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Centralizers in endomorphism rings ☆

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ABSTRACT

We prove that the centralizer $\mathrm{Cen}(\varphi) \subseteq \mathrm{End}_R(M)$ of a nilpotent endomorphism φ of a finitely generated semisimple left R-module ${}_RM$ (over an arbitrary ring R) is the homomorphic image of the opposite of a certain $Z(R)$-subalgebra of the full $m \times m$ matrix algebra $M_m(R[z])$, where m is the dimension of $\ker(\varphi)$. If R is a local ring, then we give a complete characterization of $\mathrm{Cen}(\varphi)$ and of the containment $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$, where σ is a not necessarily nilpotent element of $\mathrm{End}_R(M)$. For a K-linear map A of a finite dimensional vector space over a field K we determine the PI-degree of $\mathrm{Cen}(A)$.

1. Introduction

If S is a ring (or algebra), then the centralizer $\mathrm{Cen}(s) = \{u \in S \mid us = su\}$ of an element $s \in S$ is a subring (subalgebra) of S. The aim of this paper is to investigate the centralizer $\mathrm{Cen}(\varphi)$ of an element φ in the endomorphism ring $\mathrm{End}_R(M)$ of a left R-module ${}_RM$. In the case of finite dimensional vector spaces the study of $\mathrm{Cen}(\varphi)$ can be reduced to the nilpotent case. Thus we focus only on the

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nilpotent endomorphisms of a finitely generated semisimple ${}_RM$. We note that most of our statements are generalizations of classical linear algebra results about commuting matrices (see [2,5,6,8]).

Following observations about the nilpotent Jordan normal base in Section 2 and other preliminary results in Section 3, we prove in Theorem 3.9 that $\mathrm{Cen}(\varphi)$ is the homomorphic image of the opposite of a certain $Z(R)$-subalgebra of the full $m\times m$ matrix algebra $M_m(R[z])$ over the polynomial ring $R[z]$, where m is the dimension of $\ker(\varphi)$. If R is a local ring, then in Theorem 3.11 we present $\mathrm{Cen}(\varphi)$ as (the opposite of) a factor of a certain subalgebra of $M_m(R[z])$. The $Z(R)$-dimension of $\mathrm{Cen}(\varphi)$ is determined when R is local, $Z(R)$ is a field and $R/J(R)$ is finite dimensional over $Z(R)$.

If φ is a so-called indecomposable nilpotent element of $\mathrm{End}_R(M)$, then the elements of $\mathrm{Cen}(\varphi)$ are described in terms of an appropriate R-generating set of ${}_RM$ in Theorem 4.1. In particular, if R is commutative, then $\psi\in\mathrm{Cen}(\varphi)$ if and only if ψ is a polynomial expression of φ. If R is a local ring, φ is nilpotent and σ is an arbitrary element of $\mathrm{End}_R(M)$, then $\mathrm{Cen}(\varphi)\subseteq\mathrm{Cen}(\sigma)$ is equivalent to the existence of a certain R-generating set of ${}_RM$ (Theorem 4.3). In the commutative local case $\mathrm{Cen}(\varphi)\subseteq\mathrm{Cen}(\sigma)$ if and only if σ is a polynomial expression of φ.

For a nilpotent matrix $A\in M_n(K)$ (over a field K) the semisimple component of $\mathrm{Cen}(A)$ is determined in Theorem 5.1. Our proof of Theorem 5.1 is based on the use of Theorem 3.11. If p is the maximum number of elementary Jordan matrices of the same size and with the same eigenvalue (of a not necessarily nilpotent A), then for the T-ideals of the identities we prove that $T(\mathrm{Cen}(A))\supseteq T(M_p(K))^q$ for a suitable q. Hence the PI-degree of $\mathrm{Cen}(A)$ is equal to p.

Since all known results about matrix centralizers are closely connected with the Jordan normal form, it is not surprising that our development depends on the existence of the so-called nilpotent Jordan normal base of a semisimple module with respect to a given nilpotent endomorphism (the main theorem of [7]).

For a version of this paper containing more computational details see [1].

2. The nilpotent Jordan normal base

Throughout the paper a ring R means a (not necessarily commutative) ring with identity, $Z(R)$ and $J=J(R)$ denote the center and the Jacobson radical of R, respectively. Also, $M_m(R)$ and $R[z]$ denote the $m\times m$ matrix ring and the polynomial ring of the commuting indeterminate z, respectively. The ideal $(z^k)\lhd R[z]$ generated by z^k will be considered in the sequel, and $(z^0)=R[z]$.

A subset $X=\{x_{\gamma,i}\mid \gamma\in\Gamma,\ 1\leqslant i\leqslant k_\gamma\}$ of a (unitary) left R-module ${}_RM$ is called a nilpotent Jordan normal base with respect to $\varphi\in\mathrm{End}_R(M)$ if each R-submodule $Rx_{\gamma,i}\leqslant M$ is simple,

$$\bigoplus_{\gamma\in\Gamma,\,1\leqslant i\leqslant k_\gamma} Rx_{\gamma,i}=M$$

is a direct sum, $\varphi(x_{\gamma,i})=x_{\gamma,i+1}$, $\varphi(x_{\gamma,k_\gamma})=x_{\gamma,k_\gamma+1}=0$ for all $\gamma\in\Gamma$, $1\leqslant i\leqslant k_\gamma$, and the set $\{k_\gamma\mid \gamma\in\Gamma\}$ of integers is bounded (Γ is called the set of Jordan blocks and the size of the block $\gamma\in\Gamma$ is the integer k_γ). Obviously, the existence of a nilpotent Jordan normal base implies that ${}_RM$ is semisimple and φ is nilpotent with $\varphi^n=0\neq\varphi^{n-1}$, where $n=\max\{k_\gamma\mid\gamma\in\Gamma\}$. Clearly,

$$\mathrm{im}(\varphi)=\bigoplus_{\gamma\in\Gamma',\,2\leqslant i\leqslant k_\gamma} Rx_{\gamma,i}\quad\text{and}\quad \ker(\varphi)=\bigoplus_{\gamma\in\Gamma} Rx_{\gamma,k_\gamma},$$

where $\Gamma'=\{\gamma\in\Gamma\mid k_\gamma\geqslant 2\}$. The following is one of the main results in [7].

