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ALGEBRAS GENERATED BY TWO QUADRATIC ELEMENTS

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Let K be a field of any characteristic and let R be an algebra generated by two elements satisfying quadratic equations. Then R is a homomorphic image of $F = K\langle x, y \mid x^2 + ax + b = 0, y^2 + cy + d = 0\rangle$ for suitable $a, b, c, d \in K$. We establish that F can be embedded into the 2×2 matrix algebra $M_2(\overline{K}[t])$ with entries from the polynomial algebra $\overline{K}[t]$ over the algebraic closure of K and that F and $M_2(\overline{K})$ satisfy the same polynomial identities as K-algebras. When the quadratic equations have double zeros, our result is a partial case of more general results by Ufnarovskij, Borisenko, and Belov from the 1980s. When each of the equations has different zeros, we improve a result of Weiss, also from the 1980s.

Key Words: Algebras with polynomial identity; Embedding in matrices; Free products of algebras; Idempotents; Quadratic elements.

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INTRODUCTION

In this article, K is an arbitrary field of any characteristic. All algebras are unital and over K or over its algebraic closure $\overline{K}$.

Initially this project was motivated by results on the minimum number of idempotents needed to generate various kind of algebras, see, e.g., Roch and Silbermann [11], Krupnik [8], Kelarev et al. [7], van der Merwe and van Wyk [10], Goldstein and Krupnik [5]. In particular, Krupnik [8] exhibited three idempotent matrices which generate $M_n(K)$. He also showed that any algebra generated by two idempotents satisfies the standard identity of degree four, and hence $M_n(K)$ cannot be generated by two idempotent matrices if $n \geq 3$. Independently, Weiss [14, 15] established that if an infinite dimensional algebra R is generated by two idempotents and K is of characteristic different from 2, then R is isomorphic to a subalgebra of $M_2(K[v])$.

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It is worthy to study algebras generated by idempotents and nilpotent elements, and this was another motivation for our project. Van der Merwe and van Wyk [10, Theorem 2.1] showed that the algebra $M_n(K)$, $n \geq 2$, over an arbitrary field K can be generated by an idempotent matrix and a nilpotent matrix of index n.

Our third motivation came from the well-developed theory of (noncommutative) monomial algebras. For an ideal J of the free associative algebra $K\langle x_1, \ldots, x_d\rangle$ generated by a finite set of monomials Ufnarovskij [12] associated an oriented graph and described the growth of the factor algebra $K\langle x_1, \ldots, x_d\rangle / J$ in terms of paths in the graph. In particular, the growth of such an algebra is either polynomial or exponential and its Hilbert series is rational. Borisenko [3] showed that a finitely presented monomial algebra is of polynomial growth if and only if it satisfies a polynomial identity. Then for a suitable n it can be embedded into the matrix algebra $M_n(K[t])$ over the polynomial algebra $K[t]$. This allows to treat successfully the case when the algebra has the presentation

$$F_{p,q} \cong K\langle x, y \mid x^p = y^q = 0\rangle, \qquad p, q \geq 2,$$

i.e., $F_{p,q}$ is generated by two elements and the only relations give the nilpotency of the generators. If $q \geq 3$, then it is easy to see (and it follows also from Ufnarovskij [12]) that the monomials xyx and xy^2x generate a free subalgebra of $F_{p,q}$. If $p = q = 2$, the approach of Ufnarovskij [12] and Borisenko [3] gives that the algebra $F_{2,2}$ can be embedded into $M_n(K[t])$ for a suitable $n \leq 6$. Further, Belov [1] developed the theory of monomial algebras associated with infinite periodic words. If $w = x_{i_1} \cdots x_{i_m}$ is a monomial, then the algebra A_w has a basis consisting of all finite subwords of the infinite word

$$w^{\infty} = \ldots x_{i_1} \cdots x_{i_m} x_{i_1} \cdots x_{i_m} \ldots,$$

and all monomials which are not subwords of w^{∞} are equal to 0 in A_w. Again, the algebra A_w can be embedded into $M_n(K[t])$ and, additionally, both algebras have the same polynomial identities. Tracing step-by-step the proof of Belov, one sees that the algebras $F_{2,2}$ and A_{xy} are isomorphic and can be embedded into $M_2(K[t])$. An account of the work of Ufnarovskij, Borisenko, and Belov can be found in the survey articles by Ufnarovskij [13] and Belov et al. [2]. For a background on PI-algebras see, e.g., Drensky and Formanek [4].

