Lie algebras

Lecture 1

We will follow on from the Honours course "Lie groups and Lie algebras" last year.

Main reference: Hall, Lie Groups, Lie Algebras, and Representations - An Elementary Introduction.

1. Basic Representation Theory (chap 4 in Hall)

1.1. Definition of a representation of a group

Definition A representation of a group G on a vector space V is a group homomorphism

$$\Pi : G \longrightarrow \mathrm{Aut}(V)$$

So each abstract group element $g \in G$ is "represented" as a linear map

$$\Pi(g) : V \longrightarrow V$$

i.e. a matrix, satisfying $\Pi(e) = id_V, \quad \Pi(g)\Pi(h) = \Pi(gh) \ \forall g,h \in G$

We will mostly consider complex representations (i.e. V is a complex vector space).

Definition If (V,π) and (V',π') are representations of G, then a linear map

$$T : V \longrightarrow V'$$

is called an intertwiner (or an equivariant map) if

$$T\left(\pi(g)(v)\right) = \pi'(g)\left(T(v)\right)$$

for all $g \in G$, $v \in V$.

That is, the following diagram commutes for all $g \in G$:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\pi(g)"'] & V' \arrow[d, "\pi'(g)"] \\
V \arrow[r, "T"'] & V'
\end{tikzcd}$$

The representations of G assemble into a category $\mathrm{Rep}(G)$ where:

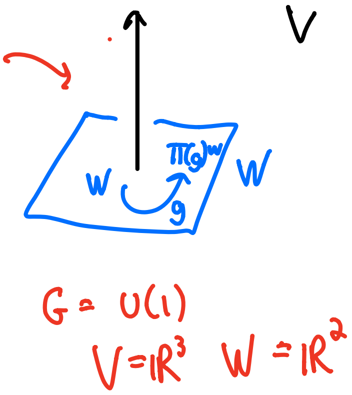
- an object is a representation (V, π) of G
- a morphism is an intertwiner.

Defn Let π be a representation of a group G on a vector space V. A subspace $W \leq V$ is called an invariant subspace if

$$\pi(g)\, w \in W$$

all $\in G$, $w \in W$.

An invariant subspace W is called <u>nontrivial</u> if $W \neq \{0\}$ and $W \neq V$.



Defn A representation (π, V) of a group is called <u>irreducible</u> if V has no nontrivial invariant subspaces.

Note: every 1-dim representation of a group is automatically irreducible.

Defn A representation π of a group G on a complex real inner product space V is <u>unitary</u> orthogonal if each $\pi(g)$ is a unitary linear map, i.e.

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$$

for all $g \in G$, $v, w \in V$.

<u>Example</u>
- For any group G, have <u>trivial representation</u> on $\mathbb{C}$, defined by $\pi(g) = \mathrm{id} : \mathbb{C} \to \mathbb{C}$ for all $-G$.

- All the matrix Lie groups have "canonical representations" on spaces of column vectors, e.g. $\underline{O(3)}$ has $\mathbb{R}^3$ as an (orthogonal) representation.

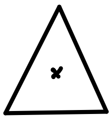
- Let $X \subseteq \mathbb{R}^n$ be any geometric figure (eg triangle, polyhedron).

Let

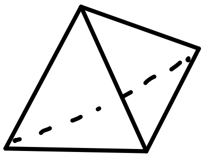
$$G := \{ g \in O(n) \text{ such that } g \cdot X = X \} \subseteq O(n)$$

Then $\mathbb{R}^n$ is an orthogonal representation of G.

eg. G = symmetries of equilateral triangle centred at origin



$D_3 \cong \mathbb{Z}/3\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{R}^2$.



G = symmetries of tetrahedron
$\cong S_4$ acts on $\mathbb{R}^3$.

"Canonically constructed" vector subspaces will in general be invariant subspaces. For instance:

<u>Lemma</u> If V and V' are representations of G, then so are:

① $\mathrm{Ker}(A) \subseteq V$, where $A: V \longrightarrow V'$ is any intertwiner

② $\mathrm{Im}(A) \subseteq V'$, where $A: V \longrightarrow V'$ is any intertwiner

③ Any eigenspace $E_\lambda \subseteq V$ of any intertwiner $A: V \longrightarrow V$.

