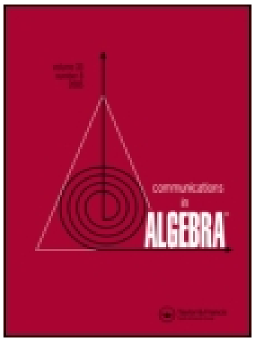

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A LINK BETWEEN A NATURAL CENTRALIZER AND THE SMALLEST ESSENTIAL IDEAL IN STRUCTURAL MATRIX RINGS

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ABSTRACT. In a structural matrix ring $\mathbb{M}_n(\rho, R)$ over an arbitrary ring R we determine the centralizer of the set of matrix units in $\mathbb{M}_n(\rho, R)$ associated with the anti-symmetric part of the reflexive and transitive binary relation ρ on $\{1, 2, \ldots, n\}$. If the underlying ring R has no proper essential ideal, for example if R is a field, then we show that the largest ideal of $\mathbb{M}_n(\rho, R)$ contained in the mentioned centralizer coincides with the smallest essential ideal of $\mathbb{M}_n(\rho, R)$.

Key words and phrases. Structural matrix ring, centralizer, essential ideal.

1. INTRODUCTION

Every ring herein is assumed to be associative with identity, subrings inherit the identity, and ideal means two-sided ideal.

The centre $Z(\mathbb{M}_n(R))$ of a full $n \times n$ matrix ring $\mathbb{M}_n(R)$ over a commutative ring R comprises the scalar matrices. Therefore $Z(\mathbb{M}_n(R)) \cong R$. Similarly, with e_{kl} denoting the matrix unit with 1 in position (k, l) and zeroes elsewhere, and by viewing a direct sum $\oplus_{i=1}^{m} \mathbb{M}_{n_i}(R)$ of full matrix rings over R as a subring of the full matrix ring $\mathbb{M}_{n_1+n_2+\cdots+n_m}(R)$, the center $Z(\oplus_{i=1}^{m} \mathbb{M}_{n_i}(R))$ of $\oplus_{i=1}^{m} \mathbb{M}_{n_i}(R)$ comprises the "piecewise" scalar matrices

$$\sum_{i=1}^{m} \sum_{k=n_1+n_2+\cdots+n_{i-1}+1}^{n_1+n_2+\cdots+n_i} u_i e_{kk},$$

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$u_i \in R$, $i = 1, 2, \ldots, m$ (and with $n_0 := 0$). Therefore $Z(\oplus_{i=1}^m \mathbb{M}_{n_i}(R)) \cong \oplus_{i=1}^m R$. In neither of these cases does the centre really yield anything new regarding substructures of the particular matrix ring. Furthermore, in both cases, assuming that the base ring R is commutative, the centre is also obtained as the centralizer of all the matrix units, i.e. for example, for the full matrix ring case,

$$\begin{aligned} C_{\mathbb{M}_n(R)}(\{e_{kl}\colon 1 \leq k, l \leq n\}) &:= \{U = [u_{ij}] \in \mathbb{M}_n(R)\colon e_{kl}U = Ue_{kl},\ 1 \leq k, l \leq n\} \\ &= \{\sum_{i=1}^n ue_{ii}\colon u \in R\} = Z(\mathbb{M}_n(R)). \end{aligned}$$

The situation regarding substructures of matrix rings becomes much more interesting and provides a link with the smallest essential ideal if one considers the class of structural matrix rings. A structural matrix ring $\mathbb{M}_n(\rho, R)$ has, unlike a full matrix ring $\mathbb{M}_n(R)$, a much richer lattice of ideals than the base ring R. We focus in this note on the centralizer of a natural set of matrix units in a structural matrix ring over a suitable ring.

Recall that, for a reflexive and transitive binary relation ρ on the set $\{1, 2, \ldots, n\}$, the subset $\mathbb{M}_n(\rho, R)$ of $\mathbb{M}_n(R)$ comprising all matrices with (i, j)th entry equal to 0 if $(i, j) \notin \rho$, forms a subring of $\mathbb{M}_n(R)$, called a *structural matrix ring* (over R), the properties of which are determined by the structure of the underlying binary relation ρ and the base ring R. Apart from being a rich source of examples and counterexamples and playing a role in the structure theory of rings, various kinds of incidence matrix rings, including structural matrix rings, have in recent years been the object of study in their own right. See, for example, [1], [2], [4], [6], [7] and [9].

