Stellenbosch Number Theory Conference 2023: Titles and abstracts

All talks in room 1005 of Mathematical Sciences / Industrial Psychology building

Universal Skolem Sets

Florian Luca
(joint work with Joël Ouaknine (Max–Planck Saabrücken), James Worrell (Oxford))

Wits University, South Africa

Abstract: The celebrated Skolem–Mahler–Lech theorem asserts that if $\mathbf{u} := (u_n)_{n\geq 0}$ is a linearly recurrent sequence of integers then the set of its zeros, that is the set of positive integers n such $u_n = 0$, form a union of finitely many infinite arithmetic progressions together with a (possibly empty) finite set. Except for some special cases, is not known how to bound effectively all the zeros of $\mathbf{u}$. This is called *the Skolem problem*. In this talk we present the notion of a *universal Skolem set*, which an infinite set of positive integers $\mathcal{S}$ such that for every linearly recurrent sequence $\mathbf{u}$, the solutions $u_n = 0$ with $n \in \mathcal{S}$ are effectively computable. We present a couple of examples of universal Skolem sets, one of which has positive lower density as a subset of all the positive integers.

Double Perfect Partitions of Higher Order

Augustine O. Munagi

Wits University, South Africa

Abstract: A partition of n is called perfect if it contains exactly one partition each positive integer not exceeding n. A partition is double-perfect if it contains two partitions of each integer between 2 and $n-2$. Both perfect and double-perfect partitions are known to be enumerated by ordered factorization functions. In this talk we pick any positive integer $r < n$ and show that the set of partitions of n that contain two partitions of each integer between r and $n-r$ are also enumerated by ordered factorizations functions.

On Y-coordinates of Pell equations which are Fibonacci numbers.

Faith Zottor

Wits University, South Africa

Abstract: Let $d \geq 2$ be an integer which is not a square. We show that if $(F_n)_{n\geq 0}$ is the Fibonacci sequence and $(X_m, Y_m)_{m\geq 1}$ is the m-th solution of the Pell equation $X^2 - dY^2 = \pm 1$, then the equation $Y_m = F_n$ has at most two positive integer solutions (m, n) except for $d = 2$ when it has three solutions $(m, n) = (1, 2), (2, 3), (3, 5)$.

Exponential sums with coefficients of modular forms

Victor C. García Hernández

Universidad Autónoma Metropolitana Azcapotzalco, Mexico

Abstract: Let f be a modular form with weight $k \in 2\mathbb{Z}$ and level $N \in \mathbb{Z}$ such that it has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}, \quad \Im(z) \geq 0,$$

with $a(n)$ be the nth Fourier coefficient. We shall restrict to the family of modular forms with rational coefficients.

Let ℓ denote a large prime number and p be a fixed prime. In this talk we consider the problem of finding effective upper bounds for exponential sums in $\mathbb{F}_\ell$ with the sequence $\{a(p^n) \pmod{\ell}\}_{n\in\mathbb{N}}$, that is

$$\max_{(\xi,\ell)=1} \left| \sum_{n\leq \tau} e^{2\pi i \xi \frac{a(p^n)}{\ell}} \right| \leq \tau\Delta, \quad \text{for some } \Delta \to 0 \text{ as } \ell \to \infty,$$

where τ denotes the period of the sequence $a(p^n)$ in $\mathbb{F}_\ell$. Together with J. Bajpai and S. Bhakta we obtained nontrivial bounds for a set of primes ℓ with density 1. Our approach is supported by new estimations of exponential sums with linear recurrence sequences. This impact on the Waring-Type problem of the representation of residue classes modulo ℓ as sum of a fixed number of terms $a(p^n) \pmod{\ell}$ for almost all primes ℓ.

Modular Polynomials

Florian Breuer

Newcastle University, Australia

Abstract: Classical modular polynomials are polynomials in two variables which vanish at pairs of j-invariants of elliptic curves which are linked by a cyclic isogeny of degree N, for a fixed N. These have important uses in cryptography. This talk is a survey of some results on modular polynomials, related concepts and function field analogues.

Universal families of Eulerian multiple zeta values in positive characteristic

Tuan Ngo Dac

Normandie Université, Université de Caen Normandie - CNRS, Caen, France.

Abstract: In this talk we study positive characteristic multiple zeta values associated to general curves over a finite field together with a rational point at infinity as introduced by Thakur. For the case of the projective line these values were defined as analogues of classical multiple zeta values. We first establish a general non-commutative factorization of exponential series associated to certain lattices of rank one. Next we introduce universal families of multiple zeta values of Thakur and show that they are Eulerian in full generality. In particular, we prove a conjecture of Lara Rodriguez and Thakur. One of the main ingredients of the proofs is the notion of L-series in Tate algebras introduced by Pellarin in 2012. This is a joint work with K. Chung and F. Pellarin.