The purpose of this article is to describe algebras generated by two elements satisfying quadratic equations. We prove that the algebra with presentation

$$F = K\langle x, y \mid x^2 + ax + b = 0, y^2 + cy + d = 0\rangle$$

for suitable $a, b, c, d \in K$ can be embedded into the 2×2 matrix algebra $M_2(\overline{K}[t])$ with entries from the polynomial algebra $\overline{K}[t]$ over the algebraic closure of K and that F and $M_2(\overline{K})$ satisfy the same polynomial identities as K-algebras. Our embedding is similar to the embedding obtained by the methods of Belov [1]. As in Weiss [14, 15] we show that F is a free module of rank 4 over its centre and all proper homomorphic images of F are finite dimensional.

Our algebra F is a free product of two two-dimensional algebras,

$$F = K[x \mid x^2 + ax + b = 0] * K[y \mid y^2 + cy + d = 0].$$

There is an obvious analogy of F in group theory. The free product of two cyclic groups

$$G = \langle x \mid x^p = 1 \rangle * \langle y \mid y^q = 1 \rangle, \qquad p, q \geq 2,$$

contains a free subgroup if $q \geq 3$, and is metabelian (solvable of class 2) if $p = q = 2$, see, e.g., Magnus et al. [9, Problem 19, p. 195].

MAIN RESULTS

Let S be any K-algebra. Then S may be considered as a K-subalgebra of the $\overline{K}$-algebra $\overline{S} = \overline{K} \otimes_K S$ via the embedding $s \to 1 \otimes s$, $s \in S$. In particular, we assume that the free associative algebra $K\langle x_1, x_2, \ldots \rangle$ is naturally embedded into $\overline{K}\langle x_1, x_2, \ldots \rangle$. For a K-algebra R we denote by $T_K(R)$ the T-ideal of $K\langle x_1, x_2, \ldots \rangle$ consisting of all polynomial identities of R. The corresponding notation when R is a $\overline{K}$-algebra is $T_{\overline{K}}(R)$. The following fact is well known, see, e.g., Drensky and Formanek [4, p. 12, Remark 1.2.9(ii)].

Lemma 1. *If the field K is infinite, C is a commutative (unital) K-algebra, and the K-algebra S satisfies a polynomial identity, then S and $C \otimes_K S$ have the same polynomial identities. In particular, S and $\overline{S} = \overline{K} \otimes_K S$ have the same polynomial identities and $T_K(S) = T_{\overline{K}}(\overline{S}) \cap K\langle x_1, x_2, \ldots \rangle$.*

We fix $\delta, \varepsilon = 0$ or 1 and elements $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \overline{K}$, $\alpha_1, \beta_1 \neq 0$. We define the following 2×2 matrices with entries in the polynomial algebra $\overline{K}[t]$:

$$X = \begin{pmatrix} \alpha_2 + \delta\alpha_1 & \alpha_1 t \\ 0 & \alpha_2 \end{pmatrix}, \qquad Y = \begin{pmatrix} \beta_2 & 0 \\ \beta_1 t & \beta_2 + \varepsilon\beta_1 \end{pmatrix}, \qquad U = \begin{pmatrix} \delta & t \\ 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 0 \\ t & \varepsilon \end{pmatrix},$$

i.e., $X = \alpha_1 U + \alpha_2 I$, $Y = \beta_1 V + \beta_2 I$, where I is the identity 2×2 matrix. Direct computations give

$$U^2 = \delta U, \qquad V^2 = \varepsilon V,$$

$$XY = \begin{pmatrix} (\alpha_2 + \delta\alpha_1)\beta_2 + \alpha_1\beta_1 t^2 & \alpha_1 t(\beta_2 + \varepsilon\beta_1) \\ \alpha_2\beta_1 t & \alpha_2(\beta_2 + \varepsilon\beta_1) \end{pmatrix},$$

