An introduction to pseudo-isotopy and Morse theory

⓪ Introducing the pseudo-isotopy problem

Question: What tools do we have to study questions about smooth manifolds?

Two very influential interplays h e been:

Differential topology

Alg topology of manifolds

Smooth manifold M

Pontryagin Thom

$\downarrow$ construct

"Space" that captures some aspects of the topology (ideally smoothness) of M

$\downarrow$ compute

Algebraic invariants

Transversality

Dynamics on manifolds

Smooth manifold M

Morse-Smale (e.t)

$\downarrow$ choose

Smooth real-valued function (or family thereof) on M

$\downarrow$ study

Critical points and gradient flows

In order to appreciate any kind of interplay, it is helpful to bear in mind some key motivating problems it helps solve.

Steenrod's realization roblem

Given a topological space X and $\alpha \in H^n(X; \mathbb{F}_2)$, does there exist a smooth n-manifold M and continuous $f: M \to X$ s.t. $f_*([M]) = \alpha$?

$\rightsquigarrow$ Thom's theory on Cobordism

Pseudo-isotopy problem

- <u>Def</u>: Let M be a smooth, closed $(n-1)$-dim manifold and let $\mathrm{Diff}(M)$ denote the diffeomorphism group of M equipped with the Whitney C^∞-topology.

An <u>isotopy</u> of M is a smooth path

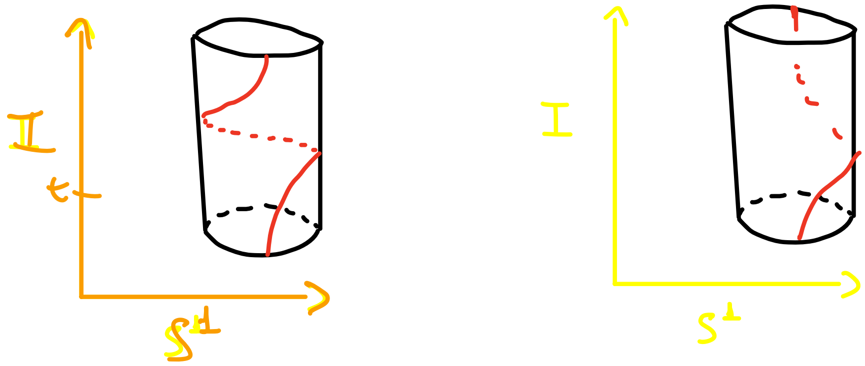
$$\begin{aligned} F: I &\longrightarrow \mathrm{Diff}(M) \\ t &\longmapsto f_t \end{aligned}$$

such that

- $f_0(x) = x \quad \forall x \in M$
- the map $\begin{aligned} M \times I &\longrightarrow M \\ (x,t) &\longmapsto f_t(x) \end{aligned}$ is smooth

Example(s) to keep in mind:

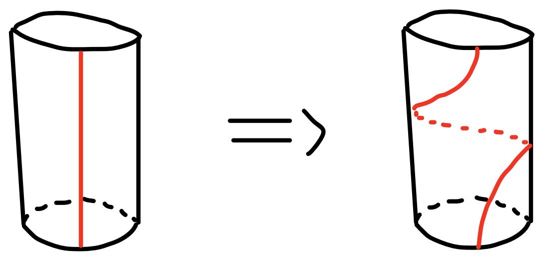
The (half) Dehn twist on the cylinder is an isotopy from the identity on S^1 to itself (rotation by π)



Def: An isotopy F induces a diffeomorphism of the cylinder

$$M \times I \longrightarrow M \times I$$
$$(x, t) \longmapsto (f_t(x), t)$$

Example to keep in mind:



Let $\mathcal{I}(M) \subset \mathrm{Diff}(M \times I)$ be the subgroup of diffeomorphisms induced by isotopies, i.e.