<u>Proof</u> ① Let $A: V \longrightarrow V'$ be an intertwiner, and let $v \in \mathrm{Ker}(A)$. Then

$$\begin{aligned} A(\pi(g)v) &= \pi'(g)(\underbrace{Av}_{=0}) \\ &= \pi'(g)(0) \\ &= 0 \end{aligned}$$

$$\therefore \quad \pi(g)v \in \mathrm{Ker}(A).$$

② Let $v' \in \mathrm{Im}(A)$, i.e. $v' = Av$ for some $v \in V$. Then,

$$\begin{aligned} \pi'(g)v' &= \pi'(g)Av \\ &= A(\pi(g)v) \end{aligned}$$

$A: V \longrightarrow V$

so $\pi'(g)v' \in \mathrm{Im}(A)$.

③ $E_\lambda = \mathrm{Ker}(\underbrace{A - \lambda \mathrm{id}}_{\text{an intertwiner}})$ so this is a special case of ①. $\square$

finite-dim Lie finite-dim

Definition A ^ representation of a ^ group G on a ^ vector space V is a Lie group homomomorphism (i.e. a continuous group homomomorphism)

$$\pi : G \longrightarrow \underbrace{\mathrm{Aut}(V)}$$

also called $GL(V)$, "general linear group" of V

$= \{$ invertible linear maps $V \to V \}$

Choosing a basis for V gives identification

$$GL(V) \cong GL(\mathbb{R}^n) \subseteq \mathrm{Mat}_{n \times n} = \mathbb{R}^{n^2}$$

which defines the topology on $GL(V)$.

1.2. Basic examples and non-examples of reps of Lie groups

1.2.1. The trivial representation

For any Lie group G, the map

$$\begin{aligned} \Pi : G &\longrightarrow GL(\mathbb{C}) = \mathbb{C}^* \\ g &\longmapsto 1 \qquad \forall g \in G \end{aligned}$$

is a representation of G. (continuous ✓ group homomorphism ✓)

1.2.2. Representations of $U(1)$

For any $k \in \mathbb{Z}$, the map

$G \leftarrow$ abelian

$$\begin{tikzcd} G \arrow[d, bend right] \\ V \end{tikzcd}$$

$v \in V$

$\mathbb{C}v$ is invariant

$$\begin{aligned} \Pi_k : U(1) &\longrightarrow GL(\mathbb{C}) = \mathbb{C}^* \\ z &\longmapsto z^k \end{aligned}$$

Fourier analysis

$$e^{i\theta} \longmapsto \boxed{e^{ik\theta}}$$

irreps of $U(1)$.

is a 1-dim representation of $U(1) = \{ z \in \mathbb{C} : |z| = 1 \}$.

1.2.3. Representations of $\mathbb{R}$

For any $k \in \mathbb{C}$, the map

$$\begin{aligned} \Pi_k : (\mathbb{R}, +) &\longrightarrow \mathbb{C}^* \\ x &\longmapsto e^{kx} \end{aligned}$$

$$\Pi_k(x) : \mathbb{C} \longrightarrow \mathbb{C}$$
$$\lambda \longmapsto e^{kx}\lambda$$

is a 1-dim rep of $(\mathbb{R}, +)$, as

$$\begin{aligned} \Pi(x \underset{\uparrow}{+} y) &= e^{k(x+y)} \\ &= e^{kx} e^{ky} \\ &= \Pi(x)\Pi(y) \end{aligned}$$

group operation in $(\mathbb{R}, +)$

multiplication is the group operation in $GL(\mathbb{C}) = \mathbb{C}^*$.

1.2.4. Discontinuous representation of $\mathbb{R}$

Let $\Lambda \subseteq \mathbb{R}$ be a Hamel basis for $\mathbb{R}$ as a vector space over $\mathbb{Q}$ (always exists by axiom of choice). That is, every real number $x \in \mathbb{R}$ can be uniquely written as a finite sum

$$x = \underbrace{x_{\lambda_1}\lambda_1 + \cdots + x_{\lambda_n}\lambda_n} \qquad x_{\lambda_0}\lambda_0$$

where

$$\lambda_1, \dots, \lambda_n \in \Lambda$$
$$x_{\lambda_1}, \dots, x_{\lambda_n} \in \mathbb{Q}$$

$$f: \mathbb{R} \longrightarrow \mathbb{Q}, \quad x \longmapsto x_{\lambda_0} \quad \text{is not continuous}$$

Choose some fixed $\lambda_0 \in \Lambda$. Then the map

$$\pi: (\mathbb{R}, +) \longrightarrow \mathbb{C}^*$$
$$x \longmapsto e^{x_{\lambda_0}}$$

is a group homomorphism but is not continuous, hence not a representation.

