An introduction to Cerf paths

Generic maps to the plane

Given a smooth map $f: M \longrightarrow N$, we defined the sets $S_r(f)$ and we defined the submanifolds $S_r \subset J^1(M,N)$.

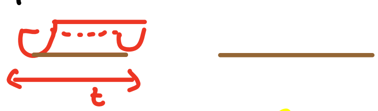
Def: A smooth map $f: M \longrightarrow N$ is called *one-generic* if $j^1 f \pitchfork S_r \; \forall r$.

Remark: Morse functions are then the one-generic maps with target $\mathbb{R}$. And immersions are the one-generic maps with high enough dimensional target.

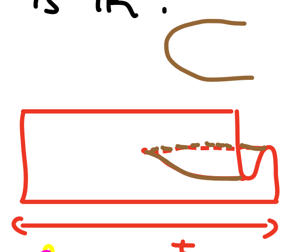
Recall that $S_r(f) = j^1 f^{-1}(S_r)$, so for one-generic maps $S_r(f)$ is a submanifold of M. When the target is $\mathbb{R}$, it contains essentially all the information needed.

But that's not the case when the target is $\mathbb{R}^2$.

For example, consider:



- $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



- $g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,t) \longmapsto (\overset{f_1}{x^2}, \overset{f_2}{t})$

$$df_{(x,t)} = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}$$

It drops rank when $x = 0$

$S_1(f) = \{(0,t)\}$

$(x,t) \longmapsto (x^3 - tx, t)$

$$dg_{(x,t)} = \begin{pmatrix} 3x^2 - t & 0 \\ -x & 1 \end{pmatrix}$$

It drops rank when $t = 3x^2$

$S_1(g) = \{(x, 3x^2)\}$

In both cases we get submanifolds of $\mathbb{R}^2$ along which the differentials drop rank by 1. The S_r classification won't give us more information which is of course not surprising – we require 3-jet information in order to distinguish the germs above. But the question still remains, in what form should we get it?

Since the $S_1(f)$, $S_1(g)$ are submanifolds we can consider

- $f|_{S_1(f)} : \mathbb{R} \longrightarrow \mathbb{R}^2$
  $t \longmapsto (0,t)$

- $g|_{S_1(g)} : \mathbb{R} \longrightarrow \mathbb{R}^2$
  $x \longmapsto (x^3 - 3x^3, 3x^2)$

$d\left(f|_{S_1(f)}\right)_t = \begin{pmatrix} 0 & 1 \end{pmatrix}$

so it never drops rank.

$d\left(g|_{S_1(g)}\right)_x = \begin{pmatrix} -6x^2 & 6x \end{pmatrix}$

so it drops rank by 1 at 0.

We should keep this in mind as a motivating example for Thom - Boardman.

<u>Def</u>: Suppose that a smooth map $f: M \longrightarrow N$ is one generic. We define $S_{r,s}(f)$ to be the locus of points in $S_r(f) \subset M$ where the differential of $f|_{S_r(f)}$ drops rank by s.

In the above example, the point $(0,0)$ lives in $S_{1,1}(g)$ and all other points $(x, 3x^2)$ live in $S_{1,0}(g)$. $S_{1,1}(f)$ is empty and $S_{1,0}(f)$ consists of all points in $S_1(f)$.

Thom - Boardman theory now proceeds as in the 1-jet case. For each submanifold $S_r \subset J^1(M,N)$, we consider $S_r^{(2)}$ the preimage of S_r under the evident map $J^2(M,N) \longrightarrow \underset{\underset{S_r}{\cap}}{J^1(M,N)}$.

We then find a stratification of $S_r^{(2)}$ by submanifolds $S_{r,0}, S_{r,1}, \dots, S_{r,s}, \dots$ such that:

if $f \pitchfork W$, then $f^{-1}(W)$

$$\boxed{x \in S_{r,s}(f) \iff j^2 f(x) \in S_{r,s}}$$

$x \in S_r(f) \iff j^1 f(x) \in S_r$

$f: X \longrightarrow \underset{\underset{W}{\cup}}{Y}$

$j^2 f: M \longrightarrow \underset{\underset{S_{r,s}}{\cup}}{J^2(M,N)}$

We then define a smooth map $f: M \to N$ to be 2-generic if $j^2 f \pitchfork S_{r,s}$ for all r, s. And we proceed iteratively.

