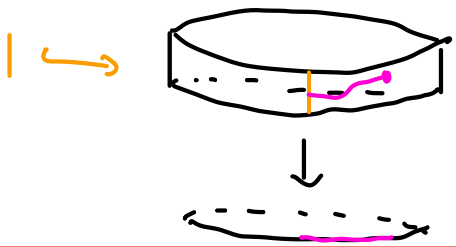
Morse functions and Thom's jet transversality theorem

Recall from last time

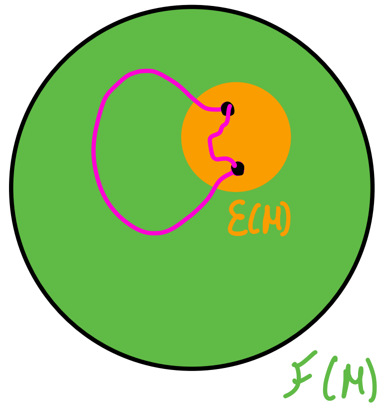
We constructed a fibration

$$\begin{tikzcd} I(M) \arrow[r, hook] & P(M) \arrow[d, "p"] & D: M\times I \to M\times I \arrow[d, mapsto] \\ & \mathcal{E}(M) & \pi_2 \circ D^{-1} \end{tikzcd}$$



Proposition: Let $D: M\times I \longrightarrow M\times I$ be a pseudo-isotopy. Then there is a path of elements $f_s \in \mathcal{E}(M)$, $0 \leq s \leq 1$, connecting $p(D) = \pi_2 \circ D^{-1}$ to the projection $\pi_2: M\times I \to I$, if and only if there is a one-parameter family of pseudo-isotopies connecting D to an isotopy.

Theorem: If $\pi_1(\mathcal{F}(M), \mathcal{E}(M)) = \{*\}$, i.e., if every path in $\mathcal{F}(M)$ with end points in $\mathcal{E}(M)$ deforms relative to its endpoints to a path in $\mathcal{E}(M)$, then every pseudo-isotopy of M deforms through pseudo-isotopies to the identity. In particular, in this case any diffeomorphism $\varphi: M \to M$ pseudo-isotopic to the identity is isotopic to the identity.



Proof: Since $\mathcal{F}(M)$ is contractible, $\pi_1(\mathcal{F}(M), \mathcal{E}(M)) = \{*\} \Rightarrow \pi_0(\mathcal{E}(M)) \simeq *$ and the result then follows from the above Proposition. $\lrcorner$

$\boxed{\text{Cerf's theorem}}$ Suppose that $\pi_1(M) = 0$ and the dimension of is at least 5. Then $\pi_1(\mathcal{F}(M), \mathcal{E}(M)) = \{*\}$, i.e., every path in $\mathcal{F}(M)$ with end points in $\mathcal{E}(M)$ deforms relative to its end points to a path in $\mathcal{E}(M)$.

Morse functions

We have "reduced" the pseudo-isotopy problem to a statement about the relative fundamental group of the pair $(\mathcal{F}(M), \mathcal{E}(M))$. This is still quite hard!

$\mathcal{F}(M)$ might be contractible but that doesn't help too much with the pair $(\mathcal{F}(M), \mathcal{E}(M))$. $\mathcal{F}(M)$ is quite complicated, but thankfully there are subspaces $\mathcal{F}^0(M)$, $\mathcal{F}^1(M)$, $\mathcal{F}^2(M)$ that we can explicitly understand with the properties:

- $\mathcal{F}^0(M)$ is dense in $\mathcal{F}(M)$
- $\mathrm{Maps}(I, \mathcal{F}^0(M) \cup \mathcal{F}^1(M))$ is dense in $\mathrm{Maps}(I, \mathcal{F}(M))$
- $\mathrm{Maps}(I^2, \mathcal{F}^0(M) \cup \mathcal{F}^1(M) \cup \mathcal{F}^2(M))$ is dense in $\mathrm{Maps}(I^2, \mathcal{F}(M))$

Using those results, we will be able to substitute $\mathcal{F}(M)$ for $\mathcal{F}'(M) = \mathcal{F}^0(M) \cup \mathcal{F}^1(M) \cup \mathcal{F}^2(M)$ and explicitly study this space.

