the fact that we call $\mathrm{radj}_{(k)}(A) = nP_1 \cdots P_k$ the kth right adjoint and $\mathrm{ladj}_{(k)}(A) = nQ_k \cdots Q_1$ the kth left adjoint of A.

THEOREM 4.2 (see [2,4]) *The product matrices $A\mathrm{radj}_{(1)}(A)$ and $\mathrm{ladj}_{(1)}(A)A$ in $\mathrm{M}_n(R)$ can be written as*

$$A\mathrm{radj}_{(1)}(A) = nAA^* = \mathrm{tr}(AA^*)I + C' = \mathrm{rdet}_{(1)}(A)I + C'$$

and

$$\mathrm{ladj}_{(1)}(A)A = nA^*A = \mathrm{tr}(A^*A)I + C'' = \mathrm{ldet}_{(1)}(A)I + C'',$$

where I is the identity matrix, $\mathrm{tr}(C') = \mathrm{tr}(C'') = 0$ and all entries of the matrices C' and C'' are in the additive commutator subgroup $[R, R]$ of R generated by all elements of the form $[u, v] = uv - vu$, $u, v \in R$.

THEOREM 4.3 (see [2]) *If the ring R satisfies the polynomial identity*

$$[[[\ldots[[x_1, x_2], x_3], \ldots], x_k], x_{k+1}] = 0$$

(R is Lie nilpotent of index k), then the products $A\mathrm{radj}_{(k)}(A)$ and $\mathrm{ladj}_{(k)}(A)A$ are scalar matrices in $\mathrm{M}_n(R)$ such that

$$A\mathrm{radj}_{(k)}(A) = nAP_1 \cdots P_k = \mathrm{rdet}_{(k)}(A)I, \mathrm{ladj}_{(k)}(A)A = nQ_k \ldots Q_1 A = \mathrm{ldet}_{(k)}(A)I.$$

If R is commutative, then $\mathrm{radj}_{(1)}(A) = \mathrm{ladj}_{(1)}(A) = nA^* = n!\mathrm{adj}(A)$ and

$$\mathrm{rdet}_{(k)}(A) = \mathrm{ldet}_{(k)}(A) = n\left\{(n-1)!\right\}^{1+n+n^2+\cdots+n^{k-1}}\left\{\det(A)\right\}^{n^{k-1}}.$$

If R is Lie nilpotent of index 2 and $\frac{1}{n} \in R$, then in Theorem 4.2
$C' \in \mathrm{M}_n([R, R]) \subseteq \mathrm{M}_n(\mathrm{Z}(R))$ implies that

$$AA^* = \frac{1}{n}(\mathrm{tr}(AA^*)I + C') \in \mathrm{M}_n(\mathrm{Z}(R)[\mathrm{tr}(AA^*)]),$$

where the subring $\mathrm{Z}(R)[\mathrm{tr}(AA^*)]$ generated by $\mathrm{Z}(R)$ and $\mathrm{tr}(AA^*)$ is commutative. Thus

$$\mathrm{rdet}_{(2)}(A) = \mathrm{rdet}_{(1)}(AA^*) = n!\det(AA^*).$$

Let $1 \leq t \leq n-1$ be an integer and $R = R_0 \oplus R_1$ be a $\mathbb{Z}_2$-grading of R. Now $A \in \mathrm{M}_n(R)$ is called an (n, t) supermatrix if

$$a_{i,j} \in R_0 \text{ for all } 1 \leq i, j \leq t \text{ and } t+1 \leq i, j \leq n,$$

and

$$a_{i,j} \in R_1 \text{ for all } 1 \leq i \leq t, t+1 \leq j \leq n \text{ and } t+1 \leq i \leq n, 1 \leq j \leq t.$$

Thus, an (n, t) supermatrix can be partitioned into square and rectangular blocks as follows:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ is a $t \times t$ and $A_{2,2}$ is an $(n-t) \times (n-t)$ square matrix over R_0 and $A_{1,2}$ is a $t \times (n-t)$ and $A_{2,1}$ is an $(n-t) \times t$ rectangular matrix over R_1. Clearly, the set of all (n, t) supermatrices $\mathrm{M}_{n,t}(R)$ is a subring (algebra) of $\mathrm{M}_n(R)$.