Thom and Boardman constructed all such submanifolds $S_{r_1, r_2, \dots, r_k}$ of $J^k(M,N)$ and computed (I think?) their codimensions.

The upshot for us is that for a 2-generic map to the plane (using Thom's jet transversality theorem we know that the collection of such maps is dense) two kinds of singularities may occur:

- $S_{1,0}$ type: these are the fold points
- $S_{1,1}$ type: these are the elementary cusp points

This is especially important to us, as we want to study generic paths of smooth functions $M \longrightarrow \mathbb{R}$ which can be thought of as (level preserving) smooth maps $M \times I \longrightarrow \mathbb{R} \times I$.
(though there is a bit of a caveat here)

A coordinate independent definition of Morse and cusp critical points

Let $f: M \longrightarrow \mathbb{R}$ be a smooth function.
We denote by $Crit(f)$ the set of critical points of f.
If $p \in Crit(f)$, we can define its intrinsic Hessian

$$d^2 f_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$

as follows:

Given $v_1, v_2 \in T_pM$, choose $\gamma : \mathbb{R}^2 \longrightarrow M$ with $\gamma(0) = p$ and $\frac{\partial \gamma}{\partial x_i}(0) = v_i \in T_pM$ for $i = 1,2$ and set

$$d^2 f_p (v_1, v_2) = \left. \frac{\partial^2 (f \circ \gamma)}{\partial x_1 \partial x_2} \right|_{x_1 = x_2 = 0}$$

Claim: $d^2 f_p$ is well-defined (i.e. independent of γ)

The kernel of $d^2 f_p$ consists of all $v \in T_pM$ s.t. $d^2 f_p(w, v) = 0$, $\forall w \in T_pM$.

We can now similarly define the intrinsic third derivative at points $p \in Crit(f)$:

$$d^3 f_p : \ker d^2 f_p \times \ker d^2 f_p \times \ker d^2 f_p \longrightarrow \mathbb{R}$$

For $v_1, v_2, v_3 \in \ker d^2 f_p$, choose $\eta : \mathbb{R}^3 \longrightarrow M$ such that $\eta(0) = p$ and $\frac{\partial \eta}{\partial x_i} = v_i$. Define $d^3 f_p$ via the formula

$$d^3 f_p(v_1, v_2, v_3) = \left. \frac{\partial^3 (f \circ \eta)}{\partial x_1 \partial x_2 \partial x_3} \right|_{x_1 = x_2 = x_3 = 0}$$

Just as above, we can show that $d^3 f_p$ is well-defined (i.e. not dependent on η) by passing to a local chart.

Def: Given a function $f : M \longrightarrow \mathbb{R}$ on a manifold, we say that p_0 is a Morse critical point, if we have:

(i) $p_0 \in Crit(f)$

(ii) $\dim_{\mathbb{R}} (\ker(d^2 f_{p_0})) = 0$

Def: Given a function $f : M \longrightarrow \mathbb{R}$ on a manifold, we say that p_0 is a *cusp* critical point, if we have:

(i) $p_0 \in Crit(f)$

(ii) $\dim_{\mathbb{R}}(\ker(d^2 f_{p_0})) = 1$, say $\mathbb{R}v = \ker(d^2 f_{p_0})$, $v \in T_{p_0}M$

(iii) $d^3 f_{p_0}(v,v,v) \neq 0$

Example:
$$\begin{aligned} \mathbb{R}^{n+1} &\longrightarrow \mathbb{R} \\ (x_1, \dots, x_{n+1}) &\longmapsto x_{n+1}^3 - x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_n^2 \end{aligned}$$

$$\begin{pmatrix} 0 & & & & & & \\ & -2 & & & & & \\ & & \ddots & & & & \\ & & & -2 & & & \\ & & & & 2 & & \\ & & & & & 2 & \\ & & & & & & \ddots \end{pmatrix}$$

Pre-Cerf paths

Let $\mathcal{F} := \mathcal{C}^\infty(M; \mathbb{R})$ and denote by $\mathcal{F}^0 \subset \mathcal{F}$ the subspace of $\mathcal{F}$ consisting of excellent Morse functions.