On adelic surjectivity of Galois representation for Drinfeld modules

Chien-Hua Chen

National Center for Theoretical Sciences, Taiwan

Abstract: In 2009, Pink and Rütsche proved the Drinfeld module analogue of the "open image theorem". Based on the open image theorem, it is natural to ask whether there is a Drinfeld module whose adelic Galois representation is surjective. This question is the function field analogue of a question mentioned by Serre in the context of elliptic curves. In this talk, I will discuss my result on a certain class of Drinfeld modules having adelic surjective Galois representation

Zagier-Hoffman's conjectures in positive characteristic

Khac-Nhuan LE

Normandie Université, Université de Caen Normandie - CNRS, Caen, France.

Abstract: In 2015, G. Todd proposed in his PhD thesis a conjecture toward the dimension of the vector space generated by multiple zeta values in positive characteristic of a fixed weight. Soon after, Dinesh S. Thakur proposed a refinement of Todd's conjecture by giving an explicit basis. These conjectures could be regarded as an analog of Zagier-Hoffman's conjectures in the classical setting.

The aim of the talk is to present our method to completely solve the Zagier-Hoffman's conjectures in positive characteristic. We shall focus on the algebraic part of the conjectures and give some results for the transcendental part. This is a joint work with B-H. Im, H. Kim, T. Ngo Dac and L-H. Pham.

Isogeny volcanoes: an ordinary inverse problem

Fabien Pazuki

University of Copenhagen, Denmark

Abstract: We prove that any abstract ℓ-volcano graph can be realized as a connected component of the ℓ-isogeny graph of an ordinary elliptic curve defined over $\mathbb{F}_p$, where ℓ and p are two different primes. This is joint work with Henry Bambury and Francesco Campagna.

Isogenies of Elliptic Curves over Function Fields

Richard Griffon

Université Clermont-Auvergne, France

Abstract: This talk is based on a joint work with Fabien Pazuki, in which we study elliptic curves over function fields and the isogenies between them. Specifically, we prove analogues in the function field setting of two famous theorems about isogenous elliptic curves over number fields. The function field versions of these theorems, though having a similar flavour to their number field counterparts, display some striking differences. The first of these results completely describes the variation of the Weil height of the j-invariant of elliptic curves within an isogeny class. In particular, we show that the modular height remains constant under an isogeny of degree prime to the characteristic. Our second main theorem is an "isogeny estimate" in the spirit of theorems by Masser–Wüstholz and by Gaudron–Rémond. Unavoidable inseparability issues aside, we prove a uniform isogeny bound in this setting. After stating our results and giving quick sketches of their proof I will, time permitting, mention a few Diophantine applications.

Some unlikely intersections between curves in certain algebraic groups

David Smith

Stellenbosch University, South Africa

Abstract: Given two varieties X, Y of dimension r, $s \geq 0$ in a space of dimension n, in the absence of a special reason, one would expect their intersection to be empty given that $r + s \leq n$. If such an intersection is non-empty, we then refer to it as an "unlikely intersection". These intersections have been subject to much study with a number of results and conjectures formulated to show that certain (infinite) families of unlikely intersections have finite union. Specifically, many of these results and conjectures consider an algebraic group as the ambient space with algebraic subgroups playing an important role. An important example is Zilber's conjecture on intersections of tori (otherwise known as Zilber's CIT) which is a conjecture on unlikely intersections of subvarieties of algebraic tori. One statement known to follow from Zilber's CIT is that, if $N \geq 3$ and C_1 and C_2 are two irreducible curves in $\mathbb{G}_m^N(\mathbb{C})$, then, provided C_1 is not contained in a proper algebraic subgroup of $\mathbb{G}_m^N(\mathbb{C})$, at most finitely many $x \in C_1$ have the property that there is a positive integer n such that $x^n \in C_2$ and $[n]C_1 \nsubseteq C_2$, where $[n]C_1 = \{x^n : x \in C_1\}$. In a recent paper, Boxall showed this to be true provided that at least one of the curves is not defined over a number field. Now, due to the structure of algebraic subgroups in general commutative algebraic groups, it is natural to investigate the same result by replacing the algebraic torus $\mathbb{G}_m^N(\mathbb{C})$ with a power of an elliptic curve. For the purpose of this talk, we will outline elements of Boxall's result, discuss analogous concepts in the elliptic curve setting and consider how one might extend such results the result to powers of elliptic curves. In particular, we will highlight some model theoretic ideas that play a role in the argument.

