Linear Algebra and its Applications 434 (2011) 1018–1026

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Automorphism groups of generalized triangular matrix rings ☆

P.N. Ánh [a], L. van Wyk [b,*]

[a] Rényi Institute of Mathematics, Hungarian Academy of Sciences, 1364 Budapest, Pf. 127, Hungary
[b] Department of Mathematical Sciences, Stellenbosch University, P/Bag X1, Matieland 7602, Stellenbosch, South Africa

A R T I C L E I N F O

Article history:
Received 10 June 2009
Accepted 14 October 2010
Available online 11 November 2010

Submitted by T. Laffey

AMS classification:
16S50
15A33
16D20

Keywords:
Automorphism group
Upper triangular matrix ring
Strongly indecomposable ring
Bimodule isomorphism

A B S T R A C T

We call a ring strongly indecomposable if it cannot be represented as a non-trivial (i.e. $M \neq 0$) generalized triangular matrix ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, for some rings R and S and some R-S-bimodule ${}_RM_S$. Examples of such rings include rings with only the trivial idempotents 0 and 1, as well as endomorphism rings of vector spaces, or more generally, semiprime indecomposable rings. We show that if R and S are strongly indecomposable rings, then the triangulation of the non-trivial generalized triangular matrix ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is unique up to isomorphism; to be more precise, if $\varphi : \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \to \begin{pmatrix} R' & M' \\ 0 & S' \end{pmatrix}$ is an isomorphism, then there are isomorphisms $\rho : R \to R'$ and $\psi : S \to S'$ such that $\chi := \varphi_{|M} : M \to M'$ is an R-S-bimodule isomorphism relative to ρ and ψ. In particular, this result describes the automorphism groups of such upper triangular matrix rings $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$.

☆ The research was carried out in accordance with the Hungary/South Africa Agreement on Cooperation in Science and Technology. In particular, the first author was supported by the Hungarian National Foundation for Scientific Research under Grant Nos. K61007 and NK72523, and the second author was supported by the National Research Foundation of South Africa under Grant No. UID 61857. Any opinion, findings and conclusions or recommendations expressed in this material are those of the authors and therefore the National Research Foundation does not accept any liability in regard thereto.

* Corresponding author.
E-mail addresses: anh@renyi.hu (P.N. Ánh), LvW@sun.ac.za (L. van Wyk).

0024-3795/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2010.10.007

1. Introduction

Triangular matrix rings appear naturally in Lie theory of both nilpotent and solvable Lie algebras. Since then they have become an important ring construction; indeed a main tool in the description of semiprimary hereditary rings (see, for example, [7]). For this reason the isomorphism of triangular matrix rings, in particular, the automorphism group of a triangular matrix ring, is an interesting field of study. In this paper we compute the automorphism groups of certain generalized triangular matrix rings. Recall that a ring R is a *generalized upper triangular matrix ring of size $n \times n$* if there is a decomposition $1 = e_1 + \cdots + e_n$, where the e_i's are pairwise orthogonal idempotents such that $e_iRe_j = 0$ for all $i > j$. For further information on this topic we refer to [3,8,10,11].

The isomorphism problem, in the other words, the recovery of the underlying Boolean matrix involved in a structural matrix ring, or equivalently, the underlying preordered set in an incidence ring, is an essential aim in the investigation of generalized matrix rings, and was studied in [1,4,5]. The uniqueness of the tile in the non-diagonal position in a 2×2 upper triangular tiled matrix ring, i.e. in position $(1, 2)$, was considered in [6]. However, as was shown by an example in [4], additional conditions are necessary in order to obtain a positive result.