Exercise 1 Check this.

1.2.5. Standard representation of a linear Lie group If $G \subseteq GL(V)$ is a linear Lie group, then it carries the standard representation on V,

$$\begin{aligned} id : G &\longrightarrow GL(V) \\ g &\longmapsto g \end{aligned}$$

Exercise 2 Determine whether the standard n-dimensional representation of $SO(n)$ on $\mathbb{R}^n$ is irreducible.

1.2.6. Representations on functions If G is any group (not necessarily Lie) and X is any left G-set, then we get a representation π of G on the vector space $Fun(X, \mathbb{C})$ by

$$\underbrace{\pi(g)(f)}_{} \ \underbrace{(x)}_{} \quad := \underbrace{f}_{}\underbrace{(g^{-1} \cdot x)}_{}$$

Exercise a) Check this is a group representation.

b) Why is the inverse on the RHS necessary?

c) Is this a finite-dim representation? Explain.

Exercise Let G be a finite group and X a finite left G-set.

*a) What are the irreducible subrepresentations of $\mathbb{C}[X]$?

b) Let $G = S_3$, and $X = S_3$, with left action of G by multiplication. What are the irreducible subrepresentations of $\mathbb{C}[X]$?

1.2.6. Irreducible representations of SU(2)

Let

$$V_n = \left\{ \begin{array}{l} \text{complex homogenous polynomials in } z, w \text{ of} \\ \text{degree } n \end{array} \right\}$$

For example, the function

$$f(z,w) = 2z^2w + 3w^3$$

$3x^4y^2$

is in V_3. Indeed

$$V_n = \text{span}_{\mathbb{C}} \{ \overset{z^n}{p_0}, \overset{z^{n-1}w}{p_1}, \overset{\cdots}{\cdots}, \overset{w^n}{p_n} \}, \quad p_k = \underbrace{z^{n-k} w^k} \qquad ⊛$$

So $\text{Dim}(V_n) = n+1$. Note that we can consider

$$V_n \subseteq \text{Fun}(\mathbb{C}^2, \mathbb{C}).$$

By identifying $\mathbb{C}^2$ with the space of 2-dim column vectors, we see that $GL(2,\mathbb{C})$ acts from the left on $\mathbb{C}^2$ via

$$\underbrace{\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}}_{g \in G} \cdot \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{\in \mathbb{C}^2} = \begin{pmatrix} g_{11}u + g_{12}v \\ g_{21}u + g_{22}v \end{pmatrix}$$

and hence (Example 1.2.6), for each n, we get a representation of $GL(2,\mathbb{C})$ on V_n.

Exercise There is something to check here (I'm not talking about continuity). Explain what it is, and check it.

Exercise Using the basis ⊛ for V_n, determine explicitly the resulting matrix representation for $n=2$, i.e.

$$\Pi_2 : GL(2,\mathbb{C}) \longrightarrow GL(3,\mathbb{C})$$

Since $SU(2) \leq GL(2,\mathbb{C})$, the V_n are also representations of $SU(2)$, by restriction. In fact, the V_n are a complete list of the irreducible representations of $SU(2)$, up to isomorphism, as we will see.

Lemma V_n is an irreducible representation of $SU(2)$.

Proof Since $SU(2)$ is compact, V_n is unitarizable. So by Schur's Lemma, we just need to show that any intertwiner

$$A : V_n \longrightarrow V_n$$

is a scalar multiple of the identity. For $t \in \mathbb{R}$, consider

$$t_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad , \quad r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

in $SU(2)$. Observe :

$$\left[\Pi(t_\theta)\, p_k\right]\begin{pmatrix} z \\ w \end{pmatrix} = p_k\left(t_\theta^{-1} \begin{pmatrix} z \\ w \end{pmatrix}\right)$$

$p_k\begin{pmatrix} z \\ w \end{pmatrix} = z^{n-k} w^k$

$$= p_k\begin{pmatrix} e^{-i\theta} z \\ e^{i\theta} w \end{pmatrix}$$

$$= e^{i[-(n-k)+k]\theta} z^{n-k} w^k$$

$$= e^{i[2k-n]\theta} p_k\begin{pmatrix} z \\ w \end{pmatrix}$$

i.e. $$\Pi(t_\theta) p_k = e^{i[2k-n]\theta} p_k$$

So every p_k is an eigenvector of $\Pi(t_\theta)$. We can choose θ s.t. all these eigenvalues are different. So, the eigenspaces of $\Pi(t_\theta)$ are

also eigenspace of A $\longrightarrow$ $\boxed{\mathbb{C}p_0}$, $\mathbb{C}p_1$, $\dots$, $\mathbb{C}p_k$

Since A commutes with $\Pi(t_\theta)$, it leaves all these eigenspaces invariant. Hence

$$Ap_k = \lambda_k p_k \qquad 0 \leq k \leq n$$

for some $\lambda_0, \dots, \lambda_n \in \mathbb{C}$. We will show all these eigenvalues λ_i must be equal, so that A is a scalar multiple of the identity.