THEOREM 4.4 (see [3]) *If $R = R_0 \oplus R_1$ is a $\mathbb{Z}_2$-grading of R and $A \in \mathrm{M}_{n,t}(R)$, then $A^* \in \mathrm{M}_{n,t}(R)$ and $\mathrm{rdet}_{(k)}(A), \mathrm{ldet}_{(k)}(A) \in R_0$ for all $1 \leq k$.*

Let $R[z]$ denote the ring of polynomials of the single commuting indeterminate z, with coefficients in R. The kth right (left) characteristic polynomial of A is the kth right (left) determinant of the $n \times n$ matrix $zI - A$ in $\mathrm{M}_n(R[z])$:

$$p_{A,k}(z) = \mathrm{rdet}_{(k)}(zI - A) \quad \text{and} \quad q_{A,k}(z) = \mathrm{ldet}_{(k)}(zI - A).$$

THEOREM 4.5 (see [3]) *If $R = R_0 \oplus R_1$ is a $\mathbb{Z}_2$-grading of R and $A \in \mathrm{M}_{n,t}(R)$, then $p_{A,k}(z), q_{A,k}(z) \in R_0[z]$ for all $1 \leq k$.*

The above characteristic polynomials appear in the following Cayley–Hamilton theorems.

THEOREM 4.6 (see [4]) *The first right characteristic polynomial* $p_{A,1}(z) \in R[z]$ *of a matrix* $A \in \mathrm{M}_n(R)$*is of the form*

$$p_{A,1}(z) = \lambda_0^{(1)} + \lambda_1^{(1)} z + \ldots + \lambda_{n-1}^{(1)} z^{n-1} + \lambda_n^{(1)} z^n$$

with $\lambda_0^{(1)}, \lambda_1^{(1)}, \ldots, \lambda_{n-1}^{(1)}, \lambda_n^{(1)} \in R$ *and* $\lambda_n^{(1)} = n!$*. The product matrix* $n(zI - A)(zI - A)^*$ *can be written as*

$$n(zI - A)(zI - A)^* = p_{A,1}(z)I + C_0 + C_1 z + \cdots + C_n z^n,$$

where the matrices $C_i \in \mathrm{M}_n(R)$ *are uniquely determined by* A*,* $\mathrm{tr}(C_i) = 0$ *and each entry of* C_i *is in* $[R, R]$*, i.e.* $C_i \in \mathrm{M}_n([R, R])$ *for all* $0 \le i \le n$*. The right*

$$\left(\lambda_0^{(1)} I + C_0\right) + A\left(\lambda_1^{(1)} I + C_1\right) + \cdots + A^{n-1}\left(\lambda_{n-1}^{(1)} I + C_{n-1}\right) + A^n(n!I + C_n) = 0$$

and a similar left

$$\left(\mu_0^{(1)} I + D_0\right) + \left(\mu_1^{(1)} I + D_1\right)A + \cdots + \left(\mu_{n-1}^{(1)} I + D_{n-1}\right)A^{n-1} + (n!I + D_n)A^n = 0$$

Cayley–Hamilton identity with right and left matrix coefficients hold for A*.*

THEOREM 4.7 (see [2]) *If the ring* R *satisfies the polynomial identity*

$$[[[\ldots[[x_1, x_2], x_3], \ldots], x_k], x_{k+1}] = 0$$

(R *is Lie nilpotent of index* k*), then the* k*th right characteristic polynomial* $p_{A,k}(z) \in R[z]$ *of a matrix* $A \in \mathrm{M}_n(R)$ *is of the form*

$$p_{A,k}(x) = \lambda_0^{(k)} + \lambda_1^{(k)} z + \cdots + \lambda_{n^k-1}^{(k)} z^{n^k-1} + \lambda_{n^k}^{(k)} z^{n^k},$$

with $\lambda_0^{(k)}, \lambda_1^{(k)}, \ldots, \lambda_{n^k-1}^{(k)}, \lambda_{n^k}^{(k)} \in R$ *and* $\lambda_{n^k}^{(k)} = n\,\{(n-1)!\}^{1+n+n^2+\cdots+n^{k-1}}$*. The right*

$$(A)p_{A,k} = I\lambda_0^{(k)} + A\lambda_1^{(k)} + \cdots + A^{n^k-1}\lambda_{n^k-1}^{(k)} + A^{n^k}\lambda_{n^k}^{(k)} = 0$$

and a similar left

$$q_{A,k}(A) = \mu_0^{(k)} I + \mu_1^{(k)} A + \cdots + \mu_{n^k-1}^{(k)} A^{n^k-1} + \mu_{n^k}^{(k)} A^{n^k} = 0$$