$$YX = \begin{pmatrix} \beta_2(\alpha_2 + \delta\alpha_1) & \beta_2\alpha_1 t \\ \beta_1 t(\alpha_2 + \delta\alpha_1) & \beta_1\alpha_1 t^2 + (\beta_2 + \varepsilon\beta_1)\alpha_2 \end{pmatrix},$$

$$[X, Y] = XY - YX = \alpha_1\beta_1 t \begin{pmatrix} t & \varepsilon \\ -\delta & -t \end{pmatrix}, \qquad [X, Y]^2 = \alpha_1^2\beta_1^2 t^2(t^2 - \delta\varepsilon)I.$$

Proposition 2. *The K-subalgebra R generated by X and Y in $M_2(\overline{K}[t])$ satisfies the same polynomial identities as the K-algebra $M_2(\overline{K})$.*

Proof. Since the algebra R is a K-subalgebra of $M_2(\overline{K}[t])$, it satisfies all polynomial identities of $M_2(\overline{K}[t])$. By Lemma 1 the $\overline{K}$-algebras $M_2(\overline{K}[t])$ and $M_2(\overline{K})$ have the same polynomial identities. Hence $T_K(M_2(\overline{K})) \subseteq T_K(R)$.

First, let the base field K be infinite. By Lemma 1, the K-algebras R and $\overline{R}$ have the same polynomial identities. Hence, the inclusion in the opposite direction $T_K(R) \subseteq T_K(M_2(\overline{K}))$ would follow if we show that the $\overline{K}$-algebra $M_2(\overline{K})$ is a homomorphic image of the $\overline{K}$-algebra $\overline{R}$. It is sufficient to see that the matrices $I, X_0 = X(t_0), Y_0 = Y(t_0), X_0Y_0$ are linearly independent for some $t_0 \in \overline{K}$. Equivalently, we may consider the matrices $I, U_0 = U(t_0), V_0 = V(t_0), U_0V_0$. Let $t_0 \in \overline{K}$, $t_0 \neq 0, \pm 1$, and let for some $\xi_0, \xi_1, \xi_2, \xi_3 \in \overline{K}$

$$0 = \xi_0 I + \xi_1 U_0 + \xi_2 V_0 + \xi_3 U_0 V_0 = \begin{pmatrix} \xi_0 + \delta\xi_1 + t_0^2\xi_3 & t_0(\xi_1 + \varepsilon\xi_3) \\ t_0\xi_2 & \xi_0 + \varepsilon\xi_2 \end{pmatrix}.$$

We derive consecutively

$$\xi_2 = 0, \qquad \xi_0 = 0, \qquad \xi_1 + \varepsilon\xi_3 = 0, \qquad \delta\xi_1 + t_0^2\xi_3 = 0.$$

Since $\delta, \varepsilon = 0, 1$ and $t_0 \neq 0, \pm 1$, we obtain $t_0^2 - \delta\varepsilon \neq 0$. Hence $\xi_1 = \xi_3 = 0$ and I, X_0, Y_0, X_0Y_0 are linearly independent and span $M_2(\overline{K})$ for any $t_0 \neq 0, \pm 1$. Therefore, $M_2(\overline{K})$ is a homomorphic image of $\overline{R}$.

Now, let the field K be finite. We mimic the standard Vandermonde arguments used in theory of PI-algebras. Let $f(x_1, \ldots, x_n) \in K\langle x_1, x_2, \ldots \rangle$ be a polynomial identity for the algebra R. Hence $f(r_1, \ldots, r_n) = 0$ for any $r_1, \ldots, r_n \in R$. We are interested in the case when $r_1 \in R$ has the form

$$r_1 = h_0 I + h_1 X + h_2 Y + h_3 XY,$$

where $h_i = h_i([X, Y]^2)$, $i = 0, 1, 2, 3$, are polynomials in $[X, Y]^2 = \alpha_1^2\beta_1^2 t^2(t^2 - \delta\varepsilon)I$. In this way, $f(r_1, \ldots, r_n) = 0$ is an evaluation of $g = f(x_{10} + x_{11} + x_{12} + x_{13}, x_2, \ldots, x_n)$. We write g as

