Thom's jet transversality theorem - II:

A short introduction to Thom-Boardman theory

Recall from last time

Last time we introduced jet bundles, Thom's jet transversality theorem and sketched how the latter can be used to derive the density of Morse functions.

Today, we will expand upon these ideas more, talk about generic maps to the plane and introduce Thom-Boardman theory. Tomorrow, we will talk about generic paths of smooth functions (Cerf paths).

Recall our broader goal is to calculate $\pi_1(\mathcal{F}(M), \mathcal{E}(M))$.

Def: A *k-jet* with source M and target N is an equivalence class of triples (x, σ, y), where:

- $x \in M$, $y \in N$
- σ is a germ of smooth functions with $\sigma(x) = y$
- two such triples (x, σ, y), (x, σ', y) are

equivalent if all partial derivatives up to order k of σ, σ' at x agree.

Def: $\underline{J^k(M,N)}$ is the set of all k-jets from M to N. It has the structure of a smooth manifold and the projections

$$\begin{array}{ccc} J^k(M,N) & J^k(M,N) & J^k(M,N) \\ \downarrow \pi_M & \downarrow \pi_N & \downarrow \pi_{M\times N} \\ M & N & M\times N \end{array}$$

are fiber bundles.

We have $J^k(\mathbb{R}^m, \mathbb{R}^n) \cong \mathbb{R}^m \times \mathbb{R}^n \times P^k(m,n)$, which we then use to obtain an atlas for $J^k(M,N)$.

<u>Correction from last time regarding the structure of $J^k(M,N)$:</u>

- $\left(\pi_{M\times N}: J^k(M,N) \longrightarrow M\times N,\ P^k(m,n)\right)$ is a fiber bundle with structure group $GL^{(k)}(m,\mathbb{R}) \times GL^{(k)}(n,\mathbb{R})$, where $GL^{(k)}(r,\mathbb{R}) := GL(r,\mathbb{R}) \times \prod_{j=2}^{k} P^j_{Sym}(r,r)$, where the group action is given by truncated composition.

In particular: If $k=1$, we get the structure of a vector bundle.

- $(\pi_M : J^k(M, \mathbb{R}^n) \longrightarrow M,\ \mathbb{R}^n \times P^k(m,n))$ is a vector bundle

A good reference where you can see all relevant computations is Chapter 1 of the book "Manifolds of Differentiable Mappings" by P W Michor.

We ended last time with:

Theorem <u>Thom's jet transversality theorem</u>

Let M and N be smooth manifolds and W a smooth submanifold of $J^k(M,N)$. Let

$$T_W := \{ f \in C^\infty(M,N) \mid j^k f \pitchfork W \}$$

Then T_W is a dense subset of $C^\infty(M,N)$. Moreover, if W is closed, T_W is open.

Morse functions are generic

We'll focus on $J^1(M,N)$ first. Consider (x, σ, y) a representative of a class in $J^1(M,N)$. Then,

$$d\sigma : T_x M \longrightarrow T_y N .$$

The maximum rank such a linear map can have is $q = \min(m,n)$.

Def: Let $f: V^m \to W^n$ be a linear and let $q = \min(m,n)$. We define the corank of f by the formula

$$\operatorname{corank}(f) = q - \operatorname{rank}(f)$$

We say that f "drops rank by r" if $\operatorname{corank}(f) = r$.

Def: Let $f: M \to N$ be a smooth map. We define

$$S_r(f) := \{ x \in M \mid df_x : T_xM \to T_{f(x)}N \text{ drops rank by } r \}$$

Def We define $S_r \subset J^1(M,N)$ to be the subset of $J^1(M,N)$ consisting of those jets that drop rank by r.

Theorem S_r is a submanifold of $J^1(M,N)$ of codimension $r^2 + r \cdot |m-n|$.