$h \in \mathcal{I}(M) \Rightarrow$ 1) $h(x, 0) = x$

2) $\pi_2 \circ h = \pi_2$

There is a left group action $\mathcal{I}(M) \curvearrowright \mathrm{Diff}(M)$ given by "restricting to the top diffeomorphism", i.e

$$h \cdot f(x) = h(f(x), 1)$$

Def: Two diffeomorphisms of M are called <u>isotopic</u> if they are in the same orbit under the action of $\mathcal{I}(M)$

Remark: Unpacking definitions: f isotopic g if $\exists$ level-preserving $H: M \times I \to M \times I$ s.t. $H|_{M \times \{0\}} = f$, $H|_{M \times \{1\}} = g$.

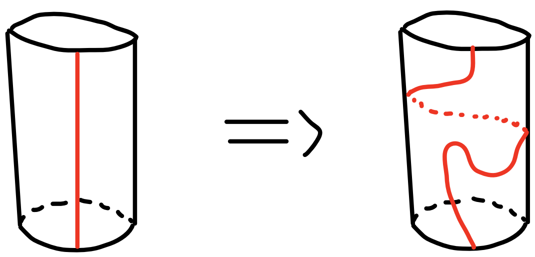
For the definition of pseudo-isotopies, we relax the level preserving condition of isotopies:

Def: A <u>pseudo-isotopy</u> of M is a diffeomorphism

$$M \times I \longrightarrow M \times I$$

$$\text{s.t.} \quad (m, 0) \longmapsto (m, 0)$$

Example to keep in mind:



Let $P(M) \subset \mathrm{Diff}(M \times I)$ be the subgroup of pseudo-isotopies. We similarly have a left group action $P(M) \curvearrowright \mathrm{Diff}(M)$ and two diffeomorphisms of M are called pseudo-isotopic if they are in the same orbit under this action.

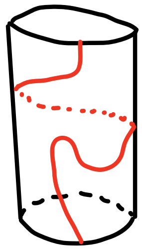
Remark: $\mathcal{I}(M) \subset P(M)$ by definition, hence isotopic $\Rightarrow$ pseudo-isotopic

Question: Under which conditions do we also have that

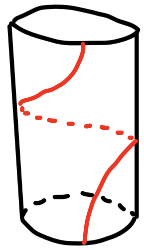
pseudo-isotopic $\Longleftrightarrow$ isotopic ?

Observation on examples we keep in mind:

There is an isotopy on the cylinder that sends



to



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(II) Cerf's results on the pseudo-isotopy problem and its applications

Theorem Let M be a smooth, closed manifold. If $\pi_1(M) = 0$ and $\dim(M) \geq 5$, then the group $\mathcal{P}(M)$ of pseudo-isotopies is connected.

Corollary 1 Under the assumptions above, pseudo-isotopy $\Longleftrightarrow$ isotopy, i.e. two diffeomorphisms of M are pseudo-isotopic iff they are isotopic.

Proof:

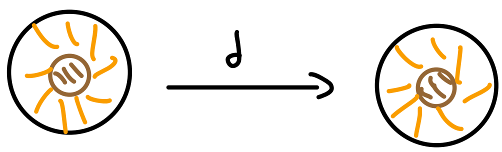
The theorem implies that for any $F \in \mathcal{P}(M) \subset \mathrm{Diff}(M \times I)$,

there exists
$$\begin{aligned}\Gamma : I &\longrightarrow \mathcal{P}(M)\\ 0 &\longmapsto F\\ 1 &\longmapsto \mathrm{Id}\end{aligned}$$

Suppose that $f, g \in \mathrm{Diff}(M)$ are pseudo-isotopic, i.e. there exists $F \in P(M)$ s.t. $f(x) = F(g(x), 1)$.