$$g = g(x_{10}, x_{11}, x_{12}, x_{13}, x_2, \ldots, x_n) = \sum_{j=0}^{m} g_j(x_{10}, x_{11}, x_{12}, x_{13}, x_2, \ldots, x_n),$$

where $g_i(x_{10}, x_{11}, x_{12}, x_{13}, x_2, \ldots, x_n)$ is the homogeneous component of g of degree j in x_{10}. Evaluating f on

$$r_1 = [X, Y]^{2i} I + h_1 X + h_2 Y + h_3 XY, \qquad r_2, \ldots, r_n \in R,$$

we obtain

$$\begin{aligned} f(r_1, \ldots, r_n) &= g([X, Y]^{2i} I, h_1 X, h_2 Y, h_3 XY, r_2, \ldots, r_n) \\ &= \sum_{j=0}^{m} [X, Y]^{2ij} g_j(I, h_1 X, h_2 Y, h_3 XY, r_2, \ldots, r_n) \end{aligned}$$

$$= \sum_{j=0}^{m} (\alpha_1^2\beta_1^2 t^2(t^2 - \delta\varepsilon))^{ij} g_j(I, h_1X, h_2Y, h_3XY, r_2, \ldots, r_n)$$
$$= 0.$$

For $i = 0, 1, \ldots, m$, we obtain $m+1$ equations which form a linear homogeneous system with unknowns $g_j(I, h_1X, h_2Y, h_3XY, r_2, \ldots, r_n)$, $j = 0, 1, \ldots, m$. Its determinant Δ is equal to the Vandermonde determinant with entries

$$v_{ij} = (\alpha_1^2\beta_1^2 t^2(t^2 - \delta\varepsilon))^{ij}, \qquad i, j = 0, 1, \ldots, m,$$

and is different from zero because the elements v_{1j} are pairwise different. Therefore, the zero solution

$$g_j(I, h_1X, h_2Y, h_3XY, r_2, \ldots, r_n) = 0, \qquad j = 0, 1, \ldots, m,$$

is the only solution of the system over the field of rational functions $K(t)$ and hence also over $K[t]$. This means that the homogeneous component g_j of degree j in x_{10} of the polynomial identity $g(x_{10}, x_{11}, x_{12}, x_{13}, x_2, \ldots, x_n)$ of R vanishes when evaluated for

$$x_{10} = I, x_{11} = h_1X, x_{12} = h_2Y, x_{13} = h_3XY, x_2 = r_2, \ldots, x_n = r_n.$$

With the same arguments, we conclude that the multihomogeneous components $g_{(i_0,i_1,i_2,i_3)}$ of g of degree (i_0, i_1, i_2, i_3) in $(x_{10}, x_{11}, x_{12}, x_{13})$ vanish when evaluated for

$$x_{10} = I, x_{11} = X, x_{12} = Y, x_{13} = XY, x_2 = r_2, \ldots, x_n = r_n.$$

Since I, X, Y, XY are linearly independent in $M_2(K(t))$, they form a basis over $K(t)$, and every element s_1 in $M_2(K(t))$ has the form

$$s_1 = h_0I + h_1X + h_2Y + h_3XY, \qquad h_0, h_1, h_2, h_3 \in K(t).$$

Hence

$$\begin{aligned} f(s_1, r_2, \ldots, r_n) &= f(h_0I + h_1X + h_2Y + h_3XY, r_2, \ldots, r_n) \\ &= g(h_0I, h_1X, h_2Y, h_3XY, r_2, \ldots, r_n) \\ &= \sum g_{(i_0,i_1,i_2,i_3)}(h_0I, h_1X, h_2Y, h_3XY, r_2, \ldots, r_n) \\ &= \sum h_0^{i_0}h_1^{i_1}h_2^{i_2}h_3^{i_3} g_{(i_0,i_1,i_2,i_3)}(I, X, Y, XY, r_2, \ldots, r_n) \\ &= 0 \end{aligned}$$