2.1. Theorem. *For $\varphi\in\mathrm{End}_R(M)$ the following are equivalent*:

1. *${}_RM$ is semisimple and φ is nilpotent.*
2. *There exists a nilpotent Jordan normal base of ${}_RM$ with respect to φ.*

2.2. Proposition. *Let $\varphi \in \mathrm{End}_R(M)$ be nilpotent, with ${}_RM$ finitely generated semisimple. If $\{x_{\gamma,i} \mid \gamma \in \Gamma,\ 1 \leqslant i \leqslant k_\gamma\}$ and $\{y_{\delta,j} \mid \delta \in \Delta,\ 1 \leqslant j \leqslant l_\delta\}$ are nilpotent Jordan normal bases of ${}_RM$ with respect to φ, then there exists a bijection $\pi : \Gamma \to \Delta$ such that $k_\gamma = l_{\pi(\gamma)}$ for all $\gamma \in \Gamma$. Thus the sizes of the blocks of a nilpotent Jordan normal base are unique up to a permutation of the blocks.*

Proof. We apply induction on the index of the nilpotency of φ. If $\varphi = 0$, then we have $k_\gamma = l_\delta = 1$ for all $\gamma \in \Gamma$, $\delta \in \Delta$, and $\bigoplus_{\gamma\in\Gamma} Rx_{\gamma,1} = \bigoplus_{\delta\in\Delta} Ry_{\delta,1} = M$ implies the existence of a bijection $\pi : \Gamma \to \Delta$ (Krull–Schmidt, Kurosh–Ore). Assume that our statement holds for any R-endomorphism $\phi : N \to N$ with ${}_RN$ a finitely generated semisimple left R-module and $\phi^{n-1} = 0 \neq \phi^{n-2}$. Consider the situation described in the proposition with $\varphi^n = 0 \neq \varphi^{n-1}$. Then

$$\mathrm{im}(\varphi) = \bigoplus_{\gamma\in\Gamma',\, 2\leqslant i\leqslant k_\gamma} Rx_{\gamma,i}$$

ensures that $\{x_{\gamma,i} \mid \gamma \in \Gamma',\ 2 \leqslant i \leqslant k_\gamma\}$ is a nilpotent Jordan normal base of the left R-submodule $\mathrm{im}(\varphi)$ of ${}_RM$ with respect to the restricted R-endomorphism $\varphi : \mathrm{im}(\varphi) \to \mathrm{im}(\varphi)$. The same holds for $\{y_{\delta,j} \mid \delta \in \Delta',\ 2 \leqslant j \leqslant l_\delta\}$, where $\Delta' = \{\delta \in \Delta \mid l_\delta \geqslant 2\}$. Since $\phi^{n-1} = 0 \neq \phi^{n-2}$ for $\phi = \varphi \restriction \mathrm{im}(\varphi)$, our assumption gives a bijection $\pi : \Gamma' \to \Delta'$ such that $k_\gamma - 1 = l_{\pi(\gamma)} - 1$ for all $\gamma \in \Gamma'$. In view of $\ker(\varphi) = \bigoplus_{\gamma\in\Gamma} Rx_{\gamma,k_\gamma} = \bigoplus_{\delta\in\Delta} Ry_{\delta,l_\delta}$ we have $|\Gamma| = |\Delta|$ and so $|\Gamma \setminus \Gamma'| = |\Delta \setminus \Delta'|$. Thus we have a bijection $\pi^* : \Gamma \setminus \Gamma' \to \Delta \setminus \Delta'$ and the natural map $\pi \sqcup \pi^* : \Gamma' \cup (\Gamma \setminus \Gamma') \to \Delta' \cup (\Delta \setminus \Delta')$ is a bijection with the desired property. $\square$

We call a nilpotent element s of a ring S *decomposable* if $es = se$ for some idempotent $e \in S$ with $0 \neq e \neq 1$. A nilpotent element which is not decomposable is called *indecomposable*. In the case of finite dimensional vector spaces an indecomposable nilpotent endomorphism is nonderogatory (or 1-regular) in the sense of [4].

2.3. Proposition. *Let $\varphi : M \to M$ be a non-zero nilpotent R-endomorphism of the semisimple left R-module ${}_RM$. Then the following are equivalent*:

1. *There is a nilpotent Jordan normal base $\{x_i \mid 1 \leqslant i \leqslant n\}$ of ${}_RM$ with respect to φ consisting of one block (thus $|\Gamma| = 1$ for any nilpotent Jordan normal base $\{x_{\gamma,i} \mid \gamma \in \Gamma,\ 1 \leqslant i \leqslant k_\gamma\}$ of ${}_RM$ with respect to φ).*
2. *φ is an indecomposable nilpotent element of the ring $\mathrm{End}_R(M)$.*
3. *${}_RM$ is finitely generated and $\varphi^{d-1} \neq 0$, where $d = \dim_R(M)$.*

Proof. (1) $\Leftrightarrow$ (3) is straightforward.

(1) $\Rightarrow$ (2): If $\varepsilon \circ \varphi = \varphi \circ \varepsilon$ for some idempotent $\varepsilon \in \mathrm{End}_R(M)$ with $0 \neq \varepsilon \neq 1$, then $\mathrm{im}(\varepsilon) \oplus \mathrm{im}(1-\varepsilon) = M$ for the non-zero (semisimple) R-submodules $\mathrm{im}(\varepsilon)$ and $\mathrm{im}(1-\varepsilon)$ of ${}_RM$, and $\varphi : \mathrm{im}(\varepsilon) \to \mathrm{im}(\varepsilon)$ and $\varphi : \mathrm{im}(1-\varepsilon) \to \mathrm{im}(1-\varepsilon)$. Since these restricted R-endomorphisms are nilpotent, we have nilpotent Jordan normal bases of $\mathrm{im}(\varepsilon)$ and $\mathrm{im}(1-\varepsilon)$ with respect to $\varphi \restriction \mathrm{im}(\varepsilon)$ and $\varphi \restriction \mathrm{im}(1-\varepsilon)$ respectively. The union of these two bases gives a nilpotent Jordan normal base of M with respect to φ consisting of more than one block, a contradiction (the direct sum property of the new base is a consequence of the modularity of the submodule lattice of ${}_RM$).