Consequently a search was initiated in [9] for a possible recovery. It was shown there for a generalized triangular matrix ring $A := \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ that if R and S have only the two trivial idempotents 0 and 1, for example, if R and S are indecomposable commutative rings or local rings (not necessarily commutative), then R and S can be recovered up to isomorphism, and although it turns out (see Lemma 2.2) that M is independent of the choice of isomorphic copies of R and S in A, in general M cannot be recovered in the ordinary sense of an R-S-bimodule, but it can indeed be recovered relative to an automorphism of R and an automorphism of S.

It is the purpose of this paper to extend the latter result from the class of rings with only the two trivial idempotents to the class of strongly indecomposable rings, which also includes the endomorphism ring of a vector space, or more generally, semiprime indecomposable rings. Our result provides a description of the automorphism group of such a generalized triangular matrix ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$. For commutative rings the concepts of indecomposability and strongly indecomposability coincide.

2. Generalized triangular matrix rings

We say that a ring A with identity 1 (say) *admits a generalized (upper) triangular matrix ring decompostion* if there is a non-trivial idempotent e (i.e. $e \neq 0$ and $e \neq 1$) in A such that $fAe = 0$, with $f := 1 - e$. Then e and f are orthogonal idempotents, i.e. $ef = 0 = fe$, and $e + f = 1$. Therefore,

$$a = (e + f)a(e + f) = eae + eaf + faf \tag{1}$$

for every $a \in A$. Consequently

$$A = eAe \oplus eAf \oplus fAf,$$

and thus the following assertion follows immediately.

Lemma 2.1. *A is isomorphic to the generalized triangular matrix ring*

$$\begin{pmatrix} eAe & eAf \\ 0 & fAf \end{pmatrix} \quad \text{via} \quad \varphi : a \mapsto \begin{pmatrix} eae & eaf \\ 0 & faf \end{pmatrix}, \quad a \in A.$$

If R and S are rings with identity and $_RM_S$ is an arbitrary unitary R-S-bimodule, then one can form a generalized triangular matrix ring

$$A := \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : \ r \in R, \ s \in S, \ m \in M \right\}$$

with the usual matrix addition and the multiplication induced by the bimodule actions and the usual rule for matrix multiplication. If we denote the idempotent $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in A by e, then it follows that

$$eAe = \left\{ \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} : \; r \in R \right\} \cong R, \tag{2}$$

and so we can identify eAe with R, and similarly, eAf and fAf with M and S respectively, where $f = 1 - e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. It is worth noting that if one starts with commutative rings R and S and a bimodule ${}_RM_S$, then, although there is no need to distinguish between left and right modules for commutative rings, the action of S on M from the left, induced by multiplication in A, is trivial, i.e. $(fAf)(eAf) = 0$.

We point out that if a ring A admits a generalized triangular matrix ring decomposition, then such a decomposition is, in general, not unique. In fact, for any element $x \in eAf$, set

$$e_x := e + x \quad \text{and} \quad f_x := 1 - e_x = f - x.$$

Since $x = exf$, we have that

$$ex = x = xf,$$

and similarly,

$$xe = 0 = fx, \;\text{ and }\; x^2 = 0. \tag{3}$$

Therefore,

$$e_x^2 = e^2 + ex + xe + x^2 = e + x = e_x \;\text{ and }\; f_x^2 = f.$$

Also,

$$e_x e = (e + x)e = e, \; ee_x = e_x, \; ff_x = f \;\text{ and }\; f_x f = f_x. \tag{4}$$

Lemma 2.2. *(i)* $e_xAe_x = \{u + ux : \; u \in eAe\} \cong eAe$
(ii) $e_xAf_x = eAf$
(iii) $f_xAf_x = \{v - xv : \; v \in fAf\} \cong fAf$
(iv) $f_xAe_x = 0$.