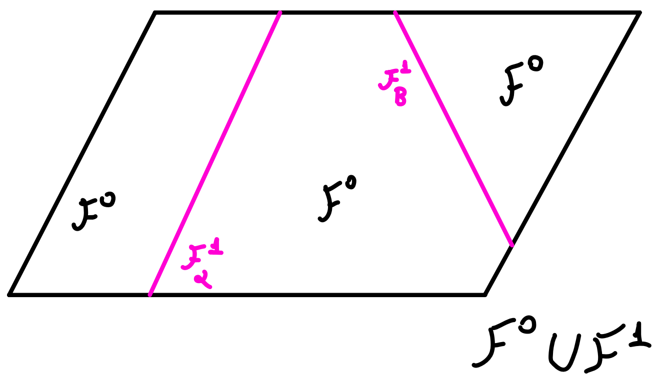
Let $\mathcal{F}_1^1 \subset \mathcal{F}$ be the subspace of $\mathcal{F}$ consisting of smooth functions f such that:

- f has a single cusp critical point
- all other critical points of f are of Morse type
- all critical values of f have multiplicity 1

Let $\mathcal{F}^1_B \subset \mathcal{F}$ be the subspace of $\mathcal{F}$ consisting of smooth functions f such that:

- all critical points of f are of Morse type
- there is exactly one critical value of f with multiplicity 2 and all other critical values of f have multiplicity 1.

Let $\mathcal{F}^1 = \mathcal{F}^1_\alpha \cup \mathcal{F}^1_B$. In this lecture, we will exclusively deal with the subspace $\mathcal{F}^0 \cup \mathcal{F}^1 \subset \mathcal{F}$.



A pre-Cerf path in $\mathcal{F}$ is a smooth family

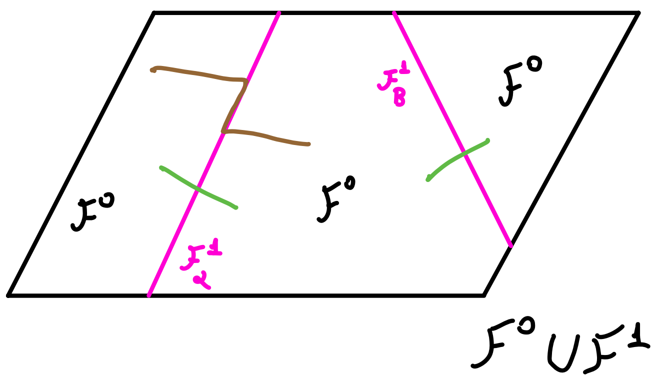
$$F : \mathbb{R} \times M \longrightarrow \mathbb{R}$$
$$(\lambda, p) \longmapsto F(\lambda, p) := f_\lambda(p)$$

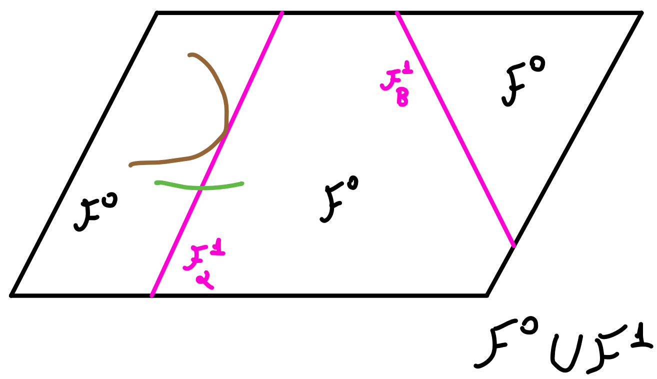
for which there is a discrete set $(t_i)_{i \in I} \in \mathbb{R}$ such that:

- $f_\lambda \in \mathcal{F}^0$ for $\lambda \notin \{t_i \mid i \in I\}$
- $f_\lambda \in \mathcal{F}^1$ for $\lambda \in \{t_i \mid i \in I\}$
- If $f_\lambda \in \mathcal{F}^1$, there exist $\varepsilon > 0$, so that

$f_{\lambda'} \in \mathcal{F}_0 \quad \forall \lambda' \in [\lambda-\varepsilon, \lambda) \cup (\lambda, \lambda+\varepsilon]$ and $f_{\lambda-\varepsilon}$, $f_{\lambda+\varepsilon}$ lie in different connected components of $\mathcal{F}_0$.