Definable sets in finite structures

Dugald MacPherson

University of Leeds, UK

Abstract: Chatzidakis, van den Dries and Macintyre proved in a 1992 paper a strong uniformity result on the approximate cardinalities of definable sets in finite fields. This generalises (and uses) the Lang-Weil estimates on the number of rational points in a finite field of an absolutely irreducible variety defined over the field. The work yields an associated dimension-measure pair for definable sets in pseudofinite fields (infinite fields which satisfy every first-order sentence that holds of all finite fields). The results, suitably modified, carry across to any family of finite simple groups of fixed Lie type.

Over the last 20+ years, Charles Steinhorn, I, and others have developed various model-theoretic frameworks for which the CDM results provide a special and motivating case. I will give an overview of the framework of 'multi-dimensional asymptotic classes' of finite structures that we are currently developing (joint work with Anscombe, Steinhorn, Wolf).

The Loughran–Smeets conjecture for some Châtelet type varieties

Kevin Destagnol

Université Paris-Saclay, France

Abstract: Serre in 1990 started a research program aiming to understand the probability that a randomly chosen diophantine equation has a solution over Q. Most cases of the problem are still open today, even when the equations satisfy the Hasse principle but the Loughran–Smeets conjectures give predictions for that probability in certain cases. I will report on joint work with Efthymios Sofos regarding this problem for $x^2 - Dy^2 = P_1(t)...P_R(t)z^2$ where D is a square-free integer and P_i are fixed integer polynomials of any degree in n variables, with n relatively large compared to the degrees of the P_i.

Further remarks on some unlikely intersections between curves

Gareth Boxall

Stellenbosch University

Abstract: Given two irreducible curves in the algebraic torus of dimension 3 (over the complex numbers), if the first curve is not contained in a proper algebraic subgroup then it is conjectured that at most finitely many points on that curve have an integer power on the second curve, excluding those integers with the property that every point on the first curve has an integer power on the second. I shall discuss various partial results towards this conjecture in the case where both curves are defined over a number field.

The motivic zeta functions of Calabi-Yau varieties and the monodromy conjecture

Luigi Pagano

Max Planck Institute for Mathematics in Bonn,
guest from University of Copenhagen

Abstract: In this talk we deal with the motivic zeta function attached to Calabi-Yau varieties defined over a field K endowed with an ultrametric absolute value. I will explain what it means for a formal series with coefficients in the Grothendieck ring of varieties to be rational and how poles are defined. I will finally discuss the monodromy conjecture that relates those poles with the action of the absolute Galois group of K on the (étale) cohomology of X, with a particular focus on the case of Hilbert schemes of points on a surface.

Near-linear algebra

Daniella Moore

Stellenbosch University, South Africa

Abstract: In this talk, we will present some fundamental results for near-vector spaces toward extending classical linear algebra to near-linear algebra. We will discuss key results on the subspace span of a near-vector space, which leads to the proof that any near-vector space subspace is itself a near-vector space. We will also present how any quotient of a near-vector space by a subspace is a near-vector space and the First Isomorphism Theorem for near-vector spaces.

On linear independence of special values of generalized polylogarithm functions

Sinnou David

Institut de mathématiques de Jussieu — Paris Rive Gauche, France

Abstract: For any set of algebraic numbers in a fixed number field K satisfying standard metric conditions in the theory (close enough to zero), we prove that the values of Lerch functions (essentially polylogarithms with "shifts") are linearly independent over K.

Stellenbosch Number Theory Conference 2023 Program

Venue: Mathematical Sciences / Industrial Psychology building, room 1005

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<tr><td>9:00</td><td rowspan="2">Florian Luca</td><td rowspan="2">Florian Breuer
(via stream)</td><td rowspan="2">Richard Griffon</td><td rowspan="13">FREE DAY</td><td rowspan="2">Luigi Pagano</td></tr>
<tr><td>9:30</td></tr>
<tr><td>10:00</td><td>Augustine Munagi</td><td>Tuan Ngo Dac</td><td>David Smith</td><td rowspan="2">Daniella Moore</td></tr>
<tr><td>10:30</td><td colspan="3">COFFEE</td></tr>
<tr><td>11:00</td><td rowspan="2">Faith Zottor</td><td rowspan="2">Chien-Hua Chen</td><td rowspan="2">Dugald MacPherson</td><td>COFFEE</td></tr>
<tr><td>11:30</td><td rowspan="2">Sinnou David</td></tr>
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<tr><td>14:00</td><td rowspan="2">Victor Garcia</td><td rowspan="2">Fabien Pazuki</td><td rowspan="2">Kevin Destagnol</td></tr>
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