Proof. (i) For $a \in A$ we obtain from (1) and (3) that

$$\begin{aligned} e_xae_x &= (e + x)(eae + eaf + faf)(e + x) \\ &= (eae + eaf + xaf)(e + x) \\ &= eae + eaex, \end{aligned}$$

which shows that $e_xAe_x = \{u + ux : \; u \in eAe\}$. Define $\varphi : eAe \to \{u + ux : \; u \in eAe\}$ by $\varphi(u) = u + ux, \; u \in eAe$. Clearly, φ is additive and onto. If $\varphi(u) = 0$, then, keeping in mind that $x \in eAf$ and $fe = 0$, we obtain that

$$u = ue = ue + uxe = (u + ux)e = \varphi(u)e = 0,$$

and so φ is 1-1. Next, if $u' \in eAe$, then using $xu' = 0$ it follows that $\varphi(u)\varphi(u') = (u + ux)(u' + u'x) = uu' + uu'x$, showing that φ is multiplicative.

(ii) Since $eAf, e_xAf_x \subseteq A$, we conclude from (4) that $e_xAf_x \supseteq e_x(eAf)f_x = eAf \supseteq e(e_xAf_x)f = e_xAf_x$.

(iii) Similar to the proof of (i).

(iv) By (4), $f_xAe_x = (f - x)(eAe + eAf + fAf)(e + x) = (fAf - xAf)(e + x) = 0$. $\square$

3. Isomorphism of generalized triangular matrix rings

We call a ring A with 1 *strongly (upper) indecomposable* if, whenever there is a nonzero idempotent e in A with $(1 - e)Ae = 0$, then $e = 1$. (We note that such a ring is called a semicentral reduced ring in [2].) One can dually define a generalized lower triangular matrix ring decomposition, as well as strongly (lower) indecomposability.

Remark. Note that strong indecomposability is indeed a stronger concept than indecomposability. To wit, the 2×2 upper triangular matrix ring $\mathcal{U}_2(k) := \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ over a field k is not strongly (upper) indecomposable, since $(1-e)\mathcal{U}_2(k)e = 0$ with $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. However, noting that the only non-trivial idempotents in $\mathcal{U}_2(k)$ are of the form $\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$ for an arbitrary element x in the field k, it can be verified that there is no non-trivial idempotent e in $\mathcal{U}_2(k)$ such that $(1-e)\mathcal{U}_2(k)e = 0 = e\mathcal{U}_2(k)(1-e)$, and so we obtain the well known fact that $\mathcal{U}_2(k)$ is indeed an indecomposable ring.

Clearly a ring having only the trivial idempotents 0 and 1 is strongly indecomposable, as is the endomorphism ring of a vector space, or more generally:

Proposition 3.1. *If A is a semiprime indecomposable ring, then A is strongly indecomposable.*

Proof. Let e be a nonzero idempotent in A such that $(1-e)Ae = 0$. Then it follows readily from (1) that $eA(1-e)$ is an ideal of A. Since $(eA(1-e))^2 = 0$, the semiprimeness of A implies that $eA(1-e) = 0$. We conclude from the indecomposability of A that $e = 1$, and so A is strongly indecomposable. $\square$

Theorem 3.2. *Let $A_i := \begin{pmatrix} R_i & M_i \\ 0 & S_i \end{pmatrix}$, $i = 1, 2$, be a generalized triangular matrix ring, and consider a function $\varphi : A_1 \to A_2$. Assume that one of the pairs (R_i, S_i) consists of strongly indecomposable rings.*

(I) If φ is an isomorphism, then $M_1 = 0$ if and only if $M_2 = 0$, in which case either $R_1 \cong R_2$ and $S_1 \cong S_2$, or $R_1 \cong S_2$ and $S_1 \cong R_2$.

(II) If $M_1 \neq 0$ and $M_2 \neq 0$, then φ is an isomorphism if and only if there is a quadruple (ρ, ψ, m, χ), where $\rho : R_1 \to R_2$ and $\psi : S_1 \to S_2$ are ring isomorphisms, $m \in M_2$ and $\chi : M_1 \to M_2$ is an R_1-S_1-bimodule isomorphism relative to ρ and ψ, such that

$$\varphi \begin{pmatrix} r & w \\ 0 & s \end{pmatrix} = \begin{pmatrix} \rho(r) & \rho(r)m + \chi(w) - m\psi(s) \\ 0 & \psi(s) \end{pmatrix} \text{ for all } \begin{pmatrix} r & w \\ 0 & s \end{pmatrix} \in A_1.$$

In particular, χ is the restriction $\varphi_{|M_1}$ of φ to M_1.