Consider the path
$$\begin{aligned}\gamma : I &\longrightarrow \mathrm{Diff}(M)\\ t &\longmapsto [x \mapsto \Gamma(t)(g(x), 1)]\end{aligned}$$

We have $\gamma(0) = f$ and $\gamma(1) = g$, hence f and g are isotopic. ⌟

id


Corollary 2 | For $n \geq 6$, we have $\pi_0(\mathrm{Diff}(D^n)) = 0$.
$\mathrm{Diff}(D^n, \partial D^n)$

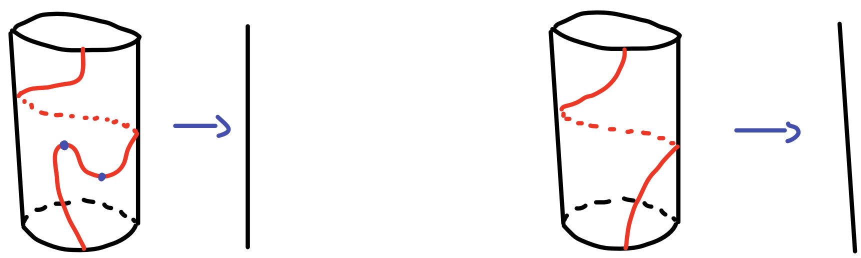
Proof: Every diffeomorphism of D^n is isotopic to a diffeomorphism that restricts to the identity on the ball of radius $1/2$.
The restriction of such a diffeomorphism to $D^n \setminus \mathring{D}^n_{1/2}$ defines a pseudo-isotopy of S^{n-1}. The theorem implies that this pseudo-isotopy must be isotopic to the identity. ⌟

Remark: This implies that D^n-disk-bundles over S^1 are trivial for $n \geq 6$ $[\pi_1(B\mathrm{Diff}(D^n)) \simeq \pi_0(\mathrm{Diff}(D^n)) \simeq 0]$.

Relation with critical points of smooth functions

How does the pseudo-isotopy problem have anything to do with critical points of smooth real-valued functions?

[We already get a climpse of that through the examples we have kept in mind:



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Def: Let $f: M \longrightarrow \mathbb{R}$ be a smooth function. The point $p \in M$ is called critical point if the differential df_p is 0. This means that for some (and thus any) chart $\varphi: U \subset M \longrightarrow \mathbb{R}^n$ around p, $\dfrac{\partial (f \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) = 0$ for $i = 1, \dots, n$.

Let $\mathcal{F}(M) := \mathcal{F}$ be the space of smooth functions $f: M \times I \to I$ having no critical points on the boundary, satisfying $f^{-1}(0) = M \times \{0\}$, $f^{-1}(1) = M \times \{1\}$ and endowed with the Whitney C^∞-topology.

Let $\mathcal{E}(M) \subset \mathcal{F}(M)$ be the open subspace of functions without critical points.

$$(A,B) \qquad \pi_i(B) \longrightarrow \cancel{\pi_i(A)} \longrightarrow \pi_i(A,B) \longrightarrow \pi_{i-1}(B) \longrightarrow \cancel{\pi_{i-1}(A)} \longrightarrow \cdots$$

Remark: $\mathcal{F}(M)$ is contractible, hence $\pi_i(\mathcal{E}(M); \pi_2) \simeq \pi_{i+1}(\mathcal{F}(M), \mathcal{E}(M); \pi_2)$, where $\pi_2 : M \times I \to I$ is the projection, by the long exact sequence for a pair.

$\rightsquigarrow$ To understand $\pi_{i+1}(\mathcal{F}(M), \mathcal{E}(M); \pi_2)$, and hence $\pi_i(\mathcal{E}(M); \pi_2)$, we need to study families of smooth real-valued functions. This is what Cerf theory is about!

Big Picture: We can use Cerf theory to understand the homotopy groups of $\mathcal{E}(M)$ (by the above remark) and we will build a fibration

$$\begin{tikzcd} \mathcal{I}(M) \arrow[r] & P(M) \arrow[d] \\ & \mathcal{E}(M) \end{tikzcd}$$

that will then give us access to the homotopy groups of $P(M)$.