As in [6], the relation ρ may be divided into two subsets, namely a symmetric part

$$\rho_S := \{(k, l) \in \rho\colon\ (l, k) \in \rho\}$$

and an anti-symmetric part

$$\rho_A := \{(k, l) \in \rho\colon\ (l, k) \notin \rho\}.$$

Note that ρ_S is an equivalence relation on $\{1, 2, \ldots, n\}$. A full matrix ring $\mathbb{M}_n(R)$ is a very special case of a structural matrix ring $\mathbb{M}_n(\rho, R)$, with ρ the universal binary relation on $\{1, 2, \ldots, n\}$. In this case $\rho = \rho_S$ and $\rho_A = \emptyset$. Even a direct sum $\oplus_{i=1}^m \mathbb{M}_{n_i}(R)$ of full matrix rings over R may be viewed as a structural matrix ring $\mathbb{M}_{n_1+n_2+\cdots+n_m}(\rho, R)$, with $\rho_S = \rho$ and $\rho_A = \emptyset$. We will in the sequel be interested in structural matrix rings $\mathbb{M}_n(\rho, R)$ for which $\rho_A \neq \emptyset$, which we call *honest* structural matrix rings.

The lattice of ideals of a structural matrix ring $\mathbb{M}_n(\rho, R)$ were characterized in [8] in terms of the lattice of ideals of R and the structure of ρ. In [3] the minimal essential ideal of $\mathbb{M}_n(\rho, R)$ was described in case R has a minimal essential ideal. Note that it is possible that a ring has essential ideals, but no minimal essential ideal, as is the case in the ring of integers. However, since the intersection of two essential ideals is again essential, it follows that if a ring has a minimal essential ideal, then it is unique, and so then it is the smallest essential ideal. A similarity between a class of commutative rings, namely certain Prüfer domains, and a class of non-commutative rings, namely the structural matrix rings, regarding the coincidence of the maximal small ideal, i.e. the Brown-McCoy radical (see [5]), and the smallest essential ideal, was obtained in [3].

We find another coincidence in this note by showing that that if the underlying ring R has no proper essential ideal, for example, if R is a field, in which case R is the smallest essential ideal of R, then the largest ideal of every honest structural matrix ring $\mathbb{M}_n(\rho, R)$ contained in the centralizer of the set of matrix units in $\mathbb{M}_n(\rho, R)$ associated with the anti-symmetric part ρ_A of ρ, i.e. the largest ideal of $\mathbb{M}_n(\rho, R)$ contained in $C_{\mathbb{M}_n(R)}(\{e_{kl}\colon (k,l) \in \rho_A\})$, is a non-trivial proper ideal and coincides with the smallest essential ideal of $\mathbb{M}_n(\rho, R)$.

2. A NATURAL CENTRALIZER AND THE SMALLEST ESSENTIAL IDEAL

We first summarize the characterization in [8] of the ideals of a structural matrix ring $\mathbb{M}_n(\rho, R)$ over an arbitrary ring R using set-inclusion preserving functions.

Let Rep denote a set of representatives of the equivalence classes induced by the equivalence relation ρ_S on $\{1, 2, \ldots, n\}$. For $x, y \in Rep$ such that $(x, y) \in \rho$, set

$$\Lambda_{xy} := \{z \in Rep\colon\ (x,z), (z,y) \in \rho\}.$$

Let

$$f\colon \{\Lambda_{xy}\colon\ x, y \in Rep \text{ and } (x,y) \in \rho\} \to \{I\colon\ I \text{ is an ideal of } R\}$$

be a set-inclusion preserving function. Then

$$\mathcal{I}_f := \{U = [u_{ij}] \in \mathbb{M}_n(\rho, R)\colon\ u_{ij} \in f(\Lambda_{xy}) \text{ if } x, y \in Rep \text{ are such that } (i,x), (j,y) \in \rho_S \text{ and } (x,y) \in \rho\}$$

is an ideal of $\mathbb{M}_n(\rho, R)$, and by considering all such set-inclusion preserving functions, it follows from [8, Proposition 1.2] that one obtains all the ideals of $\mathbb{M}_n(\rho, R)$.