(2) $\Rightarrow$ (1): Suppose we have a nilpotent Jordan normal base X of ${}_RM$ with respect to φ with $|\Gamma| \geqslant 2$. Fix $\delta \in \Gamma$ and consider the non-zero φ-invariant R-submodules

$$N'_\delta = \bigoplus_{1\leqslant i\leqslant k_\delta} Rx_{\delta,i} \quad \text{and} \quad N''_\delta = \bigoplus_{\gamma\in\Gamma\setminus\{\delta\},\, 1\leqslant i\leqslant k_\gamma} Rx_{\gamma,i}.$$

Then $M = N'_\delta \oplus N''_\delta$ and for the natural projection ε_δ of M onto N'_δ we have $\varepsilon_\delta \circ \varepsilon_\delta = \varepsilon_\delta$, $0 \neq \varepsilon_\delta \neq 1$ and $\varepsilon_\delta \circ \varphi = \varphi \circ \varepsilon_\delta$. $\square$

3. The centralizer of a nilpotent endomorphism

Note that $\varphi \in \mathrm{End}_R(M)$ defines a natural left action $*$ of $R[z]$ on M providing a left $R[z]$-module structure on M. Clearly, $\mathrm{Cen}(\varphi) = \mathrm{End}_{R[z]}(M)$ for the centralizer $Z(R)$-subalgebra of $\mathrm{End}_R(M)$.

Henceforth ${}_RM$ is semisimple and we consider a fixed nilpotent Jordan normal base

$$X = \{x_{\gamma,i} \mid \gamma \in \Gamma,\ 1 \leqslant i \leqslant k_\gamma\} \subseteq M$$

with respect to a given nilpotent $\varphi \in \mathrm{End}_R(M)$ of index n.

The Γ-copower $\coprod_{\gamma\in\Gamma} R[z]$ is an ideal of the Γ-direct power ring $(R[z])^\Gamma$ comprising all elements $\mathbf{f} = (f_\gamma(z))_{\gamma\in\Gamma}$ with a finite set $\{\gamma \in \Gamma \mid f_\gamma(z) \neq 0\}$ of non-zero coordinates. The copower (power) has a natural $(R[z], R[z])$-bimodule structure. If $f_\gamma(z) = a_{\gamma,1} + a_{\gamma,2}z + \cdots + a_{\gamma,n_\gamma+1}z^{n_\gamma}$ then

$$\Phi(\mathbf{f}) = \sum_{\gamma\in\Gamma,\,1\leqslant i\leqslant k_\gamma} a_{\gamma,i}x_{\gamma,i} = \sum_{\gamma\in\Gamma}\Bigg(\sum_{1\leqslant i\leqslant k_\gamma} a_{\gamma,i}\varphi^{i-1}(x_{\gamma,1})\Bigg) = \sum_{\gamma\in\Gamma} f_\gamma(z) * x_{\gamma,1}$$

defines a left $R[z]$-module homomorphism $\Phi : \coprod_{\gamma\in\Gamma} R[z] \to M$.

3.1. Proposition. *The function Φ is surjective, $\coprod_{\gamma\in\Gamma}(J[z] + (z^{k_\gamma})) \subseteq \ker(\Phi)$ and if R is local (R/J is a division ring), equality holds.*

Proof. The surjectivity of Φ and the containment $\coprod_{\gamma\in\Gamma}(J[z] + (z^{k_\gamma})) \subseteq \ker(\Phi)$ are clear. When R is local and $a_{\gamma,i}x_{\gamma,i} = 0$ for some $1 \leqslant i \leqslant k_\gamma$, then $a_{\gamma,i} \in J$. Thus $f_\gamma(z) = (a_{\gamma,1} + a_{\gamma,2}z + \cdots + a_{\gamma,k_\gamma}z^{k_\gamma-1}) + (a_{\gamma,k_\gamma+1}z^{k_\gamma} + \cdots + a_{\gamma,n_\gamma+1}z^{n_\gamma}) \in J[z] + (z^{k_\gamma})$ is a consequence of $\Phi(\mathbf{f}) = 0$, implying that $\mathbf{f} \in \coprod_{\gamma\in\Gamma}(J[z] + (z^{k_\gamma}))$. $\square$

From now onward we also require that ${}_RM$ be finitely generated, $\Gamma = \{1, 2, \dots, m\}$ and to ease readability we assume that $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m \geqslant 1$ for the block sizes, in which case $\dim_R(M) = \sum_{\gamma\in\Gamma} k_\gamma$ and $\dim_R(\ker(\varphi)) = |\Gamma| = m$ for the dimensions (composition lengths). Now $\coprod_{\gamma\in\Gamma} R[z] = (R[z])^\Gamma$ and an element $\mathbf{f} = (f_\gamma(z))_{\gamma\in\Gamma}$ of $(R[z])^\Gamma$ is a $1 \times m$ matrix over $R[z]$. We define the following subsets of $M_m(R[z])$:

$$\begin{aligned}
\mathcal{I}(X) &= \big\{\mathbf{P} \in M_m\big(R[z]\big) \bigm| \mathbf{P} = \big[p_{\delta,\gamma}(z)\big] \text{ and } p_{\delta,\gamma}(z) \in J[z] + \big(z^{k_\gamma}\big) \text{ for all } \delta, \gamma \in \Gamma\big\} \\
&= \begin{bmatrix} J[z] + (z^{k_1}) & J[z] + (z^{k_2}) & \cdots & J[z] + (z^{k_m}) \\ J[z] + (z^{k_1}) & J[z] + (z^{k_2}) & \cdots & J[z] + (z^{k_m}) \\ \vdots & \vdots & \ddots & \vdots \\ J[z] + (z^{k_1}) & J[z] + (z^{k_2}) & \cdots & J[z] + (z^{k_m}) \end{bmatrix}, \\
\mathcal{N}(X) &= \big\{\mathbf{P} \in M_m\big(R[z]\big) \bigm| \mathbf{P} = \big[p_{\delta,\gamma}(z)\big] \text{ and } z^{k_\delta}p_{\delta,\gamma}(z) \in J[z] + \big(z^{k_\gamma}\big) \text{ for all } \delta, \gamma \in \Gamma\big\}, \\
\mathcal{M}(X) &= \big\{\mathbf{P} \in M_m\big(R[z]\big) \bigm| \mathbf{fP} \in \ker(\Phi) \text{ for all } \mathbf{f} \in \ker(\Phi)\big\}.
\end{aligned}$$

Note that $\mathcal{I}(X)$ and $\mathcal{N}(X)$ are $(R[z], R[z])$-sub-bimodules of $M_m(R[z])$ in a natural way. For $\delta, \gamma \in \Gamma$ let $k_{\delta,\gamma} = k_\gamma - k_\delta$ when $1 \leqslant k_\delta < k_\gamma \leqslant n$ and $k_{\delta,\gamma} = 0$ otherwise.