Cayley–Hamilton identity with right and left scalar coefficients hold for A*. We also have* $(A)u = v(A) = 0$*, where* $u(z) = p_{A,k}(z)h(z)$*,* $v(z) = h(z)q_{A,k}(z)$ *and* $h(z) \in R[z]$ *is arbitrary.*

Now consider E as a base ring and observe that E is Lie nilpotent of index 2. Thus, the above Theorems 4.3 and 4.7 apply to $\mathrm{M}_n(E)$. The natural $\mathbb{Z}_2$-grading $E = E_0 \oplus E_1$ allows us to apply Theorems 4.4 and 4.5 to $\mathrm{M}_{n,t}(E)$. The most remarkable consequences of these theorems are the following: $\mathrm{M}_n(E)$ is integral over E_0 of degree $2n^2$ and $\mathrm{M}_{n,t}(E)$ is integral over E_0 of degree n^2 (see [2,3]).

For a 2×2 matrix $A = [a_{i,j}] \in \mathrm{M}_2(E)$ we have $A = A_0 + A_1$ with

$A_0 = \left[a_{i,j}^{(0)}\right] \in \mathrm{M}_2(E_0)$ and $A_1 = \left[a_{i,j}^{(1)}\right] \in \mathrm{M}_2(E_1)$. Thus

$$\operatorname{adj}(A_0 + A_1 w) = \begin{bmatrix} a_{2,2}^{(0)} + a_{2,2}^{(1)} w & -a_{1,2}^{(0)} - a_{1,2}^{(1)} w \\ -a_{2,1}^{(0)} - a_{2,1}^{(1)} w & a_{1,1}^{(0)} + a_{1,1}^{(1)} w \end{bmatrix} = B_0 + B_1 w,$$

whence

$$B_0 + B_1 = \begin{bmatrix} a_{2,2}^{(0)} + a_{2,2}^{(1)} & -a_{1,2}^{(0)} - a_{1,2}^{(1)} \\ -a_{2,1}^{(0)} - a_{2,1}^{(1)} & a_{1,1}^{(0)} + a_{1,1}^{(1)} \end{bmatrix} = \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{bmatrix} = A^*$$

and

$$2\operatorname{rgdet}(A) = 2\det(A(B_0 + B_1)) = 2\det(AA^*) = \operatorname{rdet}_{(1)}(AA^*) = \operatorname{rdet}_{(2)}(A)$$

follow. The comparison of $\operatorname{rgdet}(A)$ and $\operatorname{rdet}_{(2)}(A)$ for a 3×3 matrix $A \in \mathrm{M}_3(E)$ is a challenging problem.

5. The trace form of the symmetric determinant

If the base ring R is commutative, then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $A, B \in \mathrm{M}_n(R)$. In spite of the fact that this well-known trace identity is no longer valid for matrices over a non-commutative ring, the first left and first right determinants of A coincide (it was not recognized in [4]).

Theorem 5.1 *The traces of the product matrices A^*A and AA^* are both equal to the symmetric determinant of A:*

$$\operatorname{rdet}_{(1)}(A) = \operatorname{tr}(AA^*) = \operatorname{sdet}(A) = \operatorname{tr}(A^*A) = \operatorname{ldet}_{(1)}(A).$$