$\operatorname{Hom}(\mathbb{R}^m; \mathbb{R}^n)$

$L^r(M,N)$

Note: We define $L^r(V,W) \subset \operatorname{Hom}_{\text{Vect}}(V,W)$ to be the subset of linear maps that drop rank by r. $L^r(V,W)$ is a submanifold of $\operatorname{Hom}(V,W)$ of codimension $r^2 + r \cdot |m-n|$, which implies the theorem above. This is a linear algebra problem, one can find two detailed proofs in Guillemin-Golubitsky.

Furthermore, $\pi_{M\times N}\big|_{S_r} : S_r \longrightarrow M \times N$ is a sub-fiber bundle of $\pi_{M\times N}: J^1(M,N)_{x,y} \longrightarrow M \times N$, with fiber $L^r(\mathbb{R}^m, \mathbb{R}^n)$.

$\operatorname{Hom}(T_xM, T_yN) \cong \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}^n)$

$(x, df_x, f(x))$

Implications:

- $\boxed{x \in S_r(f) \iff j^1 f(x) \in S_r}$
- S_0 is always a codimension 0 submanifold of $J^1(M,N)$.
- If $N = \mathbb{R}$, we have:

  S_1 is of codimension $1 + 1\cdot(m-1) = m$

  $S_{r>1}$ is of codimension $r^2 + r\cdot(m-1) > m$

We thus have, for $r>1$:

$$j^1 f \pitchfork S_r \iff j^1 f \cap S_r = \emptyset$$

$$\begin{tikzcd} J^1(M,N) \arrow[d] \\ M \arrow[u, bend left, "j^1f"] \end{tikzcd}$$

Suppose $j^1 f \cap S_r = W$

$j^1 f^{-1}(W)$ is submanifold of M, call it X

$\operatorname{codim} X = \operatorname{codim} S_r$

And for $r=1$:

Claim: $j^1 f \pitchfork S_1$ at $x \iff x$ is a non-degenerate critical point of f.

Proof: We will first need a lemma that compares the transversality condition with the submersion condition.

[Recall that $f: M \longrightarrow N$ is a submersion at x if $df_x : T_x M \longrightarrow T_{f(x)} N$ is surjective (and hence of full rank).]

$f \pitchfork f(x)$ at x

Now consider the following sequence of smooth maps:

$$M \xrightarrow{f} N \xrightarrow{\phi} \mathbb{R}^k$$

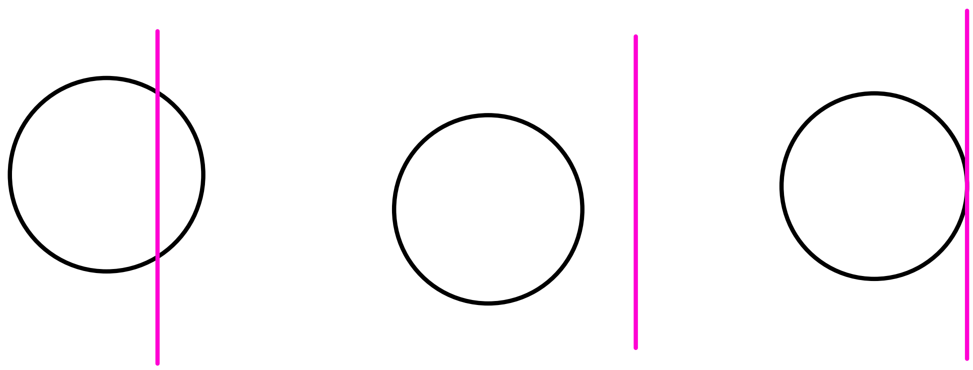
$$p \longmapsto f(p) \longmapsto 0$$

Question: If ϕ is a submersion at $f(p)$, when is

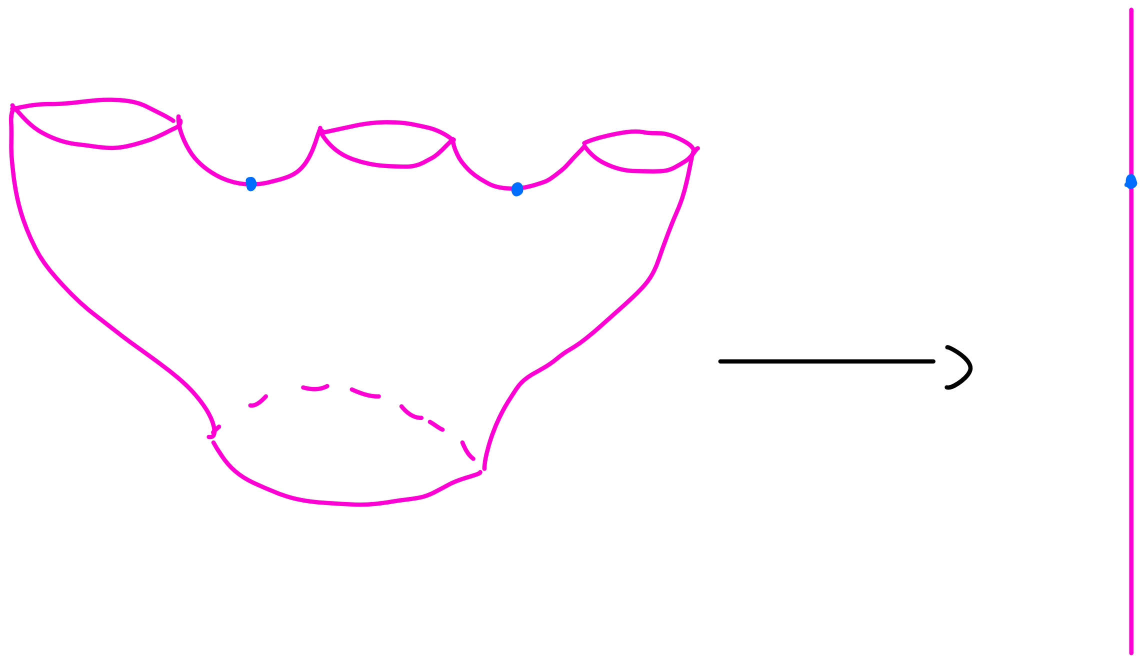
ϕ of a submersion at p?

Theorem: $f: X \longrightarrow Y$ is smooth, $W \subseteq Y$ s.t. $f \pitchfork W$, then $f^{-1}(W)$ is a sub of X and

$$\operatorname{codim}(f^{-1}(W)) = \operatorname{codim}(W)$$



$f \pitchfork W$ at p if $df(T_pX) + T_{f(p)}W = T_{f(p)}Y$



$\phi\circ f$ a submersion at p?

Lemma Given that ϕ is a submersion at $f(p)$, $\phi\circ f$ is a submersion at p iff "$f \pitchfork \phi^{-1}(0)$ at p."

Proof: Since ϕ is a submersion at $f(p)$, there exists a coordinate chart U of N centered at $f(p)$, such that $\phi(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = (x_1, \dots, x_k)$. So we can take ϕ to be a submersion in all of U and $W = \phi^{-1}(0) \cap U$ is a codimension k submanifold of $U \subset N$.

For dimension reasons we see that

$$\operatorname{Ker}(d\phi)_{f(p)} = T_{f(p)} W$$

We have that $f \pitchfork W$ at p $\Leftrightarrow$

$\Leftrightarrow$ $T_{f(p)} N = df(T_p M) \oplus T_{f(p)} W$

$\Leftrightarrow$ $T_{f(p)} N = df(T_p M) \oplus \operatorname{Ker}(d\phi)_{f(p)}$

$\Leftrightarrow$ $d(\phi\circ f)_p$ is onto given that $d\phi_{f(p)}$ is onto. $\lrcorner$

Going back to the proof of the claim, we want to apply this lemma in the case

$$M \xrightarrow{j^1 f} J^1(M, \mathbb{R}) \cong M \times \mathbb{R} \times \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}) \xrightarrow{\pi} \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}) \cong \mathbb{R}^m$$

By definition, $S_1 = \pi^{-1}(0)$, so $j^1 f \pitchfork S_1$ at x iff $\pi \circ j^1 f$ is a submersion at x.