We obtain our first proposition directly from the proof of [3, Corollary 3.2]:

Proposition 2.1. *If a ring R has no proper essential ideal, then the smallest essential ideal of a structural matrix ring $\mathbb{M}_n(\rho, R)$ over R is the ideal $\mathcal{I}_f$, where*

$$f(\Lambda_{xy}) = \begin{cases} R, & \text{if } \Lambda_{xy} \text{ is maximal with respect to set-inclusion} \\ & \text{in the set } \{\Lambda_{uv}\colon\ u, v \in Rep \text{ and } (u,v) \in \rho\} \\ \{0\}, & \text{otherwise.} \end{cases}$$

EXAMPLE 2.2. For the structural matrix ring

$$\mathbb{M}_9(\rho, R) := \begin{bmatrix} R & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R & R & R & R & R & 0 & 0 \\ 0 & 0 & 0 & R & R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R & R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R & R & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & R \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R \end{bmatrix}$$

the equivalence classes induced by the equivalence relation ρ_S on $\{1, 2, \ldots, 9\}$ are

$$\{1,2\},\ \{3\},\ \{4,5\},\ \{6\},\ \{7\},\ \{8\},\ \{9\}.$$

Set

$$Rep := \{1, 3, 4, 6, 7, 8, 9\}.$$

Then the maximal Λ_{xy}'s are

$$\Lambda_{11},\ \Lambda_{34},\ \Lambda_{37},\ \Lambda_{89},$$

and so, if R has no proper essential ideal, then the smallest essential ideal of $\mathbb{M}_9(\rho, R)$ is the ideal

$$\begin{bmatrix} R & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ R & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R & R & 0 & R & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For an honest structural matrix ring $\mathbb{M}_n(\rho, R)$ we will distinguish between the set $\{e_{kl}\colon (k,l) \in \rho_S\}$ of matrix units associated with ρ_S and the set $\{e_{kl}\colon (k,l) \in \rho_A\}$ of matrix units associated with ρ_A. The set $\{e_{kl}\colon (k,l) \in \rho_S\}$ is merely a generalization of the set $\{e_{kl}\colon 1 \le k, l \le n\}$ of all matrix units in case $\mathbb{M}_n(\rho, R)$ is a full matrix ring or a direct sum of full matrix rings over R, and so we should not expect the centralizer of $\{e_{kl}\colon (k,l) \in \rho_S\}$ in an honest structural matrix ring $\mathbb{M}_n(\rho, R)$ to be much more exciting than the scalar matrices or "piecewise" scalar matrices obtained as the centralizer of the set of all matrix units in a full matrix ring or direct sum of full matrix rings respectively:

Proposition 2.3. *In a structural matrix ring $\mathbb{M}_n(\rho, R)$ over an arbitrary ring R the centralizer of the set $\{e_{kl}\colon (k,l) \in \rho_S\}$ of matrix units associated with the symmetric part ρ_S of ρ is the subring*

$$\left\{ \sum_{x \in Rep} \sum_{\substack{k \\ (k,x) \in \rho_S}} u_{xx} e_{kk}\colon\ u_{xx} \in R \text{ for every } x \in Rep \right\} \tag{1}$$

of $\mathbb{M}_n(\rho, R)$.

Proof. Let $U = [u_{ij}] \in C_{\mathbb{M}_n(\rho,R)}(\{e_{kl}\colon\ (k,l) \in \rho_S\})$. Since $(k,k) \in \rho_S$ for every k, it follows from $e_{kk}U = Ue_{kk}$, $k = 1, 2, \ldots, n$, that U is the diagonal matrix $\sum_{k=1}^{n} u_{kk}e_{kk}$. Let $x \in Rep$ and let $(k,x) \in \rho_S$. Then $e_{kx}U = Ue_{kx}$ implies that $u_{xx}e_{kx} = u_{kk}e_{kx}$, and so $u_{xx} = u_{kk}$. Hence, $U = \sum_{x\in Rep} \sum_{k,\ (k,x)\in\rho_S} u_{xx}e_{kk}$.

Conversely, let U be a "piecewise" scalar matrix, as in (1). Let $(k,l) \in \rho_S$ and let $x \in Rep$ be such that $(k,x) \in \rho_S$. Then $(l,x) \in \rho_S$, and so $e_{kl}U = u_{xx}e_{kl} = Ue_{kl}$. □