Proof We prove that $\operatorname{tr}(AA^*) = \operatorname{sdet}(A)$. (The proof of $\operatorname{sdet}(A) = \operatorname{tr}(A^*A)$ is similar.) The trace of a matrix is the sum of the diagonal entries, hence

$$\begin{aligned}
&\operatorname{tr}(A^*A) \\
&\quad = \sum_{1 \le r,s \le n} a_{r,s}^* a_{s,r} \\
&\quad = \sum_{(\alpha,\beta,s) \in \Delta_n} \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta) a_{\alpha(1),\beta(1)} \cdots a_{\alpha(s-1),\beta(s-1)} a_{\alpha(s+1),\beta(s+1)} \ldots a_{\alpha(n),\beta(n)} a_{\alpha(s),\beta(s)} \\
&\quad = \sum_{\alpha',\beta' \in \mathrm{S}_n} \operatorname{sgn}(\alpha')\operatorname{sgn}(\beta') a_{\alpha'(1),\beta'(1)} \cdots a_{\alpha'(t),\beta'(t)} \ldots a_{\alpha'(n),\beta'(n)} = \operatorname{sdet}(A),
\end{aligned}$$

where $\Delta_n = \{(\alpha, \beta, s) \mid \alpha, \beta \in \mathrm{S}_n, 1 \le s \le n, \alpha(s) = s\}$ and the map $(\alpha, \beta, s) \longmapsto (\alpha', \beta')$ with

$$\alpha' = \begin{pmatrix} 1 & \ldots & s-1 & s & \ldots & n-1 & n \\ \alpha(1) & \ldots & \alpha(s-1) & \alpha(s+1) & \ldots & \alpha(n) & s \end{pmatrix}$$

and

$$\beta' = \begin{pmatrix} 1 & \ldots & s-1 & s & \ldots & n-1 & n \\ \beta(1) & \ldots & \beta(s-1) & \beta(s+1) & \ldots & \beta(n) & \beta(s) \end{pmatrix}$$

is a $\Delta_n \longrightarrow \mathrm{S}_n \times \mathrm{S}_n$ bijection. Since

$$\operatorname{sgn}(\alpha') = (-1)^{n-s}\operatorname{sgn}(\alpha), \quad \operatorname{sgn}(\beta') = (-1)^{n-s}\operatorname{sgn}(\beta)$$

and

$$\begin{aligned} &a_{\alpha(1),\beta(1)} \cdots a_{\alpha(s-1),\beta(s-1)} a_{\alpha(s+1),\beta(s+1)} \cdots a_{\alpha(n),\beta(n)} a_{\alpha(s),\beta(s)} \\ &\quad = a_{\alpha'(1),\beta'(1)} \cdots a_{\alpha'(t),\beta'(t)} \ldots a_{\alpha'(n),\beta'(n)}, \end{aligned}$$

the proof is complete. □

Corollary 5.2 *The first right and left characteristic polynomials of a matrix $A \in \mathrm{M}_n(R)$ coincide: $p_{A,1}(z) = q_{A,1}(z)$. Thus we have $\lambda_i^{(1)} = \mu_i^{(1)}$ for all $0 \leq i \leq n$ in the corresponding Cayley–Hamilton identities (see Theorem 4.6).*

In view of Theorem 5.1 and Corollary 5.2, for the above determinants and characteristic polynomials, it is reasonable to use the terminology 'symmetric' instead of 'first right' and 'first left'.

The following observation for 2×2 matrices over the Grassmann algebra is due to Domokos (see [16]).

Proposition 5.3 *If $A = [a_{i,j}]$ is in $\mathrm{M}_2(R)$, then*

$$\operatorname{rdet}_{(2)}(A) - \operatorname{ldet}_{(2)}(A) = \mathcal{S}_4(a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}),$$

where $\mathcal{S}_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in \mathrm{S}_4} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$ is the standard polynomial of degree four.

Proof Using

$$A^* = \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{bmatrix}$$

and the products

$$AA^* = \begin{bmatrix} a_{1,1}a_{2,2} - a_{1,2}a_{2,1} & -a_{1,1}a_{1,2} + a_{1,2}a_{1,1} \\ a_{2,1}a_{2,2} - a_{2,2}a_{2,1} & -a_{2,1}a_{1,2} + a_{2,2}a_{1,1} \end{bmatrix},$$

$$A^*A = \begin{bmatrix} a_{2,2}a_{1,1} - a_{1,2}a_{2,1} & a_{2,2}a_{1,2} - a_{1,2}a_{2,2} \\ -a_{2,1}a_{1,1} + a_{1,1}a_{2,1} & -a_{2,1}a_{1,2} + a_{1,1}a_{2,2} \end{bmatrix},$$