for any $s_1 \in M_2(K(t))$, $r_2, \ldots, r_n \in R$. Continuing in this way, we conclude that $f(s_1, s_2, \ldots, s_n) = 0$ for any $s_1, \ldots, s_n \in M_2(K(t))$ and $f(x_1, \ldots, x_n)$ is a polynomial identity for $M_2(K(t))$. Hence $T_K(R) = T_K(M_2(K(t)))$. Since the field $K(t)$ is infinite, Lemma 1 gives

$$T_{K(t)}(M_2(K(t))) = T_{K(t)}(M_2(\overline{K}(t))), \qquad T_{\overline{K}}(M_2(\overline{K}(t))) = T_{\overline{K}}(M_2(\overline{K})),$$

and this implies $T_K(R) = T_K(M_2(\overline{K}))$ because

$$\begin{aligned} T_K(R) = T_K(M_2(K(t))) &= T_{K(t)}(M_2(K(t))) \cap K\langle x_1, x_2, \dots \rangle \\ &= T_{K(t)}(M_2(\overline{K}(t))) \cap K\langle x_1, x_2, \dots \rangle \\ &= T_{\overline{K}}(M_2(\overline{K}(t))) \cap K\langle x_1, x_2, \dots \rangle \\ &= T_{\overline{K}}(M_2(\overline{K})) \cap K\langle x_1, x_2, \dots \rangle \\ &= T_K(M_2(\overline{K})). \end{aligned}$$

□

Now we shall prove the main result of our article.

Theorem 3. *Let K be a field of any characteristic, and let a, b, c, d be arbitrary elements of K. Then the algebra*

$$F = K\langle x, y \mid x^2 + ax + b = 0,\ y^2 + cy + d = 0\rangle$$

can be embedded into $M_2(\overline{K}[t])$. The algebras F and $M_2(\overline{K})$ satisfy the same polynomial identities as K-algebras.

Proof. **Step 1.** The equation $f(x) = x^2 + ax + b = 0$ has two zeros η_1, η_2 in an extension of K and $f(x) = (x - \eta_1)(x - \eta_2)$ in $\overline{K}[x]$. If $\eta_1 = \eta_2$, we change the variable x in $\overline{K}[x]$ by $x = u + \eta_1$ and obtain that $f(x) = u^2$. Similarly, if $\eta_1 \neq \eta_2$, we change x by $x = \alpha_1 u + \alpha_2$, $\alpha_1 = \eta_2 - \eta_1$, $\alpha_2 = \eta_1$ in $\overline{K}$. Then $f(x) = \alpha_1^2(u^2 - u) = (a^2 - 4b)(u^2 - u)$. Hence, working in the $\overline{K}$-algebra

$$\overline{F} = \overline{K}\langle x, y \mid x^2 + ax + b = 0,\ y^2 + cy + d = 0\rangle,$$

which contains F, we may assume that it is generated by u and v such that either $u^2 = 0$ or $u^2 = u$, and similarly, either $v^2 = 0$ or $v^2 = v$. Hence $\overline{F} = \overline{K} \otimes_K F$ has the presentation

$$\overline{F} = \overline{K}\langle u, v \mid u^2 = \delta u,\ v^2 = \varepsilon v\rangle,$$

where $\delta, \varepsilon = 0, 1$. As a $\overline{K}$-vector space $\overline{F}$ is spanned on the monomials

$$1, \qquad (uv)^p, (vu)^p, p \geq 1, \qquad (uv)^q u, (vu)^q v, \qquad q \geq 0.$$

(It follows from the general theory of free products or of Gröbner bases that these elements form a $\overline{K}$-basis of $\overline{F}$.)

Step 2. We consider the matrices

$$U = \begin{pmatrix} \delta & t \\ 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 0 \\ t & \varepsilon \end{pmatrix}$$

in $M_2(\overline{K}[t])$. As in Step 1, we choose $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \overline{K}$, $\alpha_1, \beta_1 \neq 0$, such that the matrices

$$X = \alpha_1 U + \alpha_2 I, \qquad Y = \beta_1 V + \beta_2 I$$

satisfy the equations $X^2 + aX + bI = 0$, $Y^2 + cY + dI = 0$. Let R be the K-subalgebra of $M_2(\overline{K}[t])$ generated by X and Y. The $\overline{K}$-subalgebra $\overline{R}$ of $M_2(\overline{K}[t])$ is generated also by U and V. Since $U^2 = \delta U$, $V^2 = \varepsilon V$, the mapping

$$u \to U, \qquad v \to V$$

extends to a homomorphism φ of the $\overline{K}$-algebra $\overline{F}$ to the $\overline{K}$-algebra $\overline{R}$.