▲ The fibration

$$\begin{tikzcd} I(M) \arrow[r] & P(M) \arrow[d, "p"] \\ & \mathcal{E}(M) \arrow[u, bend right, "s"'] \end{tikzcd}$$

$p \circ s = id$

$$\begin{tikzcd} M \times I \arrow[r, "D"] \arrow[d, "f"] & M \times I \arrow[d] \\ I & I \end{tikzcd}$$

First note that there is a left action of $P(M)$ on $\mathcal{F}(M)$, $P(M) \curvearrowright \mathcal{F}(M)$, given by $D \cdot f = f \circ D^{-1}$. Furthermore, this action stabilizes $\mathcal{E}(M)$.

We have: $P(M) \curvearrowright \mathcal{E}(M)$

▲ Define p as follows:

$$\begin{tikzcd} M \times I \arrow[r, "D"] \arrow[d, "\pi_2"'] & M \times I \arrow[d, "\pi_2"] \\ I \arrow[r, equal] & I \end{tikzcd}$$

$$P(M) \xrightarrow{\ p\ } \mathcal{E}(M)$$

$$(D : M \times I \to M \times I) \longmapsto (\pi_2 \circ D^{-1} : M \times I \to I)$$

Note that $\pi_2 \circ D^{-1} \in \mathcal{E}(M)$, since D is a diffeomorphism.

Note, also, that $p^{-1}(\pi_2) = I(M)$.

"Recall": A fibration is a map $p : E \to B$ that satisfies the homotopy lifting property

$$\begin{tikzcd} X \times \{0\} \arrow[r, "h"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\exists"] & B \arrow[u, bend right, "s"'] \end{tikzcd}$$

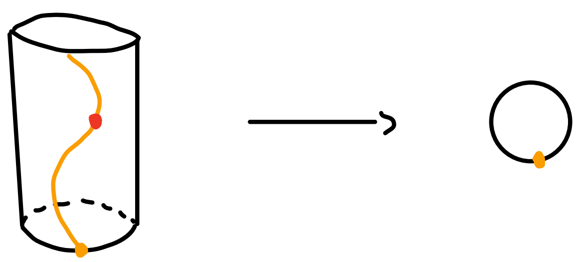
s.t. $p \circ s = id_B$

A (rather strong) way of showing that p is a fibration is by showing that it admits a section, i.e. a map $s: B \to E$ s.t. $p \circ s = \mathrm{id}$.

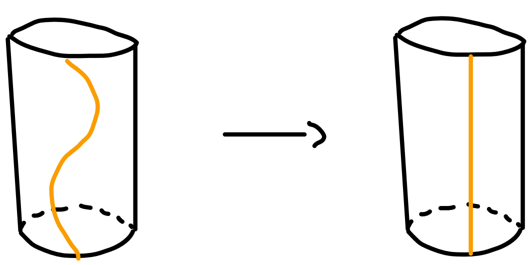
⚠ Construct a section of p, $s: \mathcal{E}(M) \to P(M)$.

To do so, we need to pick a Riemannian metric g_M on M.
Let $g = g_M \oplus dt$ be the product metric on $M \times I$.

For any $f \in \mathcal{E}(M)$, consider the gradient vector field $\nabla_g f$.
Let $\alpha_f : M \times I \longrightarrow M$ be the map that associates to any $w \in M \times I$ the initial point in $M \times \{0\}$ of the flow line of $\nabla_g f$ that passes through w.



We also get a map $\beta_f : M \times I \longrightarrow M \times I$
$w \longmapsto (\alpha_f(w), f(w))$



which lets us define the section (note: it depends on g)

$$\begin{aligned} s : \; & \mathcal{E}(M) \longrightarrow P(M) \\ & (f : M \times I \to I) \longmapsto (B_f^{-1} : M \times I \longrightarrow M \times I) \end{aligned}$$

<u>Claim</u> The above constructed map $s : \mathcal{E}(M) \longrightarrow P(M)$ is a section of $p : P(M) \longrightarrow \mathcal{E}(M)$.