Firstly, we calculate:

$$\Pi(r_\theta)\, p_0\begin{pmatrix} z \\ w \end{pmatrix} = p_0\left[\begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix}\right]$$

$$= p_0 \begin{pmatrix} \cos\theta\, z & +\sin\theta\, w \\ \sin\theta\, z & -\cos\theta\, w \end{pmatrix}$$

$$= (\cos\theta\, z + \sin\theta\, w)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta \sin^{k}\theta\, z^{n-k} w^{k}$$

$$\therefore \quad \pi(r_\theta)\, p_0 = \sum_{k=0}^{n} \binom{n}{k} \underbrace{\cos^{n-k}\theta \sin^{k}\theta}_{\text{nonzero}}\, p_k$$

Now, A is an intertwiner, so we must have

$$A\, \pi(r_\theta)\, p_0 = \pi(r_\theta) \underbrace{A p_0}_{} = \lambda_0 p_0$$

$$\therefore \quad \sum_{k=0}^{n} \underbrace{\binom{n}{k} \cos^{n-k}\theta \sin^{k}\theta}_{\text{nonzero}}\, \lambda_k\, p_k = \sum_{k=0}^{n} \underbrace{\binom{n}{k} \cos^{n-k}\theta \sin^{k}\theta}_{\text{nonzero}}\, \lambda_0\, p_k$$

$$\therefore \quad \lambda_k = \lambda_0 \quad \text{for all } k = 0 \ldots n.$$

□

Schur's Lemma Let (π, V) be a finite dimensional complex representation of a group G. Then:

a) If π is irreducible, then $\mathrm{End}_G(V) = \mathbb{C}\,\mathrm{id}_V$.

b) Conversely, if $\mathrm{End}_G(V) = \mathbb{C}\,\mathrm{id}_V$, and V is unitarizable (ie. admits an inner product s.t. π is a unitary representation), then V is irreducible.

Proof a) Suppose π is irreducible. Let $A : V \longrightarrow V$ be an intertwiner.

I want to show that $A = \lambda\,\mathrm{id}$.

A has an eigenvalue λ.

But then $E_\lambda \subseteq V$ is an invariant subspace.

Since V is irreducible, $\underbrace{E_\lambda = \{0\}}_{\text{not possible}}$ or $\underbrace{E_\lambda = V}_{\therefore\, A = \lambda\,\mathrm{id}.}$

b) Suppose a nontrivial G-invariant subspace $W \subseteq V$ exists.

Then $W^\perp$ is also G-invariant:

Let $v \in W^\perp$, and $w \in W$. Then:

$$\langle w, \pi(g)v \rangle = \langle \underbrace{\pi(g^{-1})\, w}_{\in W}, \underbrace{v}_{\in W^\perp} \rangle \qquad = 0$$

So, as representations,

$$V = \overset{\circlearrowright G}{W} \oplus \overset{\circlearrowright G}{W^\perp}$$

So the map

$$A : V \longrightarrow V$$
$$(w,w') \longmapsto (w, 2w')$$

is G-equivariant. But $A \neq$ scalar multiple of identity, so this is a contradiction. So our initial assumption is false, i.e. W does not exist, i.e. V is irreducible.

□

Lie Algebras 2020

Lecture 2

Recall A finite-dimensional real or complex Lie algebra is dimensional vector space $\mathfrak{g}$ equipped with a bilinear map

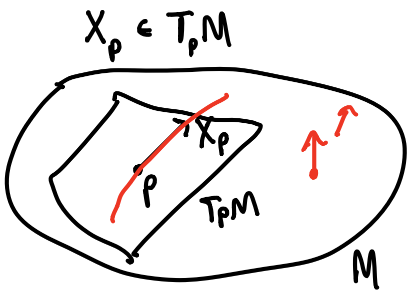
$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

a finite

Satisfying:

- $[X,Y] = -[Y,X] \quad \forall\ X,Y \in \mathfrak{g}$
- $[X,[Y,Z]] + [Y,[X,Z]] + [X,[Z,Y]] = 0$

(Jacobi identity).