Step 3. Direct computations give that

$$(UV)^p = \begin{pmatrix} t^{2p} & \varepsilon t^{2p-1} \\ 0 & 0 \end{pmatrix} = t^{2(p-1)} UV, \qquad (VU)^p = \begin{pmatrix} 0 & 0 \\ \delta t^{2p-1} & t^{2p} \end{pmatrix} = t^{2(p-1)} VU,$$

$$(UV)^q U = \begin{pmatrix} \delta t^{2q} & t^{2q+1} \\ 0 & 0 \end{pmatrix} = t^{2q} U, \qquad (VU)^q V = \begin{pmatrix} 0 & 0 \\ t^{2q+1} & \varepsilon t^{2q} \end{pmatrix} = t^{2q} V.$$

Hence the maximum degree in t of the entries of a nonzero monomial of degree k in U, V is equal to k.

Step 4. If the homomorphism $\varphi : \overline{F} \to \overline{R}$ is not an isomorphism, the kernel of φ contains a nonzero element of $\overline{F}$

$$f(u, v) = \alpha_0 + \sum_{p \geq 1} (\alpha_{1p}(uv)^p + \alpha_{2p}(vu)^p) + \sum_{q \geq 0} (\beta_{1q}(uv)^q u + \beta_{2q}(vu)^q v),$$

$\alpha_0, \alpha_{ip}, \beta_{iq} \in \overline{K}$. If $\deg_{u,v}(f) = 2k$, then at least one of the coefficients α_{1k} and α_{2k} is different from 0. The diagonal entries of the matrix $f(U, V)$ contain, respectively, the monomials $\alpha_{1k} t^{2k}$ and $\alpha_{2k} t^{2k}$. At least one of these monomials is nonzero and does not cancel with any other monomial of the corresponding entries of $f(U, V)$. Similarly, if $\deg_{u,v}(f) = 2k + 1$, then the entries of the other diagonal of the matrix $f(U, V)$ contain, respectively, the monomials $\beta_{1k} t^{2k+1}$ and $\beta_{2k} t^{2k+1}$. Again, at least one of these monomials is nonzero and does not cancel with other monomials at the same position. This implies that φ is an isomorphism and $\overline{F}$ can be embedded into $M_2(\overline{K}[t])$. This completes the proof in virtue of Proposition 2. □

As in Weiss [14, 15] the algebra F is a free module of rank 4 over its centre. We can extend Theorem 3 to arbitrary infinite dimensional algebras generated by two quadratic elements.

Theorem 4. *Let*

$$F = K\langle x, y \mid x^2 + ax + b = 0, y^2 + cy + d = 0 \rangle.$$

(i) *The centre $C(F)$ of F is isomorphic to the polynomial algebra in one variable and F is a free $C(F)$-module of rank* 4 *generated by* $1, x, y, xy$.
(ii) *All proper ideals of F are of finite codimension in F.*

Proof. (i) We work with the matrix representation of F as the K-subalgebra R of $M_2(\overline{K}[t])$ generated by the matrices

$$X = \begin{pmatrix} \alpha_2 + \delta\alpha_1 & \alpha_1 t \\ 0 & \alpha_2 \end{pmatrix}, \qquad Y = \begin{pmatrix} \beta_2 & 0 \\ \beta_1 t & \beta_2 + \varepsilon\beta_1 \end{pmatrix}.$$

The Cayley–Hamilton theorem gives

$$(X+Y)^2 - (\mathrm{tr}(X+Y))(X+Y) + \det(X+Y)I = 0,$$
$$(X+Y)^2 - (2(\alpha_2+\beta_2) + \delta\alpha_1 + \varepsilon\beta_1)(X+Y)$$
$$+ ((\alpha_2+\beta_2+\delta\alpha_1)(\alpha_2+\beta_2+\varepsilon\beta_1) - \alpha_1\beta_1 t^2)I = 0.$$