3.2. Remark. It can be verified that the condition $z^{k_\delta}p_{\delta,\gamma}(z) \in J[z] + (z^{k_\gamma})$ in the definition of $\mathcal{N}(X)$ is equivalent to $p_{\delta,\gamma}(z) \in J[z] + (z^{k_{\delta,\gamma}})$, and so

$$\mathcal{N}(X)=\begin{bmatrix} R[z] & R[z] & R[z] & \cdots & R[z] \\ J[z]+(z^{k_1-k_2}) & R[z] & R[z] & \cdots & R[z] \\ J[z]+(z^{k_1-k_3}) & J[z]+(z^{k_2-k_3}) & R[z] & \cdots & R[z] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J[z]+(z^{k_1-k_m}) & J[z]+(z^{k_2-k_m}) & J[z]+(z^{k_3-k_m}) & \cdots & R[z] \end{bmatrix}.$$

3.3. Lemma. *$\mathcal{I}(X)\lhd_l M_m(R[z])$ is a left ideal, $\mathcal{N}(X)\subseteq M_m(R[z])$ is a subring, $\mathcal{I}(X)\lhd\mathcal{N}(X)$ is an ideal and $\mathcal{M}(X)$ is a $Z(R)$-subalgebra of $M_m(R[z])$. If R is a local ring, then $\mathcal{N}(X)=\mathcal{M}(X)$.*

Proof. Since the γ-th column of the matrices in $\mathcal{I}(X)$ comes from a (left) ideal $J[z]+(z^{k_\gamma})$ of $R[z]$, we can see that $\mathcal{I}(X)$ is a left ideal.

If $\mathbf{P},\mathbf{Q}\in\mathcal{N}(X)$, then we have $p_{\delta,\tau}(z)\in J[z]+(z^{k_{\delta,\tau}})$ and $q_{\tau,\gamma}(z)\in J[z]+(z^{k_{\tau,\gamma}})$. Thus $k_{\delta,\tau}+k_{\tau,\gamma}\geqslant k_{\delta,\gamma}$ implies that $p_{\delta,\tau}(z)q_{\tau,\gamma}(z)\in J[z]+(z^{k_{\delta,\gamma}})$. It follows that $\mathbf{PQ}\in\mathcal{N}(X)$ proving that $\mathcal{N}(X)$ is a subring.

If $\mathbf{P}\in\mathcal{I}(X)$ and $\mathbf{Q}\in\mathcal{N}(X)$, then we have $p_{\delta,\tau}(z)\in J[z]+(z^{k_\tau})$ and $q_{\tau,\gamma}(z)\in J[z]+(z^{k_{\tau,\gamma}})$. Since $k_\tau+k_{\tau,\gamma}\geqslant k_\gamma$, it follows that $p_{\delta,\tau}(z)q_{\tau,\gamma}(z)\in J[z]+(z^{k_\gamma})$. Thus $\mathbf{PQ}\in\mathcal{I}(X)$ and $\mathcal{I}(X)$ is an ideal of $\mathcal{N}(X)$.

If $\mathbf{P},\mathbf{Q}\in\mathcal{M}(X)$ and $\mathbf{f}\in\ker(\Phi)$, then $\mathbf{fP}\in\ker(\Phi)$ implies that $\Phi(\mathbf{f}(\mathbf{PQ}))=\Phi((\mathbf{fP})\mathbf{Q})=0$, whence $\mathbf{f}(\mathbf{PQ})\in\ker(\Phi)$ follows. Thus $\mathbf{PQ}\in\mathcal{M}(X)$, proving that $\mathcal{M}(X)$ is a $Z(R)$-subalgebra of $M_m(R[z])$.

If R is a local ring, then Proposition 3.1 gives that $\ker(\Phi)=\coprod_{\gamma\in\Gamma}(J[z]+(z^{k_\gamma}))$. Now $\mathbf{e}_\delta\in\ker(\Phi)$, where $\mathbf{e}_\delta$ denotes the vector with z^{k_δ} in its δ-coordinate and zeros in all other places. If $\mathbf{P}\in\mathcal{M}(X)$, then $\mathbf{e}_\delta\mathbf{P}\in\ker(\Phi)$ implies that $z^{k_\delta}p_{\delta,\gamma}(z)\in J[z]+(z^{k_\gamma})$, whence $\mathbf{P}\in\mathcal{N}(X)$ follows. If $\mathbf{P}\in\mathcal{N}(X)$ and $\mathbf{f}\in\ker(\Phi)$, then $p_{\delta,\gamma}(z)\in J[z]+(z^{k_{\delta,\gamma}})$ and $f_\delta(z)\in J[z]+(z^{k_\delta})$. Thus $k_\delta+k_{\delta,\gamma}\geqslant k_\gamma$ implies that $f_\delta(z)p_{\delta,\gamma}(z)\in J[z]+(z^{k_\gamma})$, whence $\mathbf{fP}\in\ker(\Phi)$ and $\mathbf{P}\in\mathcal{M}(X)$ follows. $\square$

3.4. Lemma. *If the center $Z(R)$ of the ring R is a field such that R/J is finite dimensional over $Z(R)$, then*

$$\dim_{Z(R)}\big(\mathcal{N}(X)/\mathcal{I}(X)\big)=\big[R/J:Z(R)\big]\cdot\big(k_1+3k_2+\cdots+(2m-1)k_m\big).$$

Proof. Any $Z(R)$-base of R/J naturally leads to a $Z(R)$-base of $\mathcal{N}(X)/\mathcal{I}(X)$, and so the claim is obvious from the above matrix forms of $\mathcal{I}(X)$ and $\mathcal{N}(X)$. $\square$

The assumption $k_1\geqslant k_2\geqslant\cdots\geqslant k_m\geqslant 1$ ensures that

$$\mathcal{U}(X)=\big\{U\in M_m(R/J)\ \big|\ U=[u_{\delta,\gamma}]\text{ and }u_{\delta,\gamma}=0\text{ if }1\leqslant k_\delta<k_\gamma\big\}$$

is a block upper triangular subalgebra of $M_m(R/J)$. The T-ideal of the identities of $\mathcal{U}(X)$ is described in [3]. We note that, if $k_1>k_2>\cdots>k_m\geqslant 1$, then

$$\mathcal{U}(X)=\begin{bmatrix} R/J & R/J & \cdots & R/J \\ 0 & R/J & \ddots & \vdots \\ \vdots & \ddots & \ddots & R/J \\ 0 & \cdots & 0 & R/J \end{bmatrix}$$

is an upper triangular matrix algebra.

3.5. Lemma. *There is a natural ring isomorphism*

$$\mathcal{N}(X)/\big(\mathcal{N}(X)\cap zM_m\big(R[z]\big)\big)+\mathcal{I}(X)\cong\mathcal{U}(X)$$

which is an (R,R)-bimodule isomorphism at the same time.