But $\pi \circ j^1 f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$

The differential of this map is exactly the Hessian and $\pi \circ j^1 f$ is a submersion at x exactly when its differential is non-singular, which is the non-degeneracy condition. $\square$

Thom's jet transversality theorem along with the claim above shows that the space of Morse functions is open and dense in $C^\infty(M;\mathbb{R})$.

$S_1 := \{[x, 0, r]\}$ and is hence closed in $J^1(M, \mathbb{R})$

We will now introduce a variant of Thom's jet transversality theorem that will let us show that excellent Morse functions are open and dense in $C^\infty(M;\mathbb{R})$.

Def: Given a smooth manifold X, we will denote by X^s the s-fold Cartesian product and we define

$$\mathrm{Conf}_s(X) := \{(x_1, \dots, x_s) \in X^s \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Def Consider the fiber bundle $\pi_M : J^k(M,N) \longrightarrow M$ and let $\pi_M^s : J^k(M,N)^s \longrightarrow M^s$ be the induced map of s-fold Cartesian products. We denote

$$J_s^k(M,N) := (\pi_M^s)^{-1}(\mathrm{Conf}_s(M))$$

and call it the s-fold k-jet bundle of M and N.

A multijet bundle is some s-fold k-jet bundle.

If $f : M \longrightarrow N$ is a smooth map, we get an induced map

$$\begin{aligned} j_s^k f : \mathrm{Conf}_s(M) &\longrightarrow J_s^k(M,N) \\ (x_1, \dots, x_s) &\longmapsto (j^k f(x_1), \dots, j^k f(x_s)) \end{aligned}$$

Theorem <u>Thom's multijet transversality theorem</u>

Let M and N be smooth manifolds and let W be a submanifold of $J_s^k(M,N)$. Let

$$T_W := \{ f \in C^\infty(M,N) \mid j_s^k f \pitchfork W \}$$

Then T_W is dense in $C^\infty(M,N)$. Moreover, if W is compact, T_W is open.

$$\pi_{\mathbb{R}}^2 : J^1(M,\mathbb{R}) \times J^1(M,\mathbb{R}) \longrightarrow \mathbb{R} \times \mathbb{R}$$

Lemma The space of excellent Morse functions is an open and dense subset of $C^\infty(M,N)$.

Proof: Take $N=\mathbb{R}$ and $W = S_1 \times S_1 \cap (\pi^2_{\mathbb{R}})^{-1}(\Delta_{\mathbb{R}})$.

By the multijet transversality theorem, the collection of $f \in C^\infty(M;\mathbb{R})$ such that $j^1_2 f \pitchfork W$ is a dense subset of $C^\infty(M;\mathbb{R})$.

Since $\dim W = 2\cdot \dim S_1 - 1$, $\operatorname{codim} W = 2 \operatorname{codim} S_1 + 1 =$

$$= 2\cdot \dim M + 1 =$$

$$j^1_2 f : \underset{(x,y)}{\mathrm{Conf}_2(M)} \longrightarrow J^1_2(M,\mathbb{R})$$

$$= \dim \mathrm{Conf}_2(M) + 1$$

Thus, $j^1_2 f \pitchfork W$ iff $j^1_2 f \cap W = \emptyset$.

This means that the space of functions that don't have (two critical points with the same critical value) is dense in $C^\infty(M;\mathbb{R})$.

The intersection of an open and dense subset with a dense subset is dense, hence the space of excellent Morse functions is dense in $C^\infty(M;\mathbb{R})$.