(II′) (Alternative formulation of II) If $M_1 \neq 0$ and $M_2 \neq 0$, then φ is an isomorphism if and only if there is a triple (ρ, ψ, m), where $\rho : R_1 \to R_2$ and $\psi : S_1 \to S_2$ are ring isomorphisms, $m \in M_2$ and $\varphi_{|M_1} : M_1 \to M_2$ is an R_1-S_1-bimodule isomorphism relative to ρ and ψ, such that

$$\varphi \begin{pmatrix} r & w \\ 0 & s \end{pmatrix} = \begin{pmatrix} \rho(r) & \rho(r)m + \varphi_{|M_1}(w) - m\psi(s) \\ 0 & \psi(s) \end{pmatrix} \text{ for all } \begin{pmatrix} r & w \\ 0 & s \end{pmatrix} \in A_1.$$

We divide the proof of part I of Theorem 3.2 essentially into a number of lemmas, namely Lemmas 3.3–3.5. In these, as well as in the statements of Corollaries 3.6 and 3.7, we assume, without stating it each time, that

$\varphi : A_1 \to A_2$ is an isomorphism,

with $A_i = \begin{pmatrix} R_i & M_i \\ 0 & S_i \end{pmatrix}$ a generalized triangular matrix ring, $i = 1, 2$.

Lemma 3.3. *If R_1 is strongly indecomposable, then*

$$\varphi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha m \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & m\beta \\ 0 & \beta \end{pmatrix}$$

for some $\alpha \in R_2$, $\beta \in S_2$, $m \in M_2$.

Proof. Set $e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in A_1$ and $\varphi(e) =: \begin{pmatrix} \alpha & m \\ 0 & \beta \end{pmatrix}$ for some $\alpha \in R_2, \beta \in S_2$ and $m \in M_2$. Since

$$\begin{pmatrix} \alpha^2 & \alpha m + m\beta \\ 0 & \beta^2 \end{pmatrix} = \begin{pmatrix} \alpha & m \\ 0 & \beta \end{pmatrix}^2 = (\varphi(e))^2 = \varphi(e),$$

it follows that

$$\alpha^2 = \alpha,\ \beta^2 = \beta \text{ and } \alpha m + m\beta = m. \tag{5}$$

Hence,

$$\varphi(e) = C + D =: E, \tag{6}$$

with

$$C := \begin{pmatrix} \alpha & \alpha m \\ 0 & 0 \end{pmatrix} \text{ and } D := \begin{pmatrix} 0 & m\beta \\ 0 & \beta \end{pmatrix}. \tag{7}$$

Multiplying the last equation in (5) from the left by α, we deduce that $\alpha m \beta = 0$, and so we conclude from (5) and (7) that

$$C^2 = C,\ D^2 = D \text{ and } CD = 0 = DC. \tag{8}$$

Since $C = (C + D)C(C + D)$ and $D = (C + D)D(C + D)$, it follows that $C, D \in EA_2E = \varphi(eA_1e)$. Consequently (2) implies that there are elements $a, b \in R_1$ such that

$$\varphi \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = C \text{ and } \varphi \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = D. \tag{9}$$