Examples of paths that are not allowed:





Remark: $\mathcal{F}^0 \cup \mathcal{F}^1$ is a codimension 1 stratified space. A pre-Cerf path then is what Cerf calls a "good path" in $\mathcal{F}^0 \cup \mathcal{F}^1$. A pre-Cerf path with $I = \{*\}$ is what Cerf calls a "crossing path" in $\mathcal{F}^0 \cup \mathcal{F}^1$. Note that here we will reserve the term "crossing path" only for crossing paths in $\mathcal{F}^0 \cup \mathcal{F}^1_B$.

Birth-death critical points

Let $f \in \mathcal{F}^1_\alpha$ with cusp critical point $p_0 \in M$ and $\mathbb{R}v = \ker(d^2 f_{p_0})$.
A generic smooth family

$$\mathbb{R} \times M \xrightarrow{F} \mathbb{R} : (\lambda, p) \longmapsto F(\lambda, p) := f_\lambda(p)$$

containing f as f_{λ_0} will satisfy the condition:

$3\lambda^2 x_n$

$$d\left(\frac{\partial F}{\partial \lambda}\bigg|_{\lambda=\lambda_0}\right)_{p_0}(v) \neq 0 .$$

$$\boxed{x_n^3 - \lambda x_n + x_{n-1}^2 + \ldots + x_k^2 - x_{k-1}^2 - \ldots - x_1^2}$$

We call p_0 a birth-death critical point for the family $(f_\lambda)_{\lambda \in \mathbb{R}}$ at $\lambda = \lambda_0$.

Definition: A Cerf path is a pre-Cerf path $F: \mathbb{R} \times M \longrightarrow \mathbb{R}$ which is generic in the above sense.

Definition: A Cerf path that crosses $\mathcal{F}^1$ only once at a birth-death critical point is called a birth-death path.

Definition: A Cerf path that crosses $\mathcal{F}^1$ only once at a $\mathcal{F}^1_B$ point is called a crossing path. (not standard!)

$$3x_n^2 - t = 0$$

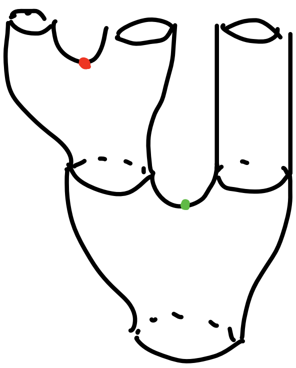
$$f : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$$

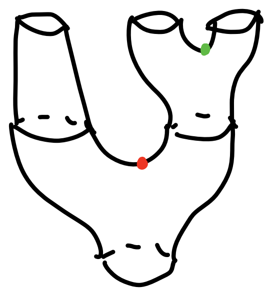
$$(x_1, \dots, x_n, t) \longmapsto x_n^3 + t x_n + x_1^2 + \dots$$

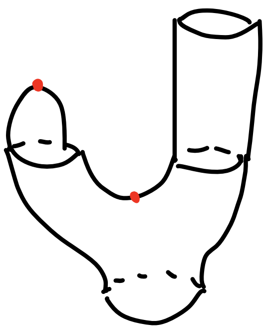
For $t < 0$, no critical points

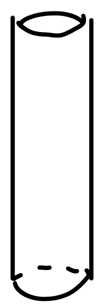
For $t = 0$, a cusp critical point

For $t > 0$, two Morse critical points







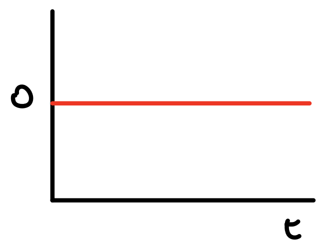


A very useful notion when working with Cerf paths is that of a Cerf graphic.

Given a Cerf path $F: \mathbb{R}\times M \longrightarrow \mathbb{R}$, we think of it as level-preserving map $F: \mathbb{R}\times M \longrightarrow \mathbb{R}\times\mathbb{R}$ and the Cerf graphic consists of the collection of critical values in $\mathbb{R}\times\mathbb{R}$.