Since Rep is a set of representatives of the equivalence classes induced by the equivalence relation ρ_S on $\{1, 2, \ldots, n\}$, it follows that ρ is a partial order relation on the set Rep. In order to determine the centralizer of the set $\{e_{kl}\colon\ (k,l) \in \rho_A\}$, we introduce another equivalence relation. Define the relation $\equiv_\rho$ on Rep by setting $(x,x) \in\ \equiv_\rho$ (for $x \in Rep$) and by setting $(x,y) \in\ \equiv_\rho$ (for $x, y \in Rep$ with $x \neq y$) if and only if there are $z_1, z_2, \ldots, z_t \in Rep$ such that $x = z_1$, $y = z_t$, and for $s = 1, 2, \ldots, t-1$, $(z_s, z_{s+1}) \in \rho_A$ or $(z_{s+1}, z_s) \in \rho_A$. Then $\equiv_\rho$ is an equivalence relation on Rep. The equivalence class of $x \in Rep$ with respect to $\equiv_\rho$ will be denoted by $[x]_{\equiv_\rho}$, and its cardinality by $|[x]_{\equiv_\rho}|$.

Note that for the structural matrix ring $\mathbb{M}_9(\rho, R)$ in Example 2.2 we have that

$$\mathbb{M}_9(\rho,R) \cong \mathbb{M}_2(R) \oplus \begin{bmatrix} R & R & R & R & R \\ 0 & R & R & 0 & 0 \\ 0 & R & R & 0 & 0 \\ 0 & 0 & 0 & R & R \\ 0 & 0 & 0 & 0 & R \end{bmatrix} \oplus \begin{bmatrix} R & R \\ 0 & R \end{bmatrix}.$$

It is thus clear that in order to determine the centralizer in a structural matrix ring $\mathbb{M}_n(\rho, R)$ associated with the anti-symmetric part ρ_A of ρ, it suffices to consider structural matrix rings for which the equivalence relation $\equiv_\rho$ induces a single equivalence class on Rep. However, since we do not run into more troublesome notation, and for the sake of generality, we state the next result for arbitrary honest structural matrix rings, i.e. for honest structural matrix rings which may be direct sums of other structural matrix rings.

Theorem 2.4. *In an honest structural matrix ring $\mathbb{M}_n(\rho, R)$ over an arbitrary ring R the centralizer of the set $\{e_{kl}\colon\ (k,l) \in \rho_A\}$ of matrix units associated with the anti-symmetric part ρ_A of ρ is the subring*

$$\Bigg\{ \sum_{\substack{x\in Rep\\ |[x]_{\equiv_\rho}|>1}} \ \sum_{\substack{k\\ (k,x)\in\rho_S}} u_{xx}e_{kk} + \sum_{\substack{y,z\in Rep\\ (y,z)\in\rho_A}} \ \sum_{\substack{k,l\\ (k,y),(l,z)\in\rho_S}} u_{kl}e_{kl} + \sum_{\substack{w\in Rep\\ |[w]_{\equiv_\rho}|=1}} \ \sum_{\substack{i,j\\ (i,w),(j,w)\in\rho_S}} u_{ij}e_{ij}\colon$$

all the u_{pq}'s are in R, y is a minimal element and z a maximal

$$\left.\textit{element in Rep with respect to } \rho, \quad \textit{and } u_{xx} = u_{x'x'} \textit{ if } (x, x') \in \equiv_\rho \right\} \qquad (2)$$

of $\mathbb{M}_n(\rho, R)$.

Proof. Let $U = [u_{ij}] \in C_{\mathbb{M}_n(\rho,R)}(\{e_{kl}\colon (k,l) \in \rho_A\})$. In order to cater for the first part

$$\sum_{\substack{x \in Rep \\ |[x]_{\equiv_\rho}|>1}} \sum_{\substack{k \\ (k,x) \in \rho_S}} u_{xx} e_{kk}$$

in (2), we have to show the following for every $x \in Rep$ with $|[x]_{\equiv_\rho}| > 1$:

(i) if $x' \in [x]_{\equiv_\rho}$, then all the diagonal entries u_{kk} of U, with $(k, x') \in \rho_S$, are equal (to u_{xx}).

(ii) if $|[x]_{\rho_S}| > 1$ (where $[x]_{\rho_S}$ denotes the equivalence class of x with respect to the equivalence relation ρ_S on $\{1, 2, \dots, n\}$), in other words, if U has non-diagonal entries $u_{k'k}$ such that $(k', x), (k, x) \in \rho_S$ (i.e. $k', k \in [x]_{\rho_S}$), then all such entries $u_{k'k}$ equal zero.