The space of excellent Morse functions is an open subset of the space of Morse functions and hence an open subset of $C^\infty(M;\mathbb{R})$.

Interlude on Whitney's immersion theorem

We are going a bit off-track here but this is a fun application of the transversality theorems that is worth mentioning.

We have extensively thought about when a map with target $\mathbb{R}$ is generic.
We will soon explain when a map with target $\mathbb{R}^2$ is generic.
In this interlude, we will think about the case of a higher dimensional target.

Recall: A smooth map $f: M \longrightarrow N$ is an immersion at $p \in M$, if $df_p : T_pM \longrightarrow T_{f(p)}N$ is injective (and thus of full rank). f is an immersion if it is an immersion at every $p \in M$.

$S_r(f) = \emptyset \ \forall r>0$

By definition, f is an immersion at $p \iff$

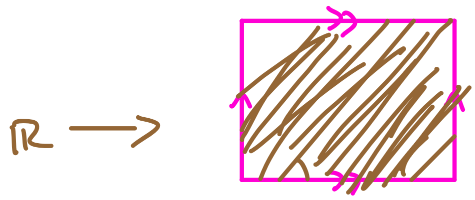
$j^1f(x) \in S_r \iff x \in S_r(f)$ $\qquad j^1 f \cap \left(\bigcup_{r>0} S_r\right) = \emptyset$

Theorem <u>Whitney's Immersion theorem</u>

Let M and N be smooth manifolds with $\dim N \geq 2 \cdot \dim M$. Then $\mathrm{Imm}(M,N)$ is an open and dense subset of $C^\infty(M,N)$.

Proof: We have $\mathrm{Imm}(M,N) = (j^1 f)^{-1}(S_0)$ which is open since S_0 is open in $J^1(M,N)$.

By Thom's jet transversality theorem, the collection of maps whose k-jet section is transverse to $S_1, S_2, \dots$ is dense in $C^\infty(M,N)$.
But $\operatorname{codim} S_r = r^2 + r(n-m) \geq r^2 + rm > m$,
hence for $r>0$, $j^1 f \pitchfork S_r \iff j^1 f \cap S_r = \emptyset$, which implies the theorem.



Theorem <u>Whitney's 1-1 immersion theorem</u>

Let M and N be smooth manifolds with $\dim N \geq 2 \cdot \dim M + 1$. Then the set of 1-1 immersions of M into N is a dense subset of $C^\infty(M,N)$.

Remark: It is also open, but we won't prove it here.

Proof: The intersection of an open and dense with a dense subset is dense. Hence, it suffices to prove that the space of 1-1 maps from M to N is dense.

We will use the multijet transversality theorem and focus on the 0-jet space. We have

$\pi_N^{(2)} : J_2^0(M,N) \longrightarrow N^2$ and set

$W = (\pi_N^{(2)})^{-1}(\Delta_N)$

$\pi_N^{(2)}$ is a submersion, so W is a submanifold of $J_2^0(M,N)$ and

$$\operatorname{codim} W = \operatorname{codim} \Delta_N = \dim N \geq 2\dim M + 1 > 2\dim M = \dim \mathrm{Conf}_2(M)$$

Thus $j_2^0 f \pitchfork W$ iff $j_2^0 f \cap W = \emptyset$ and Thom's multijet transversality theorem completes the argument. $\square$

Generic maps to the plane

Given a smooth map $f: M \longrightarrow N$, we defined the sets $S_r(f)$ and we defined the submanifolds $S_r \subset J^1(M,N)$.

Def: A smooth map $f: M \longrightarrow N$ is called *one-generic* if $j^1 f \pitchfork S_r \quad \forall r$.

Remark: Morse functions are then the one-generic maps with target $\mathbb{R}$. And immersions are the one-generic maps with high enough dimensional target.

Recall that $S_r(f) = j^1 f^{-1}(S_r)$, so for one-generic maps $S_r(f)$ is a submanifold of M. When the target is $\mathbb{R}$, it contains essentially all the information needed.