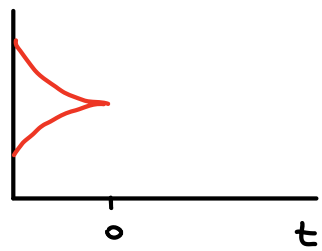
Example: $F: \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R}$

$(x, t) \longmapsto (x^2, t)$

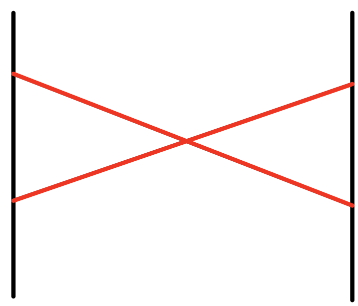


$F: \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R}$

$(x, t) \longmapsto (x^3 + tx, t)$



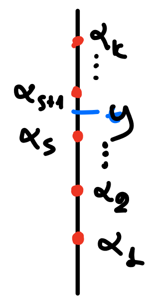
Draw a crossing example



Theorem The space of Cerf paths in $\mathcal{F}^0 \cup \mathcal{F}^1 \subset \mathcal{F}$ is an open and dense subset of the space of all paths in $\mathcal{F}$.

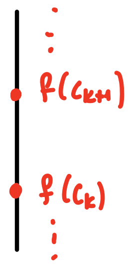
Cerf proved the following very useful theorems:

Theorem <u>Uniqueness of births</u> (Morgan's lecture notes)
Let $f \in \mathcal{F}^0(W)$ and let $\alpha_1 < \dots < \alpha_k$ be the critical values of f.



Consider the space of birth paths beginning at f where the birth appears at a level between α_s and α_{s+1}. If the level sets $W_y = f^{-1}(y)$ for $\alpha_s < t < \alpha_{s+1}$ are connected, then the space of all such birth paths is connected

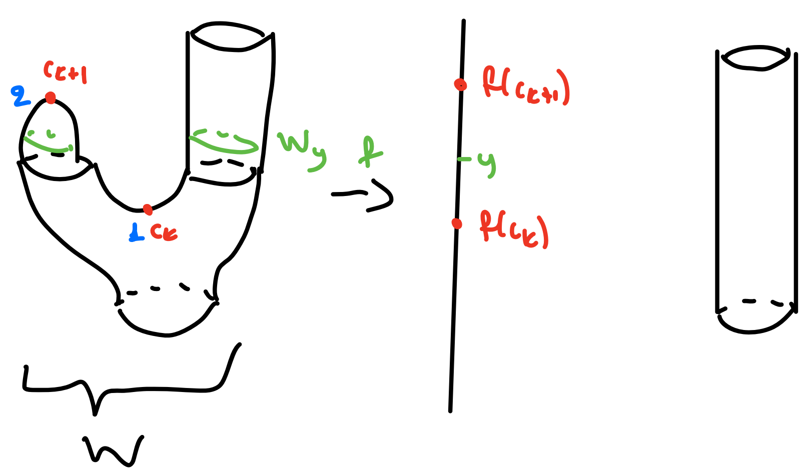
Theorem <u>Uniqueness of deaths</u> (Morgan's lecture notes)
Let $f \in \mathcal{F}^0(W)$ and suppose that $\dim W \geq 6$. Let $0 \leq i \leq n-1$ and let c_k, c_{k+1} be two consecutive critical points of f with index i and $i+1$, respectively.

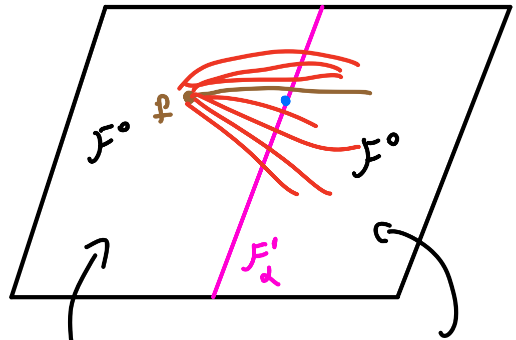


Suppose that there exists a death path starting at f that annihilates the critical points c_k, c_{k+1}. Let $y \in [f(c_k), f(c_{k+1})]$. If the level set $W_y = f^{-1}(y)$ is simply connected then the space of all death paths starting at f which annihilate c_k, c_{k+1} is connected.