$p \circ s = \mathrm{id}$

<u>Proof:</u> By construction, we have

$$\begin{tikzcd} M \times I \arrow[r, "B_f"] \arrow[d, "f"] & M \times I \arrow[d, "\pi_2"] \\ I \arrow[r, equal] & I \end{tikzcd}$$

$$\rightsquigarrow \quad p \circ s(f) = p(B_f^{-1}) = \pi_2 \circ (B_f^{-1})^{-1} = \pi_2 \circ B_f = f \qquad \lrcorner$$

Since p is a fibration, we formally know that the fiber of each point in the connected component of π_2 is homotopy equivalent to $I(M)$. A stronger statement is in fact true:

$$\begin{tikzcd} P(M) \arrow[d] & D \arrow[d, mapsto] \\ \mathcal{E}(M) & \pi_2 \circ D^{-1} \end{tikzcd}$$

<u>Claim</u> Two pseudo-isotopies D_1 and D_2 are in

the same fiber of p if and only if there is an isotopy I such that $D_1 \circ I = D_2$.

<u>Proof</u> Since D_1 and D_2 are on the same fiber of p, we must have

$$\pi_2 \circ D_1^{-1} = \pi_2 \circ D_2^{-1} \iff \pi_2 \circ (D_1^{-1} \circ D_2) = \pi_2 ,$$

which implies that $D_1^{-1} \circ D_2$ is level-preserving and hence an isotopy. We take $I = D_1^{-1} \circ D_2$. ┘

We can now state the main results underlying Cerf's approach to the pseudo-isotopy problem.

<u>Proposition</u> Let $D: M \times I \longrightarrow M \times I$ be a pseudo-isotopy. Then there is a path of elements $f_s \in \mathcal{E}(M)$, $0 \leq s \leq 1$, connecting $p(D) = \pi_2 \circ D^{-1}$ to the projection $\pi_2 : M \times I \longrightarrow I$, if and only if there is a one-parameter family of pseudo-isotopies connecting D to an isotopy.

<u>Proof</u>: ($\Leftarrow$) This direction is easy. If there is a path γ in $P(M)$ from D to an isotopy, $p(\gamma)$ is

a path in $\mathcal{E}(M)$ connecting $p(D)$ to π_2.

($\Rightarrow$) For this direction, we will use the section $S: \mathcal{E}(M) \longrightarrow P(M)$. If there is a path γ in $\mathcal{E}(M)$ connecting $p(D) = \pi_2 \circ D^{-1}$ with π_2, then $\Gamma(\gamma)$ is a path of pseudo-isotopies from a pseudo-isotopy D' that lies on the same fiber as D to an isotopy K. As we have seen, $D' = D \circ I$ for some isotopy I. Using the contractibility of the space of isotopies, there is a path from D to $D \circ I$. Composing the two, we get a path of pseudo-isotopies from D to K, which finishes the proof. $\lrcorner$

Theorem: If $\pi_1(\mathcal{F}(M), \mathcal{E}(M)) = \{*\}$, i.e., if every path in $\mathcal{F}(M)$ with end points in $\mathcal{E}(M)$ deforms relative to its endpoints to a path in $\mathcal{E}(M)$, then every pseudo-isotopy of M deforms through pseudo-isotopies to the identity. In particular, in this case any diffeomorphism $\varphi: M \to M$ pseudo-isotopic to the identity is isotopic to the identity.

Proof: $\mathcal{F}(M)$ is contractible, $\pi_1(\mathcal{F}(M), \mathcal{E}(M)) = \{*\} \Rightarrow$

$\Rightarrow \pi_0(\mathcal{E}(M)) = \{*\}$ and the result then follows from the above Proposition. $\lrcorner$

$\boxed{\text{Cerf's theorem}}$ Suppose that $\pi_1(M) = 0$ and the dimension of is at least 5. Then $\pi_1(\mathcal{F}(M), \mathcal{E}(M)) = \{*\}$, i.e., every path in $\mathcal{F}(M)$ with end points in $\mathcal{E}(M)$ deforms relative to its end points to a path in $\mathcal{E}(M)$.