We conclude from (6), (8) and (9) that

$$a^2 = a,\ b^2 = b,\ ab = 0 = ba \text{ and } a + b = 1.$$

Since

$$\begin{aligned} D\begin{pmatrix} p & q \\ 0 & r \end{pmatrix} C &= \begin{pmatrix} 0 & m\beta \\ 0 & \beta \end{pmatrix}\begin{pmatrix} p & q \\ 0 & r \end{pmatrix}\begin{pmatrix} \alpha & \alpha m \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & m\beta r \\ 0 & \beta r \end{pmatrix}\begin{pmatrix} \alpha & \alpha m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

for every $\begin{pmatrix} p & q \\ 0 & r \end{pmatrix} \in A_2$, it follows from (9) that

$$\varphi \begin{pmatrix} bR_1a & 0 \\ 0 & 0 \end{pmatrix} = \varphi \left(\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} A_1 \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) = DA_2C = 0,$$

forcing bR_1a to be 0. Since R_1 is strongly indecomposable, we deduce that $a = 0$ or $b = 0$, and so the desired result follows from (6) and (9). $\square$

Lemma 3.4. *If R_1 and S_1 are strongly indecomposable, then*

$$\varphi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & m \\ 0 & 1 \end{pmatrix}$$

for some $m \in M_2$.

Proof. By Lemma 3.3,

$$\varphi(e) = \begin{pmatrix} \alpha & \alpha m \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & m\beta \\ 0 & \beta \end{pmatrix}$$

and so

$$\varphi\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \varphi(1-e) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}\alpha & \alpha m\\ 0 & 0\end{pmatrix} = C' + D', \tag{10}$$

with

$$C' := \begin{pmatrix}1-\alpha & 0\\ 0 & 0\end{pmatrix} \text{ and } D' := \begin{pmatrix}0 & -\alpha m\\ 0 & 1\end{pmatrix}, \tag{11}$$

or

$$\varphi\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \varphi(1-e) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0 & m\beta\\ 0 & \beta\end{pmatrix} = C' + D', \tag{12}$$

with

$$C' := \begin{pmatrix}1 & -m\beta\\ 0 & 0\end{pmatrix} \text{ and } D' := \begin{pmatrix}0 & 0\\ 0 & 1-\beta\end{pmatrix}. \tag{13}$$

Without loss of generality, assume the second possibility, i.e. assume that (12) and (13) hold. Since

$$(C')^2 = C',\ (D')^2 = D' \text{ and } C'D' = 0 = D'C',$$

arguments similar to the ones above show that $C', D' \in \varphi((1-e)A_1(1-e))$, that there are elements $c, d \in S_1$ such that

$$\varphi\begin{pmatrix}0 & 0\\ 0 & c\end{pmatrix} = C' \text{ and} \varphi\begin{pmatrix}0 & 0\\ 0 & d\end{pmatrix} = D',$$

and that

$$\varphi\begin{pmatrix}0 & 0\\ 0 & dS_1c\end{pmatrix} = D'A_2C' = 0,$$

forcing dS_1c to be 0. The strongly indecomposability of S_1 implies that $c = 0$ or $d = 0$, which in turn leads to

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = \varphi\begin{pmatrix}0 & 0\\ 0 & c\end{pmatrix} = C' = \begin{pmatrix}1 & -m\beta\\ 0 & 0\end{pmatrix} \tag{14}$$

or

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = \varphi\begin{pmatrix}0 & 0\\ 0 & d\end{pmatrix} = D' = \begin{pmatrix}0 & 0\\ 0 & 1-\beta\end{pmatrix}. \tag{15}$$

Since $1 \neq 0$, we conclude that (14) cannot hold, and so (15) must be true, i.e. $\beta = 1$. □

The following result should be seen against the background of Example 1.1 in [4], where it was shown that it can happen that all possible 2×2 structural matrix rings over a ring R are isomorphic to one another; in particular, it is possible that

$$\begin{pmatrix}R & R\\ 0 & R\end{pmatrix} \cong \begin{pmatrix}R & 0\\ 0 & R\end{pmatrix}.$$