$\mathcal{F}^0(M)$:

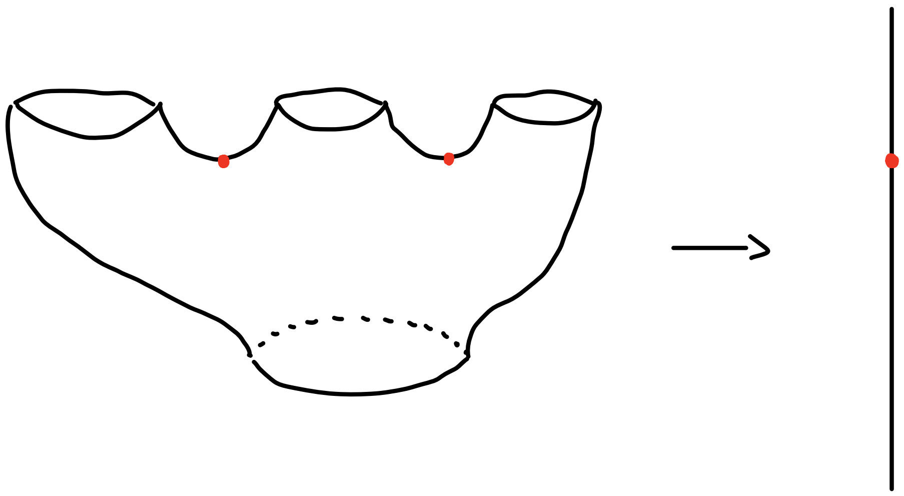
This is the space of excellent Morse functions on M.

Def: Let $f: M \longrightarrow \mathbb{R}$ be a smooth function and $p \in M$ a critical point. Recall, this means that for some $(df_p = 0)$ (and hence any) coordinate chart $(x_1, \dots, x_n)$ around p, $\left.\frac{\partial f}{\partial x_i}\right|_p = 0$. A critical point p is called non-degenerate or of Morse type if the Hessian of f at p is non-singular, i.e. $\det\left(\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_p\right)_{i,j} \neq 0$.

A smooth real-valued function f is called Morse if all its critical points are of Morse type.

A Morse function is called excellent if its critical values are pairwise distinct.

Example of a Morse function that is not excellent:



Theorem The space of excellent Morse functions $\mathcal{F}^0(N)$ is an open and dense subset of $\mathcal{F}(N) := C^\infty(N;\mathbb{R})$ in the C^∞ (in fact C^2) topology.

I also promised an explicit understanding of $\mathcal{F}^0(M)$:

Morse lemma If p is a non-degenerate critical point of f, $\exists$ a coordinate chart centered at p s.t.

$$f(x_1, \dots, x_n) = f(0) - x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_n^2$$

for some $0 \le i \le n$, called the index of p.

Proof sketch:

Step 1: Pick any coordinate system $(y_1, \dots, y_n)$ around p.

We will find functions $f_1, \dots, f_n$ s.t.

$$f(y_1,\dots,y_n) = f(0) + \sum_{i=1}^{n} y_i f_i(y_1,\dots,y_n)$$

By the fundamental theorem of calculus, we have

$$f(y_1,\dots,y_n) = f(0) + \int_0^1 \frac{d}{dt} f(ty_1,\dots,ty_n)\,dt$$

By the Chain rule, we have

$$f(y_1,\dots,y_n) = f(0) + \int_0^1 \sum_{i=1}^{n} \frac{\partial}{\partial y_i} f(ty_1,\dots,ty_n)\cdot y_i\,dt =$$

$$= f(0) + \sum_{i=1}^{n} y_i \underbrace{\int_0^1 \frac{\partial}{\partial y_i} f(ty_1,\dots,ty_n)\,dt}_{f_i}$$

Using that p is a critical point, $\frac{\partial}{\partial y_i} f(0) = 0$ and hence $f_i(0) = 0$, so we can repeat this process and find functions $h_{i,j}$ s.t. $f(y_1,\dots,y_n) = f(0) + \sum_{i,j} y_i y_j h_{i,j}(y_1,\dots,y_n)$

The proof proceeds by diagonalizing this quadratic form in a nbhd of p using non-degeneracy.

Corollary: Non-deg. critical points are isolated (finite if M compact)

Gradient-like vector fields

No introduction to Morse functions can be complete without at least a mention of gradient-like vector fields.