Examples

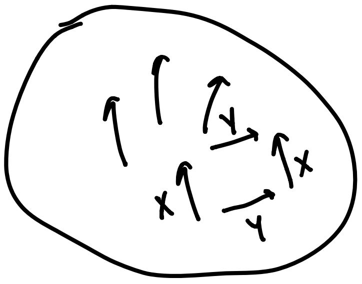
- $(\mathbb{R}^3, \ [X,Y] := X \times Y)$
- $\mathrm{Vect}(M)$
- $(\text{algebra } A, \ [a,b] := ab - ba)$

- For any matrix Lie group $G \leq GL(n,\mathbb{R})$, its Lie algebra is

$$\mathfrak{g} = \{ X \in Mat(n,\mathbb{R}) : e^{tX} \in G \text{ for all } t \in \mathbb{R} \}$$

$$= \{ \sigma'(0) \text{, where } \sigma : (\mathbb{R},+) \to G \text{ is a Lie group homomorphism} \}$$

$$= \{ \gamma'(0) \text{, where } \gamma : [-1,1] \to G \text{ is any smooth path with } \gamma(0) = I \}$$

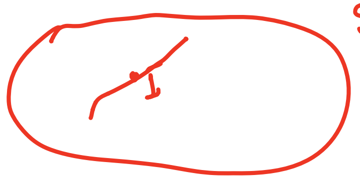


i.e. $\mathfrak{g} = T_eG$

$[X,Y] = \underbrace{XY - YX} \in End(\mathbb{R}^n)$

eg. $\underbrace{Lie(GL(n,\mathbb{R}))}_{\mathfrak{gl}(n,\mathbb{R})}$ = all real n×n matrices

$A(t) = I + tX + t^2(\dots)$



$SO(n) \leq GL(n)$

$\underbrace{Lie(SO(n))}_{\mathfrak{so}(n)}$ = antisymmetric real n×n matrices

$\underbrace{Lie(U(n))}_{\mathfrak{u}(n)}$ = antihermitian n×n matrices

$\underbrace{Lie(SU(n))}_{\mathfrak{su}(n)}$ = traceless antihermitian n×n matrices need $A^TA = I$

$$\therefore \left(I + tA + O(t^2) \right)^T \left(I + tA + O(t^2) \right) = I$$

$$\therefore I + t(A^T + A) + O(t^2) = I$$

$$\therefore A^T + A = 0$$

1.3. Lie algebra representations

All of the above definitions can be easily modified to speak ab representations of Lie algebras instead of Lie groups. So:

Definition A representation of a Lie algebra L on a vector space is a Lie algebra homomorphism

$$\pi : (L, [\cdot\,;\cdot]) \longrightarrow (\mathrm{End}(V), \text{commutator})$$

i.e.

$$\pi([X,Y]_L) = \underbrace{[\pi(X), \pi(Y)]_{\mathrm{End}(V)}}_{:= \pi(X)\pi(Y) - \pi(Y)\pi(X)}$$

A subspace $W \subseteq V$ is invariant if

$$\pi(X)w \in W \qquad \text{for all } X \in L,\ w \in W.$$

A representation of a Lie algebra is irreducible if it contains no nontrivial invariant subspaces.

1.3.1. Complexifications of real Lie algebras

Complex linear algebra is easier than real linear algebra (eg. eigenvalues always exist!) so we will mostly want our Lie algebra representations to be complex vector spaces.

But, our Lie algebras in the previous course were mostly real Lie algebras because we were working with real Lie groups, eg $SU(2)$.

$X \rightsquigarrow k[X]$

We can turn any real vector space V into a complex one by tensoring with $\mathbb{C}$:

$V \otimes W = k[V \times W] /$

$(kv) \otimes w = k(v \otimes w)$

$$\boxed{V_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V}$$

By formally writing $v \equiv 1 \otimes v$ and $Iv \equiv i \otimes v$, we can think of $V_{\mathbb{C}}$ as

$$V_{\mathbb{C}} = \{ \text{formal expressions } \underbrace{v_1 + Iv_2,\ v_1, v_2 \in V \}}_{}$$

where the complex scalar i acts via

$$i \cdot (v_1 + I v_2) = -v_2 + I v_1$$

Exercise Prove this formally.