Lemma 3.5. *If R_1 and S_1 are strongly indecomposable, then $M_1 = 0$ if and only if $M_2 = 0$, in which case either $R_1 \cong R_2$ and $S_1 \cong S_2$, or $R_1 \cong S_2$ and $S_1 \cong R_2$.*

Proof. Assume that $M_1 = 0$, and use the notation $e = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \in A_1$ and $\varphi(e) = E$, as before. By (2), $eA_1e = \begin{pmatrix}R_1 & 0\\ 0 & 0\end{pmatrix}$, and so $eA_1e \lhd A_1$ (i.e. eA_1e is an ideal of A_1), which implies that $EA_2E \lhd A_2$.

Invoking Lemma 3.4, we first consider the case $E = \begin{pmatrix}1 & m\\ 0 & 0\end{pmatrix}$ for some $m \in M_2$. If x is an arbitrary element in M_2, then using $\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}$ as an element in A_2 and the fact that $EA_2E \lhd A_2$, we conclude that

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in (EA_2E)A_2 \subseteq EA_2E. \tag{16}$$

Since E is the identity of the ring EA_2E, it follows that

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}. \tag{17}$$

However, matrix multiplication tells us that $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and so $x = 0$. Therefore, $M_2 = 0$.

Next we consider the second possibility in Lemma 3.4, i.e. $E = \begin{pmatrix} 0 & m \\ 0 & 1 \end{pmatrix}$ for some $m \in M_2$. Using $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & m \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ in (16) and (17) respectively, similar arguments as before show again that $M_2 = 0$.

Conversely, suppose $M_2 = 0$. Then by Lemma 3.4,

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{18}$$

and so $EA_2E = \begin{pmatrix} R_2 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix}$, which are both ideals of A_2 (since $M_2 = 0$). Hence, $eA_1e = \varphi^{-1}(EA_2E) \lhd A_1$, and using $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in (16) and (17) respectively, with e the identity of the ring eA_1e and x an arbitrary element of A_1, we deduce that $M_1 = 0$.

The last part of the statement of Lemma 3.5 is clear from (18). $\square$

The ideas in the proof of Lemma 3.5 serve to a large extent in proving the following result, which we state separately, since it will be the starting point of the proof of the main part of Theorem 3.2.

Corollary 3.6. *If $\varphi(e) = \begin{pmatrix} 0 & m \\ 0 & 1 \end{pmatrix}$ for some $m \in M_2$, then $M_2 = 0$.*

Proof. Since $eA_1 = \begin{pmatrix} R_1 & M_1 \\ 0 & 0 \end{pmatrix} \lhd A_1$, we have that $EA_2 \lhd A_2$. Hence, $A_2(EA_2E) = (A_2(EA_2))E \subseteq (EA_2)E$ and $(EA_2E)A_2 = E((A_2E)A_2)) \subseteq E(A_2E)$, implying that $EA_2E \lhd A_2$. The proof is concluded by adapting the appropriate arguments in the proof of Lemma 3.5. $\square$

The foregoing results give rise to

Corollary 3.7. *If $M_1 \neq 0$ and $M_2 \neq 0$, then $\varphi(e) = \begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix}$ for some $m \in M_2$.*

We are now in a position to complete the proof of Theorem 4.

Proof of part II of Theorem 3.2. Since the "if" part is direct verification, it suffices to prove the "only if" part of II.

To this end, assume that R_1 and R_2 are strongly indecomposable, that M_1 and M_2 are nonzero and that φ is an isomorphism. Let m be the fixed element in M_2 for which $\varphi(e) = \begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix}$, as in Corollary 3.7.

Let $x \in M_1$, which by the remark following (2) is isomorphic to $eA_1(1-e) = \begin{pmatrix} 0 & M_1 \\ 0 & 0 \end{pmatrix}$. By Lemma 2, $e_x := \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$ and $e_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e$ play the same role for a generalized triangular matrix ring decomposition of A_1. Hence Corollary 3.7 implies that $\varphi(e_x) = \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix}$ for some element $n \in M_2$ which is uniquely determined by x. Thus we have obtained a function

$\xi : M_1 \to M_2$ by setting $\xi(x) := n$.