To do: Recast the pseudo-isotopy problem in the language of Cerf graphics and show how the theorems above apply.

Uniqueness of deaths





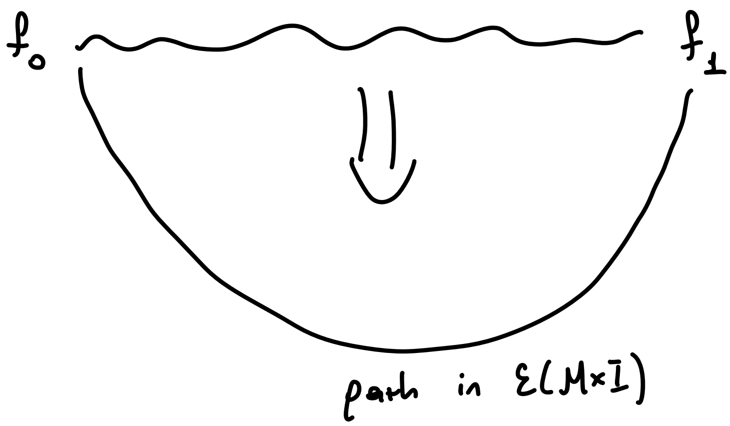
conn. comp with 2 crit points

conn. comp with 0 crit points

M

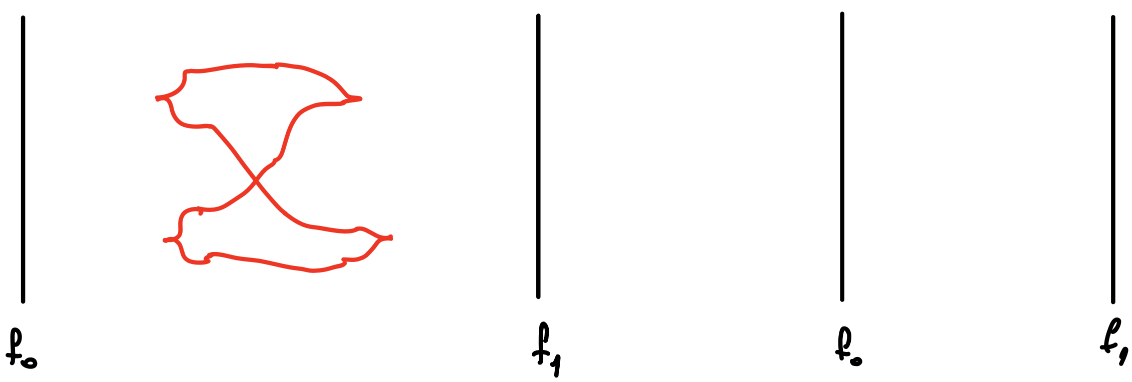
"Recall" Our objective is to calculate

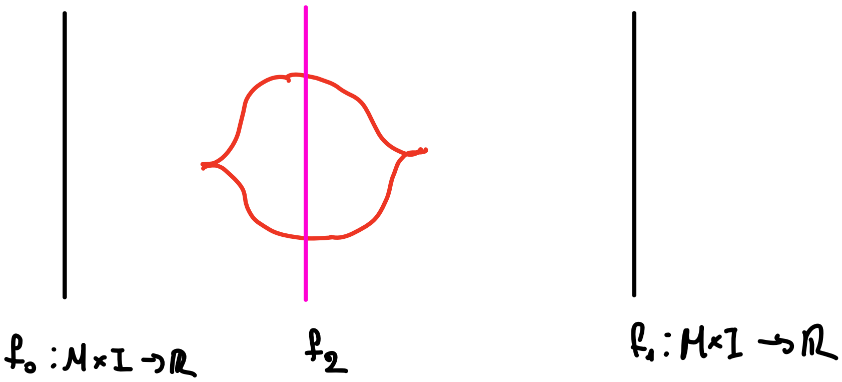
$$\pi_1(\mathcal{F}(M\times I), \mathcal{E}(M\times I)) = *$$



Cerf graphic 1

Cerf graphic 2




$f_0 : M \times I \to \mathbb{R}$
f_2
$f_1 : M \times I \to \mathbb{R}$