Using that in F

$$x^2 + ax + b = 0, \qquad y^2 + cy + d = 0,$$

which become in R

$$X^2 - (2\alpha_2 + \delta\alpha_1)X + \alpha_2(\alpha_2 + \delta\alpha_1)I = 0,$$
$$Y^2 - (2\beta_2 + \varepsilon\beta_1)Y + \beta_2(\beta_2 + \varepsilon\beta_1)I = 0,$$

we obtain

$$XY + YX + cX + aY + f(t^2)I = 0, \qquad f(t^2) \in \overline{K}[t^2].$$

Hence the element

$$Z = XY + YX + cX + aY \in R$$

belongs to the centre of R. The equations

$$X^2 = -(aX + bI), \qquad Y^2 = -(cY + dI), \qquad YX = -(cY + dI)cX - aY - XY$$

allow to express every element of R as a linear combination of I, X, Y, XY with coefficients from $K[Z]$, i.e., R is a 4-generated $K[Z]$-module. It is easy to see that this module is free and the centre of R coincides with $K[Z]$. Going back to F, we obtain that

$$C(F) = K[z], \qquad z = xy + yx + cx + ay$$

and F is a free $C(F)$-module freely generated by $1, x, y, xy$. In the special case $X = U$, $Y = V$, the above formulas are quite simple:

$$(U+V)^2 - (\delta+\varepsilon)(U+V) + (\delta\varepsilon - t^2)I = 0,$$
$$Z = XY + YX - \varepsilon U - \delta V = (t^2 - \delta\varepsilon)I \in C(\overline{R}),$$

and the elements of $\overline{R}$ and the multiplication rules between them are given by

$$W = \rho_0 Z + \rho_1 U + \rho_2 V + \rho_3 UV, \qquad Z = (t^2 - \delta\varepsilon)I,\ \rho_i \in \overline{K}[Z],$$
$$U^2 = \delta U, \qquad V^2 = \varepsilon V, \qquad VU = -(U - \delta I)(V - \varepsilon I) + t^2 I.$$

(ii) If J is an ideal of R, then $\overline{J}$ is an ideal of $\overline{R}$ and the codimensions of J in R as a K-vector space and of $\overline{J}$ in $\overline{R}$ as a $\overline{K}$-vector space are the same. Hence we may work in $\overline{R}$ and show that $\overline{J}$ is of finite codimension in $\overline{R}$. Let

$$0 \neq W = \rho_0 I + \rho_1 U + \rho_2 V + \rho_3 UV \in \overline{J}, \qquad \rho_i \in \overline{K}[Z].$$

We shall show that $\overline{J}$ contains a nonzero central element $W_0 = \tau I$, $\tau \in \overline{K}[Z]$. Hence, modulo $\overline{J}$, the elements of $\overline{R}$ have the form

$$\sigma_0 I + \sigma_1 U + \sigma_2 V + \sigma_3 UV, \qquad \sigma_i \in \overline{K}[Z], \deg_Z \sigma_i < \deg_Z \tau,$$

and the factor algebra $\overline{R}/\overline{J}$ is finite dimensional. This is true if $\rho_1 = \rho_2 = \rho_3 = 0$ in the presentation of the element $W \in \overline{J}$. Hence, we may assume that W is not central. Direct computations give

$$[W, U]^2 = \rho_2(\rho_2 + \delta\rho_3)t^2(t^2 - \delta\varepsilon)I, \qquad [W, V]^2 = \rho_1(\rho_1 + \varepsilon\rho_3)t^2(t^2 - \delta\varepsilon)I.$$

Hence, if $\rho_1, \rho_2, \rho_2 + \delta\rho_3, \rho_1 + \varepsilon\rho_3 \neq 0$, then $\overline{J}$ contains a nonzero central element. We have to consider three more cases:

(1) $\rho_1 = \rho_2 = 0$. Hence $\rho_3 \neq 0$ and $[W, U + V]^2 = \rho_3^2 t^2(\delta\varepsilon - t^2)I \neq 0$;
(2) $\rho_1 = 0$, $\delta = 1$, $\rho_2 = -\rho_3 \neq 0$. Then $[W, U + V]^2 = \rho_3^2 t^2(\varepsilon - t^2)I \neq 0$. The case $\rho_2 = 0$, $\varepsilon = 1$, $\rho_1 = -\rho_3 \neq 0$ is similar;
(3) $\delta = \varepsilon = 1$, $\rho_1 = \rho_2 = \rho_3 \neq 0$. Then $[W, U + V]^2 = \rho_3^2 t^4(1 - t^2)I \neq 0$.

In all the cases, $\overline{J}$ contains a nonzero central element and hence is of finite codimension in $\overline{R}$. □

Remark 5. An embedding similar to that in Theorem 3 appears also in group theory. For example, it is well known that for $t \in \mathbb{Z}$, $t \geq 2$, the matrices

$$U = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

generate a free subgroup of $SL_2(\mathbb{Z})$.

Remark 6. Our embedding of the monomial algebra

$$F = K\langle x, y \mid x^2 = 0, y^2 = 0\rangle$$

is the same as the embedding which follows from the proof of Belov [1]. When the characteristic of K is different from 2, the embedding of Weiss [14, 15] into $M_2(K[v])$ of the algebra

$$F = K\langle x, y \mid x^2 = x, y^2 = y\rangle$$

generated by two idempotents is given by the formulas

$$x \to \begin{pmatrix} 1+v & t \\ 1 & 1-v \end{pmatrix}, \qquad y \to \begin{pmatrix} 1+v & -t \\ -1 & 1-v \end{pmatrix},$$

where $t = 1 - v^2$. (In fact these matrices are idempotents only up to a multiplicative constant because they satisfy the equations $x^2 = 2x$ and $y^2 = 2y$. Clearly, this is not essential for the embedding of F into $M_2(K[v])$.) Then F is isomorphic to the algebra of matrices of the form

$$\begin{pmatrix} f_1 + f_3 v & t(f_2 - f_4 v) \\ f_2 + f_4 v & f_1 - f_3 v \end{pmatrix},$$

where $f_i \in K[t]$. Hence our embedding of F is simpler than that of Weiss.

Remark 7. Fixing an admissible ordering on the set of monomials $\langle x_1, \ldots, x_d\rangle$, the transfer of combinatorial results for a monomial ideal J of $K\langle x_1, \ldots, x_d\rangle$ to an arbitrary ideal of $K\langle x_1, \ldots, x_d\rangle$ with the same set of leading monomials as J does not hold automatically. For example, Irving [6] showed that the algebra with presentation

$$B = K\langle x, y \mid x^2 = 0, yxy = x\rangle$$

satisfies the polynomial identities of $M_n(K)$ (or of $M_n(\overline{K})$ if K is finite) for a suitable n but cannot be embedded into any matrix algebra $M_k(C)$ over a commutative algebra C. More precisely, it follows from his proof that B satisfies the polynomial identity $[x_1, x_2][x_3, x_4][x_5, x_6] = 0$ which as is well known generates the T-ideal of the 3×3 upper triangular matrices. Hence B satisfies all polynomial identities of $M_3(K)$ (or of $M_3(\overline{K})$ if K is finite). As a vector space B has a basis consisting of $y^a, y^a x, xy^{a+1}, xy^{a+1}x$, $a \geq 0$, which is the same as of the monomial algebra

$$B_0 = K\langle x, y \mid x^2 = 0, yxy = 0\rangle.$$

By the result of Borisenko [3] the algebra B_0 can be embedded into a matrix algebra over $K[t]$. It is interesting to mention that B_0 can be realized as a homomorphic image of a subalgebra of $M_3(K[t])$ in the following way. Let S be the algebra generated by

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $YXY = 0$. Every vector subspace of $K[t]e_{13}$ is an ideal of S. Choosing

$$J = tK[t]e_{13} = \begin{pmatrix} 0 & 0 & tK[t] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we obtain that $X^2 \in J$, and it is easy to see that $B_0 \cong S/J$.

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