In this way we can turn any real Lie algebra into a complex Lie algebra ("complexification") by complex-linearly extending the bracket:

$$[v_1 + I v_2,\ w_1 + I w_2]_{V_{\mathbb{C}}} := [v_1, w_1]_V - [v_2, w_2]_V + I\left([v_1, w_2]_V + [v_2, w_1]_V\right)$$

Proposition If L is a real Lie subalgebra of a complex Lie algebra L', and if

e.g. $\rightarrow \mathfrak{su}(2) \subseteq \mathfrak{gl}(2, \mathbb{C}) \leftarrow$ $\qquad \mathfrak{su}(2)_{\mathbb{C}}$

$$\underbrace{iX}_{} \neq 0$$

for all nonzero $X \in L$, then

$$L_{\mathbb{C}} \cong \{ X_1 \overset{\downarrow}{+} \underset{\uparrow}{i} X_2 ,\ X_1, X_2 \in L \}$$

actual complex multiplication in L', ie. not formal.

Proof We have a surjective homomorphism

$$f: L_{\mathbb{C}} \longrightarrow \{ X_1 + i X_2 ,\ X_1, X_2 \in L \} \subseteq L'$$

$$X + IY \longmapsto X + iY$$

whose kernel is

$$\begin{aligned} \mathrm{Ker}(f) &= \{ X + IY \in L_{\mathbb{C}} : X + iY = 0 \text{ in } L' \} \\ &= I\{ Y \in L : iY = 0 \text{ in } L' \} \\ &= \{0\} \end{aligned}$$

So f is injective, and hence an isomorphism. $\square$

Corollary

- $\mathfrak{gl}(n,\mathbb{R})_{\mathbb{C}} \cong \mathfrak{gl}(n,\mathbb{C})$ $\left(= \mathrm{Mat}(n,\mathbb{C})\right)$
- $\mathfrak{u}(n)_{\mathbb{C}} \cong \mathfrak{gl}(n,\mathbb{C})$
- $\mathfrak{su}(n)_{\mathbb{C}} \cong \mathfrak{sl}(n,\mathbb{C})$
- $\mathfrak{sl}(n,\mathbb{R})_{\mathbb{C}} \cong \mathfrak{sl}(n,\mathbb{C})$
- $\mathfrak{so}(n)_{\mathbb{C}} \cong \mathfrak{so}(n,\mathbb{C})$
- $\mathfrak{sp}(n,\mathbb{R})_{\mathbb{C}} \cong \mathfrak{sp}(n,\mathbb{C})$

$\{ X + iY \mid X, Y \text{ traceless antiHermitean} \}$

$= \mathfrak{sl}(n,\mathbb{C})$

$= \{ \text{traceless complex matrices} \}$

Exercise Use the proposition to check that:

(a) $\mathfrak{gl}(n,\mathbb{R}) \cong \mathfrak{gl}(n,\mathbb{C})$

(b) $\mathfrak{su}(n)_{\mathbb{C}} \cong \mathfrak{sl}(n,\mathbb{C})$.

Exercise Show that

$$\mathfrak{su}(2) \ncong \mathfrak{sl}(2,\mathbb{R})$$

even though

$$\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}$$

1.3.2. Examples of Lie algebra representations

a) For every Lie algebra $\mathfrak{g}$, have trivial representation on $\mathbb{C}$, defined by

$$\pi(X) = 0 \qquad \forall \; X \in \mathfrak{g}$$

This representation is irreducible.

b) Recall the adjoint representation of a matrix Lie group,

$$L_g, R_{g^{-1}} : G \to G$$

$$\begin{aligned} G &\longrightarrow \mathrm{Aut}(T_eG) \\ g &\longmapsto D_I C_g : \mathfrak{g} \to \mathfrak{g} \end{aligned}$$

$$\begin{aligned} \mathrm{Ad} : \underset{\subseteq GL(n,\mathbb{R})}{G} &\longrightarrow GL(n,\mathbb{R}) \\ g &\longmapsto g(\cdot)g^{-1} = L_g \circ R_{g^{-1}} = \boxed{C_g : G \to G} \end{aligned}$$

Question: what is the adjoint rep for an abstract Lie group?