Proof. If $\mathbf{P}=[p_{\delta,\gamma}(z)]$ is in $\mathcal{N}(X)$, then it is straightforward to see that there exists a matrix $[w_{\delta,\gamma}]$ in $M_m(R)\cap\mathcal{N}(X)$ such that

$$\mathbf{P}+\big(\big(\mathcal{N}(X)\cap zM_m\big(R[z]\big)\big)+\mathcal{I}(X)\big)=[w_{\delta,\gamma}]+\big(\big(\mathcal{N}(X)\cap zM_m\big(R[z]\big)\big)+\mathcal{I}(X)\big)$$

holds in $\mathcal{N}(X)/(\mathcal{N}(X)\cap zM_m(R[z]))+\mathcal{I}(X)$. The assignment

$$\mathbf{P}+\big(\big(\mathcal{N}(X)\cap zM_m\big(R[z]\big)\big)+\mathcal{I}(X)\big)\longmapsto[w_{\delta,\gamma}+J]$$

is well defined and gives the required isomorphism. □

3.6. Lemma. *For $\mathbf{P}\in\mathcal{M}(X)$ and $\mathbf{f}=(f_\gamma(z))_{\gamma\in\Gamma}$ in $(R[z])^\Gamma$ the formula $\psi_{\mathbf{P}}(\Phi(\mathbf{f}))=\Phi(\mathbf{fP})$ properly defines an R-endomorphism $\psi_{\mathbf{P}}:M\to M$ of ${}_RM$ such that $\psi_{\mathbf{P}}\circ\varphi=\varphi\circ\psi_{\mathbf{P}}$. The assignment $\Lambda(\mathbf{P})=\psi_{\mathbf{P}}$ gives a homomorphism $\mathcal{M}(X)^{\mathrm{op}}\to\mathrm{Cen}(\varphi)$ of $Z(R)$-algebras.*

Proof. Using the definition of $\mathcal{M}(X)$ and the surjectivity of Φ it is straightforward to check the claims. □

3.7. Lemma. *$\mathcal{I}(X)\subseteq\ker(\Lambda)$, and if R is local, then the equality holds.*

Proof. The containment is clear. If R is a local ring and $\mathbf{P}\in\ker(\Lambda)$, then $\Lambda(\mathbf{P})=\psi_{\mathbf{P}}=0$ implies that $\psi_{\mathbf{P}}(\Phi(\mathbf{f}))=\Phi(\mathbf{fP})=0$ for all $\mathbf{f}\in(R[z])^\Gamma$. If $\mathbf{1}_\delta$ denotes the vector in $(R[z])^\Gamma$ with 1 in its δ-coordinate and zeros in all other places, then $\mathbf{1}_\delta\mathbf{P}\in\ker(\Phi)$ and Proposition 3.1 gives that $p_{\delta,\gamma}(z)\in J[z]+(z^{k_\gamma})$. □

3.8. Lemma. *If $\psi\circ\varphi=\varphi\circ\psi$ for some $\psi\in\mathrm{End}_R(M)$, then there is a $\mathbf{P}\in\mathcal{M}(X)$ such that $\psi(\Phi(\mathbf{f}))=\Phi(\mathbf{fP})$ for all $\mathbf{f}=(f_\gamma(z))_{\gamma\in\Gamma}$ in $(R[z])^\Gamma$.*

Proof. Since $\Phi:(R[z])^\Gamma\to M$ is surjective, for each $\delta\in\Gamma$ we can find an element $\mathbf{p}_\delta=(p_{\delta,\gamma}(z))_{\gamma\in\Gamma}$ in $(R[z])^\Gamma$ such that $\Phi(\mathbf{p}_\delta)=\psi(x_{\delta,1})$. For the $m\times m$ matrix $\mathbf{P}=[p_{\delta,\gamma}(z)]$ we have

$$\begin{aligned}\psi\big(\Phi(\mathbf{f})\big)&=\sum_{\delta\in\Gamma}\psi\big(f_\delta(z)*x_{\delta,1}\big)=\sum_{\delta\in\Gamma}f_\delta(z)*\psi(x_{\delta,1})=\sum_{\delta\in\Gamma}f_\delta(z)*\Phi(\mathbf{p}_\delta)\\&=\sum_{\delta\in\Gamma}\Phi\big(f_\delta(z)\mathbf{p}_\delta\big)=\Phi\left(\sum_{\delta\in\Gamma}f_\delta(z)\mathbf{p}_\delta\right)=\Phi(\mathbf{fP})\end{aligned}$$

for all $\mathbf{f}\in(R[z])^\Gamma$. Since $\mathbf{f}\in\ker(\Phi)$ implies that $\Phi(\mathbf{fP})=\psi(\Phi(\mathbf{f}))=0$, we obtain that $\mathbf{P}\in\mathcal{M}(X)$. □

3.9. Theorem. *$\Lambda:\mathcal{M}(X)^{\mathrm{op}}\to\mathrm{Cen}(\varphi)$ is a surjective homomorphism of $Z(R)$-algebras.*

Proof. The claim directly follows from Lemma 3.6 and Lemma 3.8. □

3.10. Corollary. *$\mathrm{Cen}(\varphi)$ satisfies all the polynomial identities (with coefficients in $Z(R)$) of $M_m^{\mathrm{op}}(R[z])$.*

3.11. Theorem. *If R is a local ring, then $\mathrm{Cen}(\varphi)$ is isomorphic to the opposite of the factor $\mathcal{N}(X)/\mathcal{I}(X)$ as $Z(R)$-algebras:*

$$\mathrm{Cen}(\varphi)\cong\big(\mathcal{N}(X)/\mathcal{I}(X)\big)^{\mathrm{op}}=\mathcal{N}^{\mathrm{op}}(X)/\mathcal{I}(X).$$

If $f_i = 0$ are polynomial identities of the $Z(R)$-subalgebra $\mathcal{U}^{\mathrm{op}}(X)$ of $M_m^{\mathrm{op}}(R/J)$ with $f_i \in Z(R)\langle x_1, \ldots, x_r\rangle$, $1 \leqslant i \leqslant n$, then $f_1 f_2 \cdots f_n = 0$ is an identity of $\mathrm{Cen}(\varphi)$.