In order to prove (i), let $x \in Rep$ with $|[x]_{\equiv_\rho}| > 1$. Let $y \in [x]_{\equiv_\rho}$, with $y \neq x$. Then there are elements $z_1, \dots, z_t \in Rep$ such that $x = z_1$, $y = z_t$, and for $s = 1, \dots, t-1$, $(z_s, z_{s+1}) \in \rho_A$ or $(z_{s+1}, z_s) \in \rho_A$. Let $1 \leq s \leq t-1$ and assume, without loss of generality, that $(z_s, z_{s+1}) \in \rho_A$. Let k and l be arbitrary elements such that $(k, z_s), (l, z_{s+1}) \in \rho_S$. Then $(k, l) \in \rho_A$, and so, since we have assumed that $U \in C_{\mathbb{M}_n(\rho,R)}(\{e_{kl}\colon (k,l) \in \rho_A\})$, it follows that $e_{kl}U = Ue_{kl}$. Therefore

$$\sum_{\substack{j \\ (l,j) \in \rho}} u_{lj} e_{kj} = \sum_{\substack{i \\ (i,k) \in \rho}} u_{ik} e_{il}. \qquad (3)$$

With $j := l$ in the left hand side and $i := k$ in the right hand side of (3), we conclude that $u_{ll} = u_{kk}$. Therefore, since k and l are arbitrary elements such that $(k, z_s), (l, z_{s+1}) \in \rho_S$ and since s is an arbitrary number such that $1 \leq s \leq t-1$, we have proved (i).

In order to prove (ii), let $x, y, z_1, \dots, z_t, s$ be as in the proof of (i), and assume that $|[x]_{\rho_S}| > 1$. With s:=1, we assume, without loss of generality, as in the proof of (i), that $(x, z_2) \in \rho_A$. Let $k, k' \in [x]_{\rho_S}$, with $k' \neq k$. Then $(k', k) \in \rho$, and so, setting $i := k'$ in the right hand side of (3), it follows that $u_{k'k}e_{k'l}$ appears in the right hand side of (3). Since all the matrix units in the left hand side of (3) are of the form e_{kj}, and since $k' \neq k$, we conclude that $u_{k'k} = 0$. This concludes the proof of (ii).

The foregoing arguments cater for the first part

$$\sum_{\substack{x \in Rep \\ |[x]_{\equiv_\rho}|>1}} \sum_{\substack{k \\ (k,x) \in \rho_S}} u_{xx} e_{kk}$$

in (2).

Next, let $y, z \in Rep$, with $(y, z) \in \rho_A$, and such that y is not a minimal element or z is not a maximal element in Rep with respect to the partial order relation ρ on Rep. Assume, without loss of generality, that y is not minimal. Then there is an $x \in Rep$ such that $(x, y) \in \rho_A$. Let k and l be arbitrary elements such that $(k, y), (l, z) \in \rho_S$. Since $(y, z) \in \rho_A$, it follows that $(z, y) \notin \rho$, and so $(l, k) \notin \rho$. Therefore, $k \neq l$. Since $(x, k) \in \rho_A$, we have that $e_{xk}U = Ue_{xk}$, and so

$$\sum_{\substack{j \\ (k,j)\in\rho}} u_{kj}e_{xj} = \sum_{\substack{i \\ (i,x)\in\rho}} u_{ix}e_{ik}. \tag{4}$$

Since $(k, l) \in \rho$, it follows, with $j := l$ in the left hand side of (4), that $u_{kl}e_{xl}$ appears in the left hand side of (4). However, all the matrix units in the right hand side of (4) are of the form e_{ik}, and so, since $k \neq l$, we conclude that $u_{kl} = 0$.

The foregoing paragraphs show that if $U = [u_{ij}] \in C_{\mathbf{M}_n(\rho,R)}(\{e_{kl}\colon (k,l) \in \rho_A\})$, then U is a matrix as in (2).

Conversely, let U be a matrix as in (2). Let i and l be arbitrary elements such that $(i, l) \in \rho_A$. Let $y, z \in Rep$ be such that $(i, y), (l, z) \in \rho_S$. Then $(y, z) \in \rho_A$ and $|[z]_{\equiv_\rho}| > 1$. Since the product $e_{il}U$ focuses on the lth row of U, we have that

$$e_{il}U = \sum_{\substack{j \\ (l,j)\in\rho}} u_{lj}e_{ij} = \sum_{\substack{j \\ (l,j)\in\rho_S}} u_{lj}e_{ij} + \sum_{\substack{j \\ (l,j)\in\rho_A}} u_{lj}e_{ij}.$$