Def(1): Let $f: M \longrightarrow \mathbb{R}$ be a Morse function. A vector field ξ (recall, this means a smooth section of the tangent bundle $p: TM \longrightarrow M$) is a <u>gradient-like vector field</u> for f if:

- $\xi(f)_x > 0$ for all $x \in M \setminus Crit(M)$

(Recall, $\xi(f)_x = df_x(\xi(x)) \in T_{f(x)}\mathbb{R} \cong \mathbb{R}$)

- For any $p \in Crit(f)$, there exists a Morse coordinate chart U around p with coordinates $(x_1, \dots, x_i, x_{i+1}, \dots, x_n)$ so that

  - $f(x_1, \dots, x_n) = f(p) - x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_n^2$
  - $\xi(x_1, \dots, x_n) = (-2x_1, \dots, -2x_i, 2x_{i+1}, \dots, 2x_n)$

$g(\text{grad} f, v) = df(v)$

Remark: One could have chosen an alternative definition of a gradient-like vector field that may sound more "true to its name", namely

Def (2): A vector field on M is called gradient-like if it is the gradient vector field of f with respect to some Riemannian metric g on M.

Question: Does Def. (1) imply Def (2)? What about the other way around?

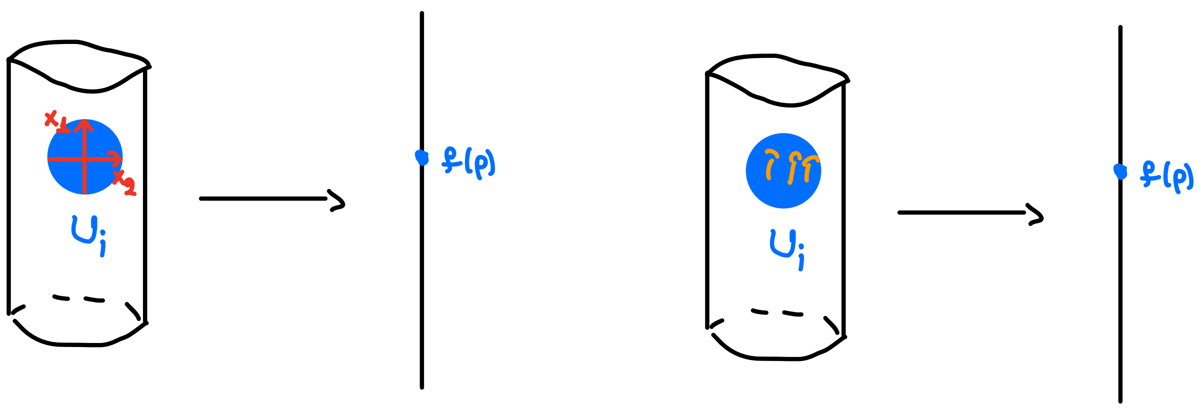
Theorem For every Morse function $f: M \to \mathbb{R}$, there exists a gradient-like vector field.

Proof: Case 1: f has no critical points

For every $p \in M$, there exists a coordinate chart $(x_1, \dots, x_n)$ centered at p s.t. f locally looks like

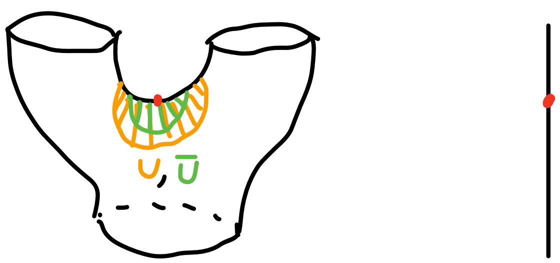
$$(x_1, \dots, x_n) \longmapsto f(p) + x_1$$

Since M is compact, there are finitely many such charts $\{U_i\}_{i \in I}$. In each U_i, choose the vector field $(1, 0, \dots, 0)$ and glue all these vector fields together using a partition of unity.



Case 2: f has a single critical point p

Choose a Morse coordinate chart U around p and an open subset $\bar{U} \subset U$. $M \setminus \bar{U}$ is compact and $f: M \setminus \bar{U} \to \mathbb{R}$ is a Morse function with no critical points. Choose a gradient-like vector field on $M \setminus \bar{U}$ using Case 1. Pull-back on U the gradient vector field of $(x_1, \ldots, x_n) \longmapsto -x_1^2 - \ldots - x_i^2 + \ldots + x_n^2$ wrt the standard Euclidean metric. Glue these vector fields together using a partition of unity.



Important take-away: Morse functions + associated gradient-like vector fields allow us to reconstruct the topology of manifolds.

For now, we will postpone discussing this to next time and concentrate instead on the genericity of Morse functions.