Similarly, the adjoint representation of a Lie algebra is

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\longrightarrow \mathrm{End}(\mathfrak{g}) \\ X &\longmapsto [X, -] \end{aligned}$$

$\longleftarrow$ just vector space endomorphisms, not Lie algebra homomorphisms

Check $\quad [\mathrm{ad}(X), \mathrm{ad}(Y)] \overset{?}{=} \mathrm{ad}([X,Y])$

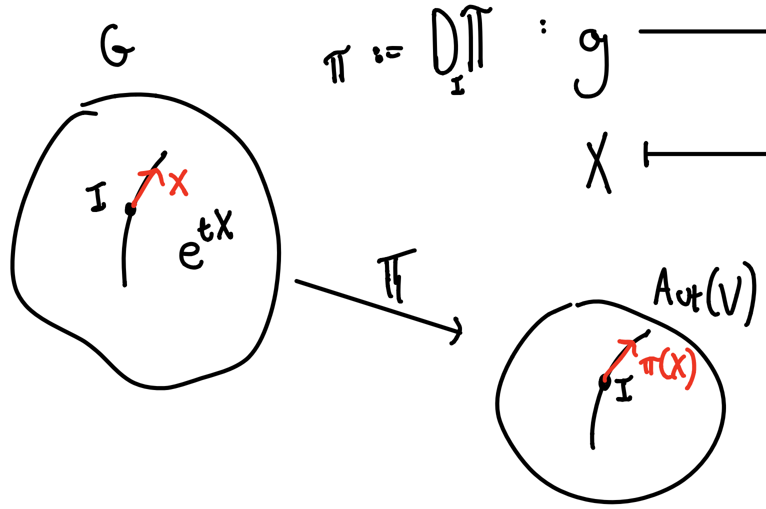
Proposition Let

$$\Pi : \quad G \longrightarrow \mathrm{Aut}(V)$$

be a finite dim rep. of a Lie group on a vector space V. Then

$$\pi := D_I \Pi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$$

$$X \longmapsto \frac{d}{dt}\Big|_{t=0} \Pi\left(e^{tX}\right)$$



is a finite dim rep of $\mathfrak{g}$ on V.

Last year: for any Lie group homomorphism

$$\Phi : \quad G \longrightarrow H$$

it holds that

$$\phi := D_I \Phi : \mathfrak{g} \longrightarrow \mathfrak{h}$$

$$X \longmapsto \frac{d}{dt}\Big|_{t=0} \Phi\left(e^{tX}\right)$$

Satisfies

$$\underbrace{[\phi(X), \phi(Y)]}_{\in \mathfrak{h}} = \phi\Big(\underbrace{[X, Y]}_{\in \mathfrak{g}}\Big).$$

Recall: a rep of a Lie algebra $\mathfrak{g}$ on a vector space V is a map $\phi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ s.t.

$$\phi([X,Y]) = \underbrace{\phi(X)\phi(Y) - \phi(Y)\phi(X)}_{[\phi(X), \phi(Y)]_{\mathrm{End}(V)}}$$

Proposition (a) $\underbrace{(V,\Pi)}_{\text{rep of } G}$ is irreducible $\iff$ $\underbrace{(V,\pi)}_{\text{rep of } \mathfrak{g}}$ is irreducible.

(b) $\underbrace{(V,\Pi) \cong (V',\Pi)}_{\text{reps of } G} \iff \underbrace{(V,\pi) \cong (V',\pi')}_{\text{reps of } \mathfrak{g}}$

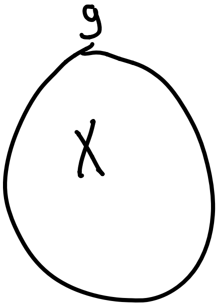
Proof (a) We will prove:

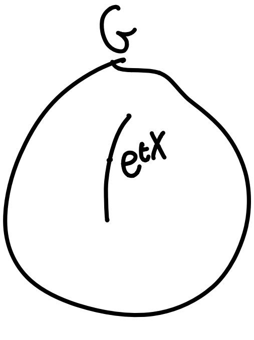
W invariant subspace for $\underbrace{(V,\Pi)}_{\text{rep of } G}$ $\iff$ W invariant subspace for $\underbrace{(V,\pi)}_{\text{rep of } \mathfrak{g}}$

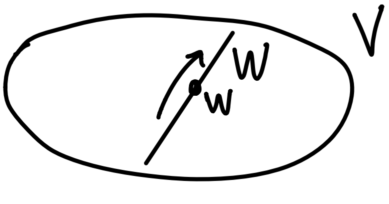
($\Rightarrow$) Let W be invariant subspace of $\underbrace{(V,\Pi)}_{\text{rep of } G}$.