If j is such that $(l, j) \in \rho_S$ and $l \neq j$, then $|[z]_{\rho_S}| > 1$, and so since $(l, z), (j, z) \in \rho_S$, we conclude from the summation

$$\sum_{\substack{x\in Rep \\ |[x]_{\equiv_\rho}|>1}} \sum_{\substack{k \\ (k,x)\in\rho_S}} u_{xx}e_{kk}$$

in (2), with $x := z$, that the non-diagonal entry u_{lj} of U equals zero. The same summation also implies that the diagonal entry u_{ll} equals the diagonal entry u_{zz}. Therefore,

$$e_{il}U = u_{zz}e_{il} + \sum_{\substack{j \\ (l,j)\in\rho_A}} u_{lj}e_{ij}. \tag{5}$$

If j is such that $(l, j) \in \rho_A$, then $(z, j) \in \rho_A$, and so if $w \in Rep$ is such that $(j, w) \in \rho_S$, then $(z, w) \in \rho_A$, from which we conclude that z is not a maximal element in Rep with respect to the partial order relation ρ on Rep. Therefore the description of the matrices in (2) implies that $u_{lj} = 0$. Consequently, $e_{il}U = u_{zz}e_{il}$. Similarly, $Ue_{il} = u_{yy}e_{il}$. Moreover, since $(y, z) \in \rho_A$, it follows directly from the definition of $\equiv_\rho$ that $(y, z) \in \equiv_\rho$. Therefore, $|[z]_{\equiv_\rho}| > 1$, and so the summation

$$\sum_{\substack{x\in Rep \\ |[x]_{\equiv_\rho}|>1}} \sum_{\substack{k \\ (k,x)\in\rho_S}} u_{xx}e_{kk}$$

in (2), with $x := z$, again implies that $u_{yy} = u_{zz}$. Consequently, $e_{il}U = Ue_{il}$. Therefore, $U \in C_{\mathbf{M}_n(\rho,R)}(\{e_{kl}\colon (k,l) \in \rho_A\})$. $\square$

For $\mathbb{M}_9(\rho, R)$ in Example 2.2 the equivalence classes induced by $\equiv_\rho$ on *Rep* are $\{1\}$, $\{3,4,6,7\}$, $\{8,9\}$, and so we conclude from Theorem 2.4 that

$$C_{\mathbb{M}_9(\rho,R)}\big(\{e_{kl}\colon\ (k,l) \in \rho_A\}\big)$$
$$= \left\{ \begin{bmatrix} u_{11} & u_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_{21} & u_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_{33} & u_{34} & u_{35} & 0 & u_{37} & 0 & 0 \\ 0 & 0 & 0 & u_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_{88} & u_{89} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_{88} \end{bmatrix} :\ u_{ij} \in R \text{ for all } i,j \right\}.$$

If y and z in Rep are such that $(y,z) \in \rho_A$, then y is a minimal element and z a maximal element in *Rep* with respect to the partial order relation ρ on *Rep* if and only if the set Λ_{yz} is a maximal element in the set $\{\Lambda_{uv}\colon\ u,v \in Rep \text{ and } (u,v) \in \rho\}$ with respect to set-inclusion. Furthermore, if $w \in Rep$ is such that $|[w]_{\equiv_\rho}| = 1$, then Λ_{ww} is also a maximal (and a minimal) element in $\{\Lambda_{uv}\colon\ u,v \in Rep \text{ and } (u,v) \in \rho\}$. Therefore Theorem 2.4 and the description of the ideals of a structural matrix ring at the beginning of Section 2 show that:

Corollary 2.5. *If $\mathbb{M}_n(\rho,R)$ is an honest structural matrix ring over an arbitrary ring R, then the largest ideal of $\mathbb{M}_n(\rho,R)$ contained in $C_{\mathbb{M}_n(\rho,R)}\big(\{e_{kl}\colon\ (k,l) \in \rho_A\}\big)$ is the ideal $\mathcal{I}_f$ in Proposition 2.1.*

Combining Proposition 2.1 and Corollary 2.5, we can now state the main result of this note:

Theorem 2.6. *Let $\mathbb{M}_n(\rho,R)$ be an honest structural matrix ring over a ring R with no proper essential ideal. Then the largest ideal of $\mathbb{M}_n(\rho,R)$ contained in the centralizer of the set of matrix units associated with the anti-symmetric part of ρ coincides with the smallest essential ideal of $\mathbb{M}_n(\rho,R)$.*

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