Define a new function

$\chi : M_1 \to M_2$ by setting $\chi(x) := \xi(x) - m = n - m$.

Since

$$\begin{aligned} \varphi\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} &= \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \varphi\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} - \varphi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & n-m \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

we deduce that

$$\varphi(x) = \chi(x) \text{ for every } x \in M_1, \tag{19}$$

i.e. χ is precisely the restriction $\varphi_{|M_1}$ of φ to M_1.

Denoting $\varphi(e) = \begin{pmatrix} 1 & m \\ 0 & 0 \end{pmatrix}$ by $\bar{e}_m$ (instead of E), and $1 - e$ by f, it follows that

$$\varphi(f) = \varphi\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \varphi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -m \\ 0 & 1 \end{pmatrix} =: \bar{f}_m.$$

Hence Lemma 2.2 implies that

$$R_1 \cong eA_1e \cong \varphi(eA_1e) = \bar{e}_m A_2 \bar{e}_m \cong R_2,$$

and

$$S_1 \cong fA_1f \cong \varphi(fA_1f) = \bar{f}_m A_2 \bar{f}_m \cong S_2.$$

For $r \in R_1, w \in M_1$ and $s \in S_1$ we have

$$\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & rws \\ 0 & 0 \end{pmatrix},$$

and so, identifying this matrix in $\begin{pmatrix} R_1 & M_1 \\ 0 & S_1 \end{pmatrix}$ again with $rws \in M_1$, we deduce from (19) that

$$\chi(rws) = \varphi(rws) = \varphi(r)\varphi(w)\varphi(s) = \varphi(r)\chi(w)\varphi(s) = \rho(r)\chi(w)\psi(s), \tag{20}$$

if

$$\rho := \varphi_{|R_1} :\to \bar{e}_m A_2 \bar{e}_m = \left\{\begin{pmatrix} a & am \\ 0 & 0 \end{pmatrix} : a \in R_2\right\} \cong R_2 \tag{21}$$

and

$$\psi := \varphi_{|S_1} :\to \bar{f}_m A_2 \bar{f}_m = \left\{\begin{pmatrix} 0 & -mb \\ 0 & b \end{pmatrix} : b \in S_2\right\} \cong S_2, \tag{22}$$

where, by Lemma 2.2, the last isomorphisms in (21) and (22) are given by

$$\begin{pmatrix} a & am \\ 0 & 0 \end{pmatrix} \mapsto a \quad \text{and} \quad \begin{pmatrix} 0 & -mb \\ 0 & b \end{pmatrix} \mapsto b$$

respectively. Therefore (20) implies that $\chi : M_1 \to M_2$ is a is an R_1-S_1-bimodule isomorphism relative to ρ and ψ, and

$$\begin{aligned} \varphi\begin{pmatrix} r & w \\ 0 & s \end{pmatrix} &= \varphi\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} + \varphi\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} + \varphi\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} \\ &= \begin{pmatrix} \rho(r) & \rho(r)m \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \chi(m) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -m\psi(s) \\ 0 & \psi(s) \end{pmatrix} \\ &= \begin{pmatrix} \rho(r) & \rho(r)m + \chi(w) - m\psi(s) \\ 0 & \psi(s) \end{pmatrix} \end{aligned}$$

for all $\begin{pmatrix} r & w \\ 0 & s \end{pmatrix} \in \begin{pmatrix} R_1 & M_1 \\ 0 & S_1 \end{pmatrix}$, which completes the proof. $\square$

Corollary 3.8. *Let* $A := \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *be a generalized triangular matrix ring, with* $M \neq 0$ *and* R *and* S *strongly indecomposable. Then the automorphism group* $Aut(A)$ *of* A *is a subgroup of the direct product of* $Aut(R)$, $Aut(S)$, M *(as an abelian group) and* $Aut(M)$, *consisting of the quadruples* (ρ, ψ, m, χ), *where* $\rho \in \mathrm{Aut}(R)$, $\psi \in \mathrm{Aut}(S)$, $m \in M$ *and* $\chi \in \mathrm{Aut}(M)$, *such that* χ *is an* $R - S$*-bimodule automorphism of* M *relative to* ρ *and* ψ.