a direct computation shows that

$$\begin{aligned} &\operatorname{rdet}_{(2)}(A) - \operatorname{ldet}_{(2)}(A) = \operatorname{rdet}_{(1)}(AA^*) - \operatorname{ldet}_{(1)}(A^*A) = \operatorname{sdet}(AA^*) - \operatorname{sdet}(A^*A) \\ &\quad = (a_{1,1}a_{2,2} - a_{1,2}a_{2,1})(-a_{2,1}a_{1,2} + a_{2,2}a_{1,1}) + (-a_{2,1}a_{1,2} + a_{2,2}a_{1,1})(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) \\ &\qquad - (-a_{1,1}a_{1,2} + a_{1,2}a_{1,1})(a_{2,1}a_{2,2} - a_{2,2}a_{2,1}) - (a_{2,1}a_{2,2} - a_{2,2}a_{2,1})(-a_{1,1}a_{1,2} + a_{1,2}a_{1,1}) \\ &\qquad - (a_{2,2}a_{1,1} - a_{1,2}a_{2,1})(-a_{2,1}a_{1,2} + a_{1,1}a_{2,2}) - (-a_{2,1}a_{1,2} + a_{1,1}a_{2,2})(a_{2,2}a_{1,1} - a_{1,2}a_{2,1}) \\ &\qquad + (a_{2,2}a_{1,2} - a_{1,2}a_{2,2})(-a_{2,1}a_{1,1} + a_{1,1}a_{2,1}) + (-a_{2,1}a_{1,1} + a_{1,1}a_{2,1})(a_{2,2}a_{1,2} - a_{1,2}a_{2,2}) \\ &\quad = \mathcal{S}_4(a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}). \end{aligned}$$

□

Corollary 5.4 *If $A = [a_{i,j}]$ is in $\mathrm{M}_2(R)$, then*

$$\begin{aligned} p_{A,2}(z) - q_{A,2}(z) &= \mathrm{rdet}_{(2)}(zI - A) - \mathrm{ldet}_{(2)}(zI - A) \\ &= \mathcal{S}_4(z - a_{1,1}, -a_{1,2}, -a_{2,1}, z - a_{2,2}) \\ &= \mathcal{S}_4(-a_{1,1}, -a_{1,2}, -a_{2,1}, -a_{2,2}) = \mathcal{S}_4(a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) \end{aligned}$$

is a constant polynomial in $R[z]$.

6. The symmetric Newton formulae for 2 × 2 and 3 × 3 matrices

If our base ring R is commutative, then the well-known Newton trace formulae for 2×2 and 3×3 matrices are the following:

$$2\det(A) = \mathrm{tr}^2(A) - \mathrm{tr}(A^2),$$

$$6\det(A) = \mathrm{tr}^3(A) - 3\mathrm{tr}(A)\mathrm{tr}(A^2) + 2\mathrm{tr}(A^3).$$

Proposition 6.1 *If R is an arbitrary ring and $A \in \mathrm{M}_2(R)$, then the symmetric analogue*

$$\mathrm{sdet}(A) = \mathrm{tr}^2(A) - \mathrm{tr}(A^2)$$

of the classical 2×2 Newton formula holds. Notice that $\mathrm{sdet}(A) = 2\det(A)$ in case of a commutative R.

Proof Using

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix},$$

we obtain that

$$\mathrm{tr}^2(A) - \mathrm{tr}(A^2) = (a + d)^2 - (a^2 + bc + cb + d^2) = ad + da - bc - cb = \mathrm{sdet}(A).$$

□

Theorem 6.2 *If R is an arbitrary ring and $A \in \mathrm{M}_3(R)$, then the following symmetric analogue of the classical 3×3 Newton formula holds:*

$$\mathrm{sdet}(A) = \mathrm{tr}^3(A) - \mathrm{tr}(A) \cdot \mathrm{tr}(A^2) - \mathrm{tr}(A \cdot \mathrm{tr}(A) \cdot A) - \mathrm{tr}(A^2) \cdot \mathrm{tr}(A) + \mathrm{tr}(A^3) + \mathrm{tr}\left((A^\top)^3\right).$$

Notice that

$$\mathrm{sdet}(A) = 6\det(A),\ \mathrm{tr}(A)\mathrm{tr}(A^2) = \mathrm{tr}(A \cdot \mathrm{tr}(A) \cdot A) = \mathrm{tr}(A^2) \cdot \mathrm{tr}(A),\ \mathrm{tr}\left((A^\top)^3\right) = \mathrm{tr}(A^3)$$

in case of a commutative R.