Proof. Theorem 3.9 ensures that $\mathrm{Cen}(\varphi) \cong \mathcal{M}(X)^{\mathrm{op}}/\ker(\Lambda)$ as $Z(R)$-algebras. In order to prove the desired isomorphism, it suffices to note that for a local ring R we have $\mathcal{M}(X) = \mathcal{N}(X)$ and $\ker(\Lambda) = \mathcal{I}(X)$ by Lemmas 3.3 and 3.7 respectively. Thus

$$L = \big(\big(\mathcal{N}(X) \cap zM_m\big(R[z]\big)\big) + \mathcal{I}(X)\big)/\mathcal{I}(X) \lhd \mathcal{N}(X)/\mathcal{I}(X)$$

can be viewed as an ideal of $\mathrm{Cen}(\varphi)$. The use of Lemma 3.5 gives

$$\mathrm{Cen}(\varphi)/L \cong \big(\mathcal{N}^{\mathrm{op}}(X)/\mathcal{I}(X)\big)/L \cong \mathcal{N}^{\mathrm{op}}(X)/\big(\mathcal{N}(X) \cap zM_m\big(R[z]\big)\big) + \mathcal{I}(X) \cong \mathcal{U}^{\mathrm{op}}(X).$$

It follows that $f_i = 0$ is an identity of $\mathrm{Cen}(\varphi)/L$. Thus $f_i(v_1, \ldots, v_r) \in L$ for all $v_1, \ldots, v_r \in \mathrm{Cen}(\varphi)$, and so $f_1(v_1, \ldots, v_r) f_2(v_1, \ldots, v_r) \cdots f_n(v_1, \ldots, v_r) \in L^n$. Since $(zM_m(R[z]))^n \subseteq \mathcal{I}(X)$ implies that $L^n = \{0\}$, the proof is complete.

3.12. Corollary. *If R is a local ring such that $Z(R)$ is a field and R/J is finite dimensional over $Z(R)$, then*

$$\dim_{Z(R)}\big(\mathrm{Cen}(\varphi)\big) = \big[R/J : Z(R)\big]\big(k_1 + 3k_2 + \cdots + (2m-1)k_m\big).$$

Proof. Since $\dim_{Z(R)}(\mathcal{N}(X)/\mathcal{I}(X))^{\mathrm{op}} = \dim_{Z(R)}(\mathcal{N}(X)/\mathcal{I}(X))$, the result follows from Lemma 3.4. $\square$

4. Further properties of the centralizers

4.1. Theorem. *Let φ be an indecomposable (nilpotent) element of $\mathrm{End}_R(M)$. Then $\psi \in \mathrm{Cen}(\varphi)$ if and only if there is an R-generating set $\{y_j \in M \mid 1 \leqslant j \leqslant d\}$ of ${}_RM$ and elements $a_1, a_2, \ldots, a_n$ in R such that*

$$a_1 y_j + a_2\varphi(y_j) + \cdots + a_n\varphi^{n-1}(y_j) = \psi(y_j) \quad \text{and}$$
$$a_1\varphi(y_j) + a_2\varphi\big(\varphi(y_j)\big) + \cdots + a_n\varphi^{n-1}\big(\varphi(y_j)\big) = \psi\big(\varphi(y_j)\big)$$

for all $1 \leqslant j \leqslant d$.

Proof. If $\psi \in \mathrm{Cen}(\varphi)$, then the first identity implies the second one. Proposition 2.3 ensures the existence of a nilpotent Jordan normal base $\{x_i \mid 1 \leqslant i \leqslant n\}$ of ${}_RM$ with respect to φ consisting of one block. Clearly, $\bigoplus_{1 \leqslant i \leqslant n} Rx_i = M$ implies that $\psi(x_1) = a_1x_1 + a_2x_2 + \cdots + a_nx_n = a_1x_1 + a_2\varphi(x_1) + \cdots + a_n\varphi^{n-1}(x_1)$ for some $a_1, a_2, \ldots, a_n \in R$. Thus $\psi(x_i) = \psi(\varphi^{i-1}(x_1)) = \varphi^{i-1}(\psi(x_1)) = \varphi^{i-1}(a_1x_1 + a_2\varphi(x_1) + \cdots + a_n\varphi^{n-1}(x_1)) = a_1\varphi^{i-1}(x_1) + a_2\varphi(\varphi^{i-1}(x_1)) + \cdots + a_n\varphi^{n-1}(\varphi^{i-1}(x_1)) = a_1x_i + a_2\varphi(x_i) + \cdots + a_n\varphi^{n-1}(x_i)$ for all $1 \leqslant i \leqslant n$.

Conversely, we have $\varphi(\psi(y_j)) = \varphi(a_1y_j + a_2\varphi(y_j) + \cdots + a_n\varphi^{n-1}(y_j)) = a_1\varphi(y_j) + a_2\varphi(\varphi(y_j)) + \cdots + a_n\varphi^{n-1}(\varphi(y_j)) = \psi(\varphi(y_j))$ for all $1 \leqslant j \leqslant d$. Thus $\varphi \circ \psi = \psi \circ \varphi$. $\square$

4.2. Corollary. *If in addition R is commutative, then $\psi \in \mathrm{Cen}(\varphi)$ if and only if there are $a_1, a_2, \ldots, a_n \in R$ such that $a_1u + a_2\varphi(u) + \cdots + a_n\varphi^{n-1}(u) = \psi(u)$ for all $u \in M$, in other words, ψ is a polynomial of φ.*

4.3. Theorem. *Let R be a local ring and $\sigma \in \mathrm{End}_R(M)$. Then $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$ if and only if there is an R-generating set $\{y_j \in M \mid 1 \leqslant j \leqslant d\}$ of ${}_RM$ and there are elements $a_1, a_2, \ldots, a_n$ in R such that*

$$a_1\psi(y_j) + a_2\varphi\big(\psi(y_j)\big) + \cdots + a_n\varphi^{n-1}\big(\psi(y_j)\big) = \sigma\big(\psi(y_j)\big)$$

for all $1 \leqslant j \leqslant d$ and all $\psi \in \mathrm{Cen}(\varphi)$.