But that's not the case when the target is $\mathbb{R}^2$.
For example, consider:

- $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
  $(x,t) \longmapsto (x^2, t)$

- $g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
  $(x,t) \longmapsto (x^3 - tx, t)$

$$df_{(x,t)} = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}$$

It drops rank when $x=0$

$$S_1(f) = \{(0,t)\}$$

$$dg_{(x,t)} = \begin{pmatrix} 3x^2-t & 0 \\ -x & 1 \end{pmatrix}$$

It drops rank when $t=3x^2$

$$S_1(g) = \{(x,3x^2)\}$$

In both cases we get submanifolds of $\mathbb{R}^2$ along which the differentials drop rank by 1. The S_r classification won't give us more information which is of course not surprising – we require 3-jet information in order to distinguish the germs above. But the question still remains, in what form should we get it?

Since the $S_1(f)$, $S_1(g)$ are submanfolds we can consider

- $f|_{S_1(f)} : \mathbb{R} \longrightarrow \mathbb{R}^2$
  $t \longmapsto (0,t)$

- $g|_{S_1(g)} : \mathbb{R} \longrightarrow \mathbb{R}^2$
  $x \longmapsto (x^3-3x^3, 3x^2)$

$d\left(f\vert_{S_1(f)}\right)_t = \begin{pmatrix} 0 & 1 \end{pmatrix}$	$d\left(g\vert_{S_1(g)}\right)_x = \begin{pmatrix} -6x & 6x \end{pmatrix}$
so it never drops rank.	so it drops rank by 1 at 0.

We should keep this in mind as a motivating example for Thom-Boardman.

Def: Suppose that a smooth map $f: M \longrightarrow N$ is one generic. We define $S_{r,s}(f)$ to be the locus of points in $S_r(f) \subset M$ where the differential of $f|_{S_r(f)}$ drops rank by s.

In the above example, the point $(0,0)$ lives in $S_{1,1}(g)$ and all other points $(x, 3x^2)$ live in $S_{1,0}(g)$. $S_{1,1}(f)$ is empty and $S_{1,0}(f)$ consists of all points in $S_1(f)$.

Thom-Boardman theory now proceeds as in the 1-jet case. For each submanifold $S_r \subset J^1(M,N)$, we consider $S_r^{(2)}$ the preimage of S_r under the evident map $J^2(M,N) \longrightarrow J^1(M,N)$.

We then find a stratification of $S_r^{(2)}$ by submanifolds $S_{r,0}, S_{r,1}, \dots, S_{r,s}, \dots$ such that:

$$\boxed{x \in S_{r,s}(f) \iff j^2 f(x) \in S_{r,s}}$$

We then define a smooth map $f: M \longrightarrow N$ to be 2-generic if $j^2 f \pitchfork S_{r,s}$ for all r, s. And we proceed iteratively.
Thom and Boardman constructed all such submanifolds $S_{r_1, r_2, \dots, r_k}$ of $J^k(M,N)$ and computed (I think?) their codimensions.

The upshot for us is that for a 2-generic map to the plane (using Thom's jet transversality theorem we know that the collection of such maps is dense) two kinds of singularities may occur:
— $S_{1,0}$ type: these are the fold points

— $S_{1,1}$ type : these are the elementary cusp points

This is especially important to us, as we want to study generic paths of smooth functions $M \longrightarrow \mathbb{R}$ which can be thought of as (level preserving) smooth maps $M \times I \longrightarrow \mathbb{R} \times I$.
(though there is a bit of a caveat here)

A coordinate independent definition of Morse and cusp

Let $f: M \longrightarrow \mathbb{R}$ be a smooth function. The point $p \in M$ is called critical point if the differential df_p is 0. This means that for any chart $\varphi: U \subset M \longrightarrow \mathbb{R}^n$ around p, $\frac{\partial (f \circ \varphi^{-1})}{\partial x_i} (\varphi(p)) = 0$ for $i = 1, \dots, n$.

We denote by $\mathrm{Crit}(f)$ the set of critical points of f.
If $p \in \mathrm{Crit}(f)$, we can define its intrinsic Hessian (intrinsic second derivative map)

$$d^2 f_p : T_p M \times T_p M \longrightarrow \mathbb{R}$$

as follows:

Given $v_1, v_2 \in T_p M$, choose $\gamma: \mathbb{R}^2 \longrightarrow M$ with $\gamma(0) = p$

and $\frac{\partial\gamma}{\partial x_i}(0) = v_i \in T_pM$ for $i=1,2$ and set

$$d^2 f_p(v_1, v_2) = \left.\frac{\partial^2 (f\circ\gamma)}{\partial x_1 \partial x_2}\right|_{x_1=x_2=0}$$

Claim: $d^2 f_p$ is well-defined (i.e. independent of γ)

Proof: Pick a chart $\varphi: U \subset M \longrightarrow \mathbb{R}^n$ and use the chain rule. We use the notation $f\circ\varphi^{-1} =: f_\varphi$, $\varphi\circ\gamma =: \gamma^\varphi$ and $\gamma^\varphi(x_1,x_2) = (\gamma_1^\varphi(x_1,x_2), \dots, \gamma_n^\varphi(x_1,x_2))$

We now have:

LHS is independent of φ!

$$\frac{\partial^2\left(\left(\overbrace{f\circ\varphi^{-1}}^{\mathbb{R}^n\to\mathbb{R}}\right)\circ\left(\overbrace{\varphi\circ\gamma}^{\mathbb{R}^2\to\mathbb{R}^n}\right)\right)}{\partial x_1\partial x_2} = \frac{\partial}{\partial x_1}\sum_{i=1}^n \frac{\partial(f\circ\varphi^{-1})}{\partial y_i}\circ(\varphi\circ\gamma)\cdot\frac{\partial\gamma_i^\varphi}{\partial x_2} =$$

$$= \frac{\partial}{\partial x_1}\sum_{i=1}^n \frac{\partial f_\varphi}{\partial y_i}\circ\gamma^\varphi\cdot\frac{\partial\gamma_i^\varphi}{\partial x_2} =$$

$$= \sum_{i=1}^n \frac{\partial}{\partial x_1}\left(\frac{\partial f_\varphi}{\partial y_i}\circ\gamma^\varphi\right)\cdot\frac{\partial\gamma_i^\varphi}{\partial x_2} + \sum_{i=1}^n \frac{\partial f_\varphi}{\partial y_i}\circ\gamma^\varphi\cdot\frac{\partial^2\gamma_i^\varphi}{\partial x_1\partial x_2} =$$

$$= \sum_{i=1}^n\left[\sum_{j=1}^n \frac{\partial^2 f_\varphi}{\partial y_j\partial y_i}\circ\gamma^\varphi\cdot\frac{\partial\gamma_j^\varphi}{\partial x_1}\right]\cdot\frac{\partial\gamma_i^\varphi}{\partial x_2} + \dots \quad =$$

$$= \sum_{i,j=1}^{n} \frac{\partial^2 f_\varphi}{\partial y_j \partial y_i} \circ \gamma^\varphi \cdot \frac{\partial \gamma_j^\varphi}{\partial x_1} \cdot \frac{\partial \gamma_i^\varphi}{\partial x_2} + \sum_{i=1}^{n} \frac{\partial f_\varphi}{\partial y_i} \circ \gamma^\varphi \cdot \frac{\partial^2 \gamma_i^\varphi}{\partial x_1 \partial x_2}$$

This is a function $\mathbb{R}^2 \to \mathbb{R}$ and we wish to evaluate it at $(0,0)$.

Since $p \in Crit(f)$, $\frac{\partial f_\varphi}{\partial y_i} \circ \gamma^\varphi (0,0) = \frac{\partial (f \circ \varphi^{-1})}{\partial y_i} (\varphi \circ \gamma (0,0)) =$

$\frac{\partial (f \circ \varphi^{-1})}{\partial y_i} (\varphi(p)) = 0 \quad \forall i$, hence the second summand vanishes

For the first summand, we have:

$$\left.\frac{\partial \gamma_j^\varphi}{\partial x_1}\right|_{x_1 = x_2 = 0} = \left.\frac{\partial (\varphi \circ \gamma)_j}{\partial x_1}\right|_{x_1 = x_2 = 0} = \left[\left.\frac{\partial (\varphi \circ \gamma)}{\partial x_1}\right|_{x_1 = x_2 = 0} \right]_j =$$

$$= \left(d\varphi_p (v_1) \right)_j$$

Similarly, $\frac{\partial \gamma_i^\varphi}{\partial x_2} = \left(d\varphi_p (v_2) \right)_i$

We thus have:

$$\left.\frac{\partial^2 (f \circ \gamma)}{\partial x_1 \partial x_2}\right|_{x_1 = x_2 = 0} = \sum_{i,j=1}^{n} \frac{\partial^2 (f \circ \varphi^{-1})}{\partial x_1 \partial x_2} (\varphi(p)) \left(d\varphi_p (v_1) \right)_j \left(d\varphi_p (v_2) \right)$$

which is independent of γ.

We have shown that the intrinsic Hessian map
$$d^2 f_p : T_p M \times T_p M \longrightarrow \mathbb{R}$$
is well-defined!
By construction, this map is symmetric and bilinear (we need to pass to a local coordinate chart to see this)

The kernel of $d^2 f_p$ consists of all $v \in T_p M$ s.t. $d^2 f_p (w, v) = 0$, $\forall w \in T_p M$.
Given the computation above, we can characterize it as:
$$\ker d^2 f_p := \Big\{ v \in T_p M \;\Big|\; \sum_{i=1}^{n} \frac{\partial^2 (f \circ \varphi^{-1})}{\partial x_i \partial x_j} (\varphi(p)) \, (d\varphi_p(v))_i = 0$$
$$\text{for some chart } \varphi \text{ and all } j = 1, \dots, n \Big\}.$$

We can now similarly define the intrinsic third derivative at points $p \in \mathrm{Crit}(f)$:

$$d^3 f_p : \ker d^2 f_p \times \ker d^2 f_p \times \ker d^2 f_p \longrightarrow \mathbb{R}$$

For $v_1, v_2, v_3 \in \ker d^2 f_p$, choose $\eta : \mathbb{R}^3 \longrightarrow M$ such that $\eta(0) = p$ and $\frac{\partial \eta}{\partial x_i} = v_i$. Define $d^3 f_p$ via the formula

$$d^3 f_p (v_1, v_2, v_3) = \frac{\partial^3 (f \circ \eta)}{\partial x_1 \partial x_2 \partial x_3} \Bigg|_{x_1 = x_2 = x_3 = 0}$$

Just as above, we can show that $d^3 f_p$ is well-defined (i.e. not dependent on η) by passing to a local chart.

Def: Given a function $f : M \longrightarrow \mathbb{R}$ on a manifold, we say that p_0 is a Morse critical point, if we have:

(i) $p_0 \in Crit(f)$

(ii) $\dim_{\mathbb{R}} (\ker(d^2 f_{p_0})) = 0$

Def: Given a function $f : M \longrightarrow \mathbb{R}$ on a manifold, we say that p_0 is a cusp critical point, if we have:

(i) $p_0 \in Crit(f)$

(ii) $\dim_{\mathbb{R}} (\ker(d^2 f_{p_0})) = 1$, say $\mathbb{R} v = \ker(d^2 f_{p_0})$, $v \in T_{p_0} M$

(iii) $d^3 f_{p_0} (v, v, v) \neq 0$