Let $X \in \mathfrak{g}$, and $w \in W$

$$\pi(X)w = \underbrace{\frac{d}{dt}\Big|_{t=0} \Big(\underbrace{\Pi(\underbrace{e^{tX}}_{\in G})\, w}_{\in W}\Big)}_{\in W}$$







$\Leftarrow$ Let W be an invariant subspace of the representation of $\mathfrak{g}$ on V.

Must show W is also an invariant subspace of rep of G on V.

Let $w \in W$, and $g \in G$.

why? See end of lecture.

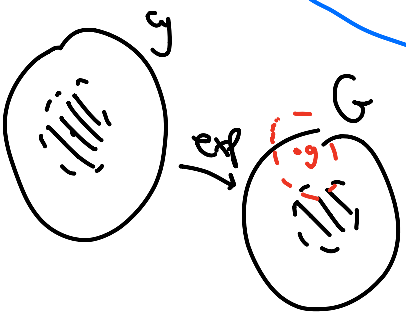
Can write

$$g = e^{X_1} \cdots e^{X_n} \qquad X_1, \dots, X_n \in \mathfrak{g}$$

(see last year: followed from theorem that

$$\exp : \mathfrak{g} \longrightarrow G$$

is a local diffeomorphism)

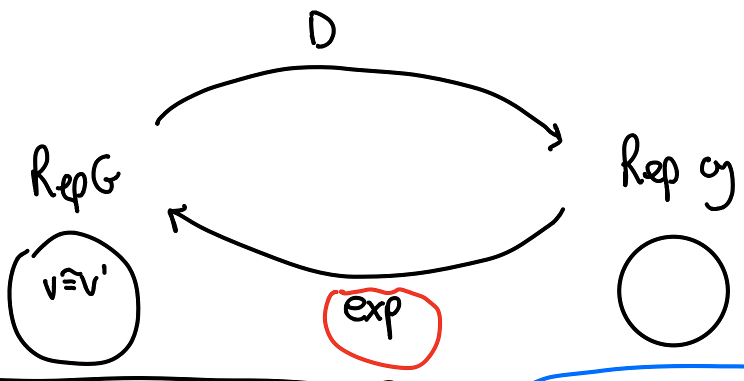


$\pi(X_n)w \in W$

$$\left(I + \pi(X_n) + \frac{\pi(X_n)^2}{2!} + \cdots\right)w$$

$$\begin{aligned}\Pi(g)w &= \Pi\left(e^{X_1}\cdots e^{X_n}\right)w \\ &= \Pi(e^{X_1})\cdots\Pi(e^{X_n})w \\ &= \underbrace{e^{\pi(X_1)} \cdots \underbrace{e^{\pi(X_n)}w}_{\in W}}_{\in W}.\end{aligned}$$

(b)



Question: Is exp well-defined here? No.

see end of lecture

Proposition Let $\mathfrak{g}$ be a real Lie algebra, and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then every f. dim complex rep (V, π) of $\mathfrak{g}$ has a unique extension to a complex-linear rep of $\mathfrak{g}_{\mathbb{C}}$, also denoted π.

Exercise Prove this.

Dario: why do we always insist on f. dim reps?

Bruce: One reason: we only know how to "take the derivative" of f. dim Lie group representations.

Also, our definition of a Lie group rep required f. dim vector spaces.

$$\pi : G \longrightarrow \mathrm{Aut}(V)$$

continuous group homomorphism

Last time: irreps of $SU(2)$:

$$V_n = \left\{ \begin{array}{c} \text{homogenous polynomials of degree } n \\ \text{in complex variables } z, w \end{array} \right\}$$

$n+1$ dim space

$SU(2)$ acts on V_n by

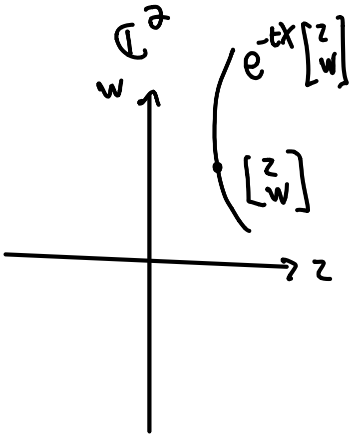
$$(g \cdot f)\begin{pmatrix} z \\ w \end{pmatrix} = f\left(g^{-1} \begin{pmatrix} z \\ w \end{pmatrix} \right)$$

What is the corresponding rep of the Lie algebra $su(2)$?

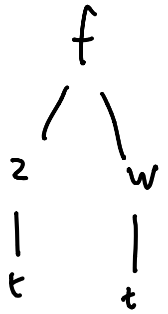