Example 3.9. 1. Let $R = S = \mathbb{Z}_p$ be the prime field of characteristic p, and let M be an elementary abelian p-group M. Then for the generalized triangular matrix ring $\begin{pmatrix} \mathbb{Z}_p & M \\ 0 & \mathbb{Z}_p \end{pmatrix}$ we have Aut $\begin{pmatrix} \mathbb{Z}_p & M \\ 0 & \mathbb{Z}_p \end{pmatrix} = M \oplus \mathrm{Aut}(M)$. (Recall that the automorphism group of a finite elementary abelian p-group is the full general linear group $\mathrm{GL}(n, \mathbb{Z}_p)$ of appropriate size (n say) over the field $\mathbb{Z}_p$ of p elements.)

2. Let $R = S = F$, with $F = \mathbb{Q}$ or $F = \mathbb{R}$, the fields of rational numbers or real numbers, and let V be a vector space over F. Then Aut $\begin{pmatrix} F & V \\ 0 & F \end{pmatrix} = V \oplus \mathrm{GL}({}_F V)$.

3. Let $R = \mathbb{Q}$ and $S = \mathbb{R}$, and let V be a vector space over $\mathbb{R}$. Then Aut$\begin{pmatrix} \mathbb{Q} & V \\ 0 & \mathbb{R} \end{pmatrix} = V \oplus \mathrm{GL}(V_{\mathbb{R}})$.

Acknowledgments

Helpful discussions with L. Márki and J. Szigeti are gratefully acknowledged. The authors also wish to thank the referee for carefully reading the paper.

References

[1] G. Abrams, J. Haefner, A. de Rio, The isomorphism problem for incidence rings, Pacific J. Math. 187 (1999) 201–214.
[2] G.F. Birkenmeier, H.E. Heatherly, J.Y. Kim, J.K. Park, Triangular matrix representations, J. Algebra 230 (2000) 558–595.
[3] S.P. Coelho, The automorphism group of a structural matrix algebra, Linear Algebra Appl. 195 (1993) 35–58.
[4] S. Dăscălescu, L. van Wyk, Do isomorphic structural matrix rings have isomorphic graphs?, Proc. Amer. Math. Soc. 124 (1996) 1385–1391.
[5] S. Dăscălescu, L. van Wyk, Complete blocked triangular matrix rings over a Noetherian ring, J. Pure Appl. Algebra 133 (1998) 65–68.
[6] S. Dăscălescu, L. van Wyk, The recovery of the non-diagonal tile in a tiled triangular matrix ring, Indian J. Math. 42 (2) (2000) 167–173.
[7] C. Faith, Algebra II: Ring Theory, Springer, 1976.
[8] S. Jondrup, Automorphisms of upper triangular matrix rings, Arch. Math. (Basel) 49 (6) (1987) 497–502.
[9] R. Khazal, S. Dăscălescu, L. van Wyk, Isomorphism of generalized triangular matrix-rings and recovery of tiles, Internat. J. Math. Math. Sci. 2003 (9) (2003) 533–538.
[10] T.P. Kezlan, A note on algebra automorphisms of triangular matrices over commutative rings, Linear Algebra Appl. 135 (1990) 181–184.
[11] M.S. Li, J.M. Zelmanowitz, Artinian rings with restricted primeness conditions, J. Algebra 124 (1989) 139–148.