Jet bundles

Def: Let M, N be two smooth manifolds. A *germ* of smooth functions between M and N based at $x \in M$ is an equivalence class of pairs (U_x, f), where:

- U_x is a neighborhood of x in M
- f is a smooth map $U_x \xrightarrow{f} N$

Two such pairs (U_x, f), (U_x', f') are equivalent if there exists $V \subset U_x \cap U_x'$ such that $f|_V = f'|_V$.

Def: A *k-jet* with source M and target N is an equivalence class of triples (x, σ, y), where:

- $x \in M$, $y \in N$
- σ is a germ of smooth functions with $\sigma(x) = y$
- two such triples (x, σ, y), (x', σ', y') are equivalent if:
  - $x = x'$, $y = y'$
  - all partial derivatives up to order k of σ, σ' at x agree.

$J^k(\mathbb{R}^m, \mathbb{R}^n)$

Def: $J^k(M,N)$ is the set of all k-jets from M to N.

It has the structure of a smooth manifold and the projections

$$\begin{array}{ccccc} J^k(M,N) & , & J^k(M,N) & , & J^k(M,N) \\ \downarrow \pi_M & & \downarrow \pi_N & & \downarrow \pi_{M\times N} \\ M & & N & & M\times N \end{array}$$

are fiber bundles.

Remark: Suppose that M is m-dimensional and N is n-dimensional. Let ${}_xJ^k(M,N)_y$ denote the fiber of $J^k(M,N) \longrightarrow M\times N$ over (x,y). Then,

$${}_xJ^1(M,N)_y \cong \mathrm{Hom}(T_xM, T_yN) \cong \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$$

So, each fiber has the structure of a vector space.

But, we cannot make these identifications in a canonical way and we still only get the structure of a fiber bundle.

The Whitney topologies

We need a definition of the topology of $C^\infty(M;N)$ in order to make precise the notion that Morse functions are generic.

Def: A subspace of a topological space is called *generic* if it is open and dense.

Let X, Y be topological spaces. We can define a topology on the space of continuous maps $\mathrm{Maps}(X,Y)$, the compact-open topology, as follows: A sub-basis is given by the collection of

$$N(K,V) := \{ f : \mathrm{Map}(X,Y) \mid f(K) \subset V, \text{ where } K \subset M \text{ is compact}, \ V \subset N \text{ is open} \}$$

Remark: If we consider the category of nice topological spaces (compactly generated, weakly Hausdorff), we see that it is an enriched category using the compact-open topology. Moreover, we get a tensor-hom adjunction

$$\mathrm{Maps}(X \times Y, Z) \cong \mathrm{Maps}(X, \mathrm{Maps}(Y,Z))$$

Upshot: This is a good topology to consider on mapping spaces.

We are looking for a topology on $C^\infty(M,N)$ and, more generally, we will construct one on $C^r(M,N)$.

1^{st} attempt: Use the subspace topology for the inclusion $C^r(M,N) \subset \mathrm{Maps}(M,N)$, with $\mathrm{Maps}(M,N)$ equipped with the compact-open topology.

Not a good idea, we don't keep any track of differentiability.

2nd attempt: Each C^r-function $f: M \to N$ defines a section

$$\begin{tikzcd} J^r(M,N) \arrow[d, "\pi_M"] \\ M \arrow[u, bend left, "j^r f"] \end{tikzcd}$$

This defines a map (of sets)

$$C^r(M,N) \xrightarrow{j^r} \Gamma(M, J^r(M,N))$$

i) $\Gamma(M, J^r(M,N)) \subset \mathrm{Maps}(M, J^r(M,N))$ and is equipped with the subspace topology

ii) There is a map

$$\begin{tikzcd} J^r(M,N) \arrow[d, "\pi_N"] \\ N \end{tikzcd}$$

$$\Gamma(M, J^r(M,N)) \longrightarrow C^r(M,N)$$
$$(s: M \to J^r(M,N)) \longmapsto \pi_N \circ s$$

which is an inverse of j^r. Consequently, j^r is an inclusion and we can equip $C^r(M,N)$ with the subspace topology. This is called the <u>weak</u>

Whitney topology on $C^r(M,N)$.