Proof Using

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & p \end{bmatrix}, \quad A^\top = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & p \end{bmatrix}$$

$$A^2 = \begin{bmatrix} a^2+bd+cg & ab+be+ch & ac+bf+cp \\ da+ed+fg & db+e^2+fh & dc+ef+fp \\ ga+hd+pg & gb+he+ph & gc+hf+p^2 \end{bmatrix}$$

and

$$A^3 = \begin{bmatrix} a^2+bd+cg & ab+be+ch & ac+bf+cp \\ da+ed+fg & db+e^2+fh & dc+ef+fp \\ ga+hd+pg & gb+he+ph & gc+hf+p^2 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & p \end{bmatrix},$$

we obtain that

$$\operatorname{tr}(A^2) = a^2+bd+cg+db+e^2+fh+gc+hf+p^2$$

and

$$\begin{aligned}\operatorname{tr}(A^3) &= (a^2+bd+cg)a+(ab+be+ch)d+(ac+bf+cp)g+(da+ed+fg)b \\ &\quad +(db+e^2+fh)e+(dc+ef+fp)h+(ga+hd+pg)c \\ &\quad +(gb+he+ph)f+(gc+hf+p^2)p.\end{aligned}$$

We obtain a similar expression for $\operatorname{tr}\left((A^\top)^3\right)$. Then the proof can be completed by direct (but annoying) computation. □

Remark 6.3 For a non-commutative ring R, the identity $\operatorname{tr}\left((A^\top)^2\right) = \operatorname{tr}(A^2)$ holds for any $A \in \mathrm{M}_n(R)$, but $\operatorname{tr}\left((A^\top)^3\right) = \operatorname{tr}(A^3)$ is not valid even in the 2×2 case.

THEOREM 6.4 *If A is a 3×3 matrix over an arbitrary ring R, then the symmetric characteristic polynomial of A in $R[z]$ is*

$$p_{A,1}(z) = q_{A,1}(z) = \operatorname{sdet}(zI-A) = 6z^3 - 6\operatorname{tr}(A)z^2 + 3(\operatorname{tr}^2(A) - \operatorname{tr}(A^2))z - \operatorname{sdet}(A).$$

Proof Using

$$\begin{aligned}
&\operatorname{tr}(zI-A) = 3z - \operatorname{tr}(A),\ \operatorname{tr}^3(zI-A) = 27z^3 - 27\operatorname{tr}(A)z^2 + 9\operatorname{tr}^2(A)z - \operatorname{tr}^3(A), \\
&(zI-A)^2 = z^2I - zA - Az + A^2,\ \operatorname{tr}((zI-A)^2) = 3z^2 - 2\operatorname{tr}(A)z + \operatorname{tr}(A^2), \\
&\operatorname{tr}(zI-A)\cdot\operatorname{tr}((zI-A)^2) = 9z^3 - 9\operatorname{tr}(A)z^2 + 2\operatorname{tr}^2(A)z + 3\operatorname{tr}(A^2)z - \operatorname{tr}(A)\operatorname{tr}(A^2), \\
&\operatorname{tr}((zI-A)^2)\cdot\operatorname{tr}(zI-A) = 9z^3 - 9\operatorname{tr}(A)z^2 + 2\operatorname{tr}^2(A)z + 3\operatorname{tr}(A^2)z - \operatorname{tr}(A^2)\operatorname{tr}(A), \\
&\operatorname{tr}\left((zI-A)\cdot\operatorname{tr}(zI-A)\cdot(zI-A)\right) \\
&\quad = \operatorname{tr}\left(3Iz^3 - 3Az^2 - \operatorname{tr}(A)Iz^2 - 3Az^2 + \operatorname{tr}(A)Az + 3A^2z + A\operatorname{tr}(A)z - A\operatorname{tr}(A)A\right) \\
&\quad = 9z^3 - 9\operatorname{tr}(A)z^2 + 2\operatorname{tr}^2(A)z + 3\operatorname{tr}(A^2)z - \operatorname{tr}(A\cdot\operatorname{tr}(A)\cdot A),
\end{aligned}$$