Corollary: For any simply-connected M of dimension at least 5, any pseudo-isotopy of M is connected via a path of pseudo-isotopies to an isotopy and hence the identity.

Morse theory

We have "reduced" the pseudo-isotopy problem to a statement about the relative fundamental group of the pair $(\mathcal{F}(M), \mathcal{E}(M))$. This is still quite hard! $\mathcal{F}(M)$ might be contractible but that doesn't help too much with the pair $(\mathcal{F}(M), \mathcal{E}(M))$. $\mathcal{F}(M)$ is quite complicated, but thankfully there are subspaces $\mathcal{F}^0(M)$, $\mathcal{F}^1(M)$, $\mathcal{F}^2(M)$ that we can explicitly understand with the properties:

- $\mathcal{F}^0(M)$ is dense in $\mathcal{F}(M)$
- $\mathrm{Maps}(I, \mathcal{F}^0(M) \cup \mathcal{F}^1(M))$ is dense in $\mathrm{Maps}(I, \mathcal{F}(M))$
- $\mathrm{Maps}(I^2, \mathcal{F}^0(M) \cup \mathcal{F}^1(M) \cup \mathcal{F}^2(M))$ is dense in $\mathrm{Maps}(I^2, \mathcal{F}(M))$

Using those results, we will be able to substitute $\mathcal{F}(M)$ for $\mathcal{F}'(M) = \mathcal{F}^0(M) \cup \mathcal{F}^1(M) \cup \mathcal{F}^2(M)$ and explicitly study this space.

$\mathcal{F}^0(M)$:

This is the space of excellent Morse functions on M.

Def: Let $f: M \to \mathbb{R}$ be a smooth function and $p \in M$ a critical point. Recall, this means that for some (and hence any) coordinate chart $(x_1, \dots, x_n)$ around p, $\left.\frac{\partial f}{\partial x_i}\right|_p = 0$. A critical point p is called <u>non-degenerate</u> or of <u>Morse type</u> if the Hessian of f at p is non-singular, i.e. $\det\left(\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_p\right)_{i,j} \neq 0$.

A smooth real-valued function f is called <u>Morse</u> if all its critical points are of Morse type.

A Morse function is called <u>excellent</u> if its critical values are pairwise distinct.

$\boxed{\text{Theorem}}$ The space of excellent Morse functions $\mathcal{F}^0(M)$ is an open and dense subset of $\mathcal{F}(M) := C^\infty(M; \mathbb{R})$ in the C^∞ (in fact C^2) topology.

I also promised an explicit understanding of $\mathcal{F}^0(M)$:

$\boxed{\text{Morse lemma}}$ If p is a non-degenerate critical point of f, $\exists$ a coordinate chart centered at p s.t.

$$f(x_1, \dots, x_n) = f(0) - y_1^2 - \dots - y_i^2 + {}_{i+1}^2 + \dots + y_n^2$$

for some $0 \le i \le n$, called the index of p.

<u>Proof sketch</u>:

Step 1: Pick any coordinate system around p.
We will find functions $f_1, \dots, f_n$ s.t.

$$f(x_1, \dots, x_n) = f(0) + \sum_{i=1}^{n} x_i f_i(x_1, \dots, x_n)$$

By the fundamental theorem of calculus, we have

$$f(x_1, \dots, x_n) = f(0) + \int_0^1 \frac{d}{dt} f(tx_1, \dots, tx_n)\, dt$$

By the Chain rule, we have

$$f(x_1, \dots, x_n) = f(0) + \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n) \cdot x_i \, dt =$$

$$= f(0) + \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n)\, dt$$

Using that p is a critical point, $\frac{\partial}{\partial x_i} f(0) = 0$ and hence $f_i(0) = 0$, so we can repeat this process and find functions $h_{i,j}$ s.t. $f(x_1, \dots, x_n) = f(0) + \sum_{i,j} x_i x_j h_{i,j}(x_1, \dots, x_n)$

The proof proceeds by diagonalizing this quadratic form in a nbhd of p using non-degeneracy.