Proof. If $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$, then $a_1 y_j + a_2\varphi(y_j) + \cdots + a_n\varphi^{n-1}(y_j) = \sigma(y_j)$ implies that $a_1\psi(y_j) + a_2\varphi(\psi(y_j)) + \cdots + a_n\varphi^{n-1}(\psi(y_j)) = \sigma(\psi(y_j))$ for all $\psi \in \mathrm{Cen}(\varphi)$. For any $\delta \in \Gamma$ we have $\varepsilon_\delta \in \mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$, where $\varepsilon_\delta : M \to N'_\delta$ is the natural projection corresponding to the direct sum $M = N'_\delta \oplus N''_\delta$, with

$$N'_\delta = \bigoplus_{1\leqslant i\leqslant k_\delta} Rx_{\delta,i} \quad \text{and} \quad N''_\delta = \bigoplus_{\gamma\in\Gamma\setminus\{\delta\},\, 1\leqslant i\leqslant k_\gamma} Rx_{\gamma,i}.$$

Thus $\sigma : \mathrm{im}(\varepsilon_\delta) \to \mathrm{im}(\varepsilon_\delta)$ and so $\sigma(x_{\delta,1}) = \sum_{1\leqslant i\leqslant k_\delta} a_{\delta,i}x_{\delta,i} = h_\delta(z) * x_{\delta,1}$ for some $h_\delta(z) = a_{\delta,1} + a_{\delta,2}z + \cdots + a_{\delta,k_\delta}z^{k_\delta-1}$ in $R[z]$. Since $\varphi \in \mathrm{Cen}(\sigma)$ implies that $\sigma \in \mathrm{Cen}(\varphi)$, it follows that $\sigma(\Phi(\mathbf{f})) = \sum_{\gamma\in\Gamma}\sigma(f_\gamma(z) * x_{\gamma,1}) = \sum_{\gamma\in\Gamma} f_\gamma(z) * \sigma(x_{\gamma,1}) = \sum_{\gamma\in\Gamma} f_\gamma(z) * (h_\gamma(z) * x_{\gamma,1}) = \sum_{\gamma\in\Gamma}(f_\gamma(z)h_\gamma(z)) * x_{\gamma,1} = \Phi(\mathbf{fH})$, where $\mathbf{f} \in (R[z])^\Gamma$ and $\mathbf{H} = \sum_{\gamma\in\Gamma} h_\gamma(z)\mathbf{E}_{\gamma,\gamma}$ is a diagonal matrix in $\mathcal{M}(X)$ ($\mathbf{H} \in \mathcal{M}(X)$ is a consequence of $\sigma(\Phi(\mathbf{f})) = \Phi(\mathbf{fH})$). By Theorem 3.9, the containment $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$ is equivalent to the condition that $\sigma \circ \psi_\mathbf{P} = \psi_\mathbf{P} \circ \sigma$ for all $\mathbf{P} \in \mathcal{M}(X)$. As a consequence, we obtain that $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$ is equivalent to $\Phi(\mathbf{fPH}) = \sigma(\Phi(\mathbf{fP})) = \sigma(\psi_\mathbf{P}(\Phi(\mathbf{f}))) = \psi_\mathbf{P}(\sigma(\Phi(\mathbf{f}))) = \psi_\mathbf{P}(\Phi(\mathbf{fH})) = \Phi(\mathbf{fHP})$ for all $\mathbf{f} \in (R[z])^\Gamma$ and $\mathbf{P} \in \mathcal{M}(X)$. Thus $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$ implies $\Phi(\mathbf{f}(\mathbf{PH} - \mathbf{HP})) = 0$ or $\mathbf{f}(\mathbf{PH} - \mathbf{HP}) \in \ker(\Phi)$. Take $\mathbf{e} = (1)_{\gamma\in\Gamma}$ and $\mathbf{E}_{1,\delta} \in \mathcal{N}(X)$ by Remark 3.2. Then the δ-coordinate of $\mathbf{e}(\mathbf{E}_{1,\delta}\mathbf{H} - \mathbf{HE}_{1,\delta}) = (h_\delta(z) - h_1(z))\mathbf{eE}_{1,\delta} \in \ker(\Phi)$ is $h_\delta(z) - h_1(z)$. Since R is local, $\mathbf{P} = \mathbf{E}_{1,\delta} \in \mathcal{M}(X)$ by the last part of Lemma 3.3. Now Proposition 3.1 gives that $h_\delta(z) - h_1(z) \in J[z] + (z^{k_\delta})$. Thus $\sigma(x_{\delta,1}) = h_\delta(z) * x_{\delta,1} = h_1(z) * x_{\delta,1}$ for all $\delta \in \Gamma$. It follows that $\sigma(x_{\gamma,i}) = \sigma(\varphi^{i-1}(x_{\gamma,1})) = \varphi^{i-1}(\sigma(x_{\gamma,1})) = \varphi^{i-1}(h_1(z) * x_{\gamma,1}) = h_1(z) * \varphi^{i-1}(x_{\gamma,1}) = h_1(z) * x_{\gamma,i} = a_1x_{\gamma,i} + a_2\varphi(x_{\gamma,i}) + \cdots + a_n\varphi^{n-1}(x_{\gamma,i})$, where $h_1(z) = a_1 + a_2z + \cdots + a_nz^{n-1}$.

Conversely, $1_M \in \mathrm{Cen}(\varphi)$ gives $a_1y_j + a_2\varphi(y_j) + \cdots + a_n\varphi^{n-1}(y_j) = \sigma(y_j)$ for all $1 \leqslant j \leqslant d$. Then $\psi(\sigma(y_j)) = a_1\psi(y_j) + a_2\varphi(\psi(y_j)) + \cdots + a_n\varphi^{n-1}(\psi(y_j)) = \sigma(\psi(y_j))$ for all $\psi \in \mathrm{Cen}(\varphi)$ and $1 \leqslant j \leqslant d$. Thus $\psi \circ \sigma = \sigma \circ \psi$ and so $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$. □

4.4. Corollary. *If in addition R is commutative, then $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$ if and only if there are $a_1, a_2, \ldots, a_n \in R$ such that $a_1u + a_2\varphi(u) + \cdots + a_n\varphi^{n-1}(u) = \sigma(u)$ for all $u \in M$, in other words, σ is a polynomial of φ.*

4.5. Remark. Since $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$ is equivalent to $\sigma \in \mathrm{Cen}(\mathrm{Cen}(\varphi))$, we may consider Theorem 4.3 as some kind of double centralizer theorem.

5. The centralizer of an arbitrary linear map

If K is an algebraically closed field and $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ is the set of all eigenvalues of $A \in M_n(K)$, then $\mathrm{Cen}(A)$ is isomorphic to the direct product of the centralizers $\mathrm{Cen}(A_i)$, where A_i denotes the block diagonal matrix consisting of all Jordan blocks of A having eigenvalue λ_i in the diagonal. The number of the diagonal blocks in A_i is $\dim(\ker(A_i - \lambda_iI_i))$, and the size of A_i is $d_i \times d_i$, where d_i is the multiplicity of the root λ_i in the characteristic polynomial of A. Since $\mathrm{Cen}(A_i) = \mathrm{Cen}(A_i - \lambda_iI_i)$ and $A_i - \lambda_iI_i$ is nilpotent in $M_{d_i}(K)$, we shall consider the case of a nilpotent matrix.