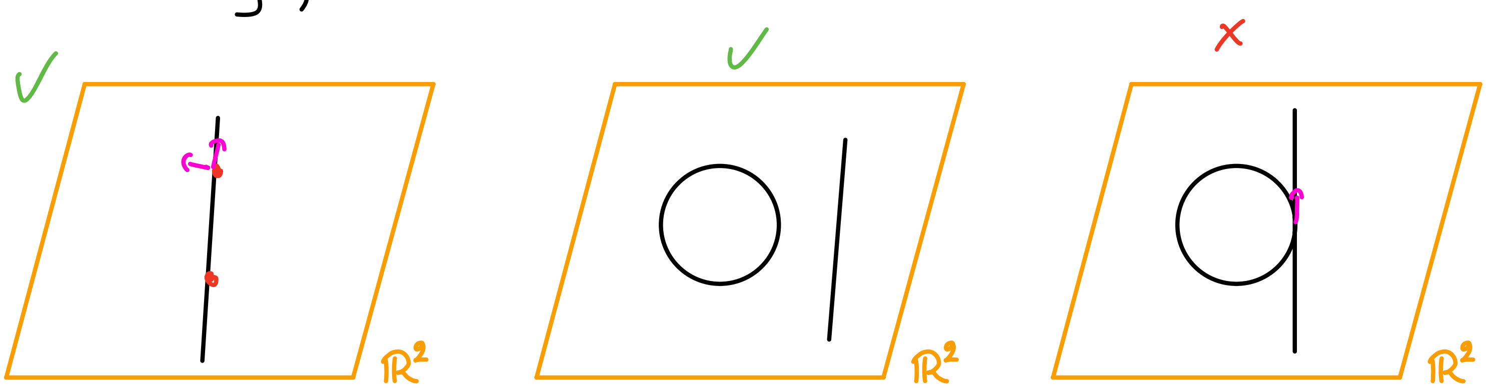
Def: For every $r \geq 0$, there exists an inclusion $C^\infty(M,N) \hookrightarrow C^r(M,N)$. We define the weak Whitney topology on $C^\infty(M,N)$ to be the coarsest topology with respect to which all these inclusions are continuous.

Now that we know with respect to which topology we try to prove that the space of Morse functions is generic, we need a tool to do so. This will be Thom's jet transversality theorem.

Thom's jet transversality theorem

A crucial notion in differential topology is transversality. It gives us a criterion for when an intersection between two smooth submanifolds is "nice".

Intuitively,



Def: Let $f: M \longrightarrow W$, $g: N \longrightarrow W$ be smooth maps and let $m \in M$, $n \in N$ be such that $f(m) = g(n) = w$. We say that f intersects g transversally at w and write $f \pitchfork g$ if

$$df(T_m M) \oplus dg(T_n N) = T_w W$$

This is the more general case of transversality for maps. By taking one or both of the maps to be inclusions of submanifolds we recover the notion of transversality between a map and a submanifold or between two submanifolds. Let's spell it out:

Def: Let $f: M \longrightarrow W$ be a smooth map and let $N \subseteq W$ be a submanifold. We say that f intersects N transversally at $x \in f^{-1}(N)$ ($f \pitchfork N$ at x) if

$$df(T_x M) \oplus T_{f(x)} N = T_{f(x)} W$$

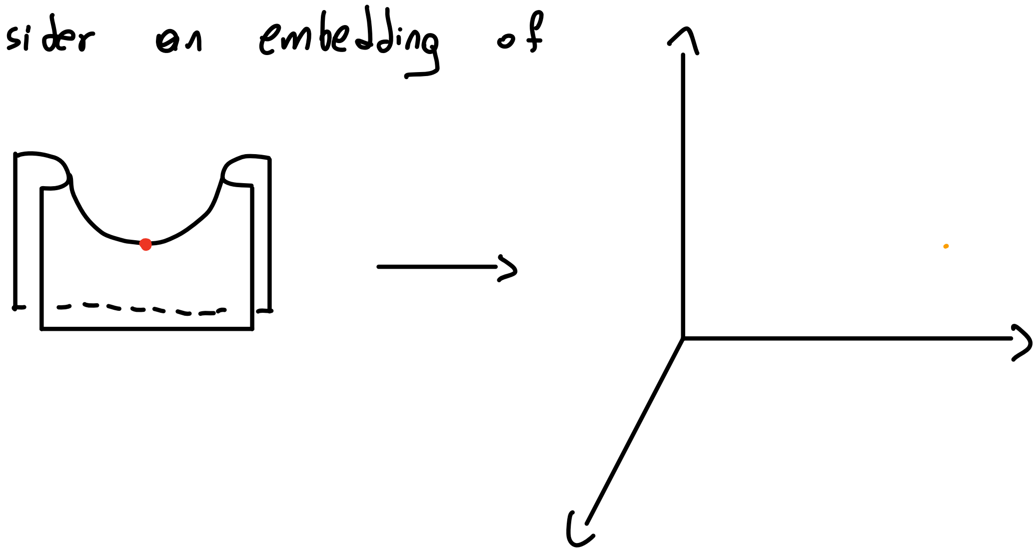
We say that f intersects N transversally ($f \pitchfork N$) if f intersects N transversally at all points $x \in f^{-1}(N)$.