$$(zI-A)^3 = z^3I - z^2A - zAz - Az^2 + A^2z + AzA + zA^2 - A^3,$$
$$\operatorname{tr}((zI-A)^3) = 3z^3 - 3\operatorname{tr}(A)z^2 + 3\operatorname{tr}(A^2)z - \operatorname{tr}(A^3),$$

$$\begin{aligned}\operatorname{tr}\left(((zI-A)^\top)^3\right) &= 3z^3 - 3\operatorname{tr}(A^\top)z^2 + 3\operatorname{tr}\left((A^\top)^2\right)z - \operatorname{tr}\left((A^\top)^3\right) \\ &= 3z^3 - 3\operatorname{tr}(A)z^2 + 3\operatorname{tr}(A^2)z - \operatorname{tr}\left((A^\top)^3\right)\end{aligned}$$

and Theorem 6.2, we obtain that

$$\begin{aligned}\mathrm{sdet}(zI-A) &= \mathrm{tr}^3(zI-A)-\mathrm{tr}(zI-A)\cdot\mathrm{tr}((zI-A)^2)\\ &\quad -\mathrm{tr}\,((zI-A)\cdot\mathrm{tr}(zI-A)\cdot(zI-A))\\ &\quad -\mathrm{tr}((zI-A)^2)\cdot\mathrm{tr}(zI-A)+\mathrm{tr}((zI-A)^3)+\mathrm{tr}\left(((zI-A)^\top)^3\right)\\ &= 6z^3-6\mathrm{tr}(A)z^2+3(\mathrm{tr}^2(A)-\mathrm{tr}(A^2))z-\mathrm{sdet}(A).\end{aligned}$$

□

Corollary 6.5 *If $A \in \mathrm{M}_3(R)$, then Theorem 4.6 gives the existence of 3×3 matrices C_i, D_i $(0 \leq i \leq 3)$ with entries in $[R,R]$ such that*

$$(-\mathrm{sdet}(A)I+C_0)+(3(\mathrm{tr}^2(A)-\mathrm{tr}(A^2))I+C_1)A+(-6\mathrm{tr}(A)I+C_2)A^2+(6I+C_3)A^3=0$$

and

$$(-\mathrm{sdet}(A)I+D_0)+A(3(\mathrm{tr}^2(A)-\mathrm{tr}(A^2))I+D_1)+A^2(-6\mathrm{tr}(A)I+D_2)+A^3(6I+D_3)=0.$$

Corollary 6.6 *If $\frac{1}{6} \in R$ and $A \in \mathrm{M}_3(R)$ such that*

$$\mathrm{tr}(A)=\mathrm{tr}(A^2)=\mathrm{tr}(A^3)=\mathrm{tr}\left((A^\top)^3\right)=0,$$

then $\mathrm{sdet}(A)=0$ *and*

$$A^3=C_0+C_1A+C_2A^2+C_3A^3=D_0+AD_1+A^2D_2+A^3D_3$$

for some 3×3 matrices C_i, D_i $(0 \leq i \leq 3)$ with entries in $[R,R]$. Thus $A^3 \in \mathrm{M}_3(T)$, where $T=R[R,R]\cap[R,R]R$ is the intersection of the left and right ideals $R[R,R]$ and $[R,R]R$ of R.

We close the paper by the following:

Problem 6.7 If R is a commutative ring, then the Newton formula for a 4×4 matrix $A \in \mathrm{M}_4(R)$ is

$$24\det(A)=\mathrm{tr}^4(A)-6\mathrm{tr}^2(A)\mathrm{tr}(A^2)+3\mathrm{tr}^2(A^2)+8\mathrm{tr}(A)\mathrm{tr}(A^3)-6\mathrm{tr}(A^4).$$

Find the symmetric analogue of the above formula for $\mathrm{sdet}(A)$ over an arbitrary ring R.

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