5.1. Theorem. *If $A \in M_d(K)$ is nilpotent of index n, then $\mathrm{Cen}(A)/J(\mathrm{Cen}(A)) \cong M_{q_1}(K) \oplus \cdots \oplus M_{q_n}(K)$, where q_e is the number of elementary Jordan matrices of size $e \times e$ and $M_{q_e}(K) = \{0\}$ if $q_e = 0$. The index of nilpotency of $J(\mathrm{Cen}(A))$ is bounded from above by $n\nu$, where ν is the number of different sizes.*

Proof. Now $A \in \mathrm{End}_K(K^d)$ has a nilpotent Jordan normal base X in K^d with block sizes $n = k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m \geqslant 1$, and Theorem 3.11 gives an isomorphism $\mathrm{Cen}(A) \cong \mathcal{N}^{\mathrm{op}}(X)/\mathcal{I}(X)$ of K-algebras. Let $T_i = K[z]/(z^{k_i})$, and to minimize the "noise" in the matrix below, we use z instead of $z + (z^{k_i})$ in T_i for the K-algebra

$$\mathcal{C}_A = \begin{bmatrix} T_1 & z^{k_1-k_2}T_1 & \cdots & \cdots & z^{k_1-k_m}T_1 \\ T_2 & T_2 & z^{k_2-k_3}T_2 & \cdots & z^{k_2-k_m}T_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ T_{m-1} & T_{m-1} & \cdots & T_{m-1} & z^{k_{m-1}-k_m}T_{m-1} \\ T_m & T_m & \cdots & T_m & T_m \end{bmatrix}.$$

Thus the map $\mathbf{P} + \mathcal{I}(X) \mapsto [p_{i,j}(z) + (z^{k_j})]^\top$ is well defined and provides an $\mathcal{N}^{\mathrm{op}}(X)/\mathcal{I}(X) \to \mathcal{C}_A$ isomorphism of K-algebras, where $\mathbf{P} = [p_{i,j}(z)]$ is in $\mathcal{N}(X)$ and $^\top$ denotes the transpose. Recall that the Jacobson radical of a finite dimensional algebra is equal to the maximal nilpotent ideal of the algebra. The $K[z]$-module

$$\mathcal{T}_A = \begin{bmatrix} T_1 & T_1 & \cdots & T_1 \\ T_2 & T_2 & \cdots & T_2 \\ \vdots & \vdots & \ddots & \vdots \\ T_m & T_m & \cdots & T_m \end{bmatrix}$$

satisfies $z^{k_1}\mathcal{T}_A = \{0\}$. The intersection $I = z\mathcal{T}_A \cap \mathcal{C}_A$ is an ideal of $\mathcal{C}_A$ and $I^n = I^{k_1} = \{0\}$, thus $I \subseteq J(\mathcal{C}_A)$. We obtain that $\mathcal{C}_A/I$ is a lower block triangular matrix algebra with diagonal blocks of size $q_{t_1} \times q_{t_1}, q_{t_2} \times q_{t_2}, \ldots, q_{t_v} \times q_{t_v}$, where $k_1 = t_1 > t_2 > \cdots > t_v = k_m \geqslant 1$ are the different block sizes (the strictly decreasing sequence of the different k_i's) appearing in X. The strictly lower triangular part

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ M_{q_{t_2} \times q_{t_1}}(K) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{q_{t_v} \times q_{t_1}}(K) & \cdots & M_{q_{t_v} \times q_{t_{v-1}}}(K) & 0 \end{bmatrix}$$

of $\mathcal{C}_A/I$ is nilpotent of index v and is equal to the radical of $\mathcal{C}_A/I$. Consequently, $(J(\mathcal{C}_A)^v)^n \subseteq I^n = \{0\}$ and the index of nilpotency of $J(\mathcal{C}_A)$ is bounded by nv. Clearly, $\mathcal{C}_A/J(\mathcal{C}_A) \cong M_{q_{t_1}}(K) \oplus \cdots \oplus M_{q_{t_v}}(K)$. $\square$

Note that the form of the centralizer $\mathcal{C}_A$ in Theorem 5.1 is a classically known object that can be found, for instance, in [2, Chapter VIII, §2, pp. 220–224] or in [8]. Hence Theorem 5.1 could have been observed without the results of this paper, even if it is a by-product of our general approach.

Recall that the PI-degree $\mathrm{PIdeg}(S)$ of a PI-algebra S is equal to the maximum p such that the multilinear polynomial identities of S follow from the multilinear polynomial identities of $M_p(K)$.

5.2. Corollary. *Let A be an $n \times n$ matrix over an algebraically closed field K and let p be the maximum number of equal elementary Jordan matrices in the canonical Jordan form of A over the algebraic closure of K. Then* $\mathrm{PIdeg}(\mathrm{Cen}(A)) = p$.

Proof. For a finite dimensional K-algebra S with Jacobson radical J the PI-degree of S is equal to the maximal size of the matrix subalgebras of S/J. Applying Theorem 5.1 one completes the proof. $\square$

5.3. Remark. Corollary 5.2 holds for all fields. If K is not algebraically closed, then a detailed argument in [1] shows how the algebraic closure of K can be used.

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References

[1] V. Drensky, J. Szigeti, L. van Wyk, Centralizers in endomorphism rings, arXiv:0910.2357v1 [math.RA], 13 October 2009.
[2] F.R. Gantmacher, The Theory of Matrices, Chelsea Publishing Co., New York, 2000.
[3] A. Giambruno, M. Zaicev, Polynomial Identities and Asymptotic Methods, Math. Surveys Monogr., vol. 122, Amer. Math. Soc., Providence, RI, 2005.
[4] M.G. Neubauer, B.A. Sethuraman, Commuting pairs in the centralizers of 2-regular matrices, J. Algebra 214 (1999) 174–181.
[5] V.V. Prasolov, Problems and Theorems in Linear Algebra, Transl. Math. Monogr., vol. 134, Amer. Math. Soc., Providence, RI, 1994.
[6] D.A. Suprunenko, R.I. Tyshkevich, Commutative Matrices, Academic Press, New York, London, 1968.
[7] J. Szigeti, Linear algebra in lattices and nilpotent endomorphisms of semisimple modules, J. Algebra 319 (2008) 296–308.
[8] H.W. Turnbull, A.C. Aitken, An Introduction to the Theory of Canonical Matrices, Dover Publications, 2004.