Let's get a first glimpse of what transversality buys us:

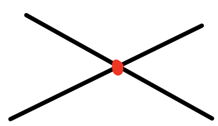
Theorem Let $f: M \longrightarrow W$ be a smooth map, $N \subset W$ a submanifold of W and $f \pitchfork N$. Then $f^{-1}(N)$ is a submanifold of M. Moreover,

$$\underbrace{\operatorname{codim}\left(f^{-1}(N)\right)}_{\dim M - \dim f^{-1}(N)} = \operatorname{codim}(N)$$

Note that this fails if we do not assume transversality. Consider $W = \mathbb{R}^3$ with $N = \{(1, x, y)\}$ and consider an embedding of

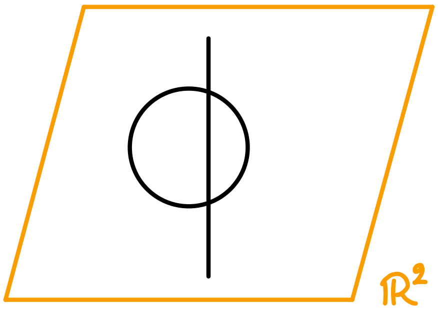


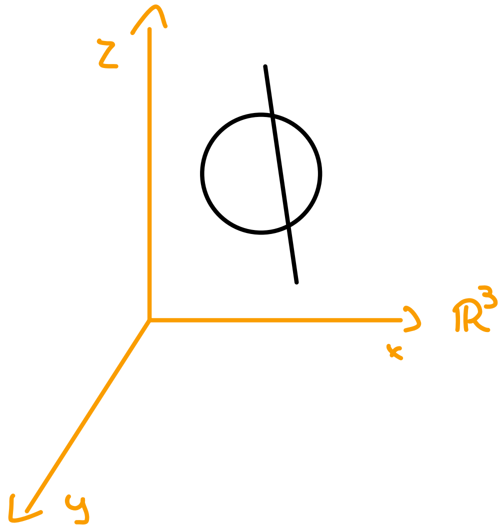
such that N is tangent to the saddle point. Then

$f^{-1}(N) =$  which is not a submanifold.

Remark: The theorem above implies that if $\dim M < \operatorname{codim}(N)$, the only way f can be transverse to N is if it doesn't intersect it at all.

Let's see how this matches intuition:





Upshot: Transversality depends crucially on the ambient manifold!

We are now ready for the main theorem:

Thom's jet transversality theorem

Let M and N be smooth manifolds and W a smooth submanifold of $J^k(M,N)$. Let

$$T_W := \{ f \in C^\infty(M,N) \mid j^k f \pitchfork W \}$$

Then, T_W is open and dense in $C^\infty(M,N)$.

For Morse functions, we get to apply this in the case $N = \mathbb{R}$ and $W = S_1 \subset J^1(M, \mathbb{R})$, where S_1 consists of those jets (x, σ, y) s.t. $d\sigma_x = 0$.

$$\dim J^1 - \dim S^1 = \dim M - \dim\left((j^1 f)^{-1}(S_1)\right)$$

It turns out that:

(1) S_1 is a submanifold of $J^1(M, \mathbb{R})$ with $\operatorname{codim}(S_1) = \dim(M)$

(2) If p is a critical point of f, then $p \in j^1 f^{-1}(S_1)$. We have:

$$p \text{ is a non-degenerate critical point} \iff j^1 f \pitchfork S_1 \text{ at } p$$

Remark: This gives an alternative proof that non-degenerate critical points are isolated.

"Def": A smooth function $f \in C^\infty(M; \mathbb{R})$ is Morse if $j^1 f \pitchfork S_1$.

⑩ References

1. John Morgan's Lectures on Morse theory
2. Guillemin - Golubitsky : Stable mappings and their singularities
3. Alexander Kupers, Lecture notes on diffeomorphism groups
4. Jean Cerf, La stratification naturelle des espaces de fonctions differentiables réelles et la théorème de la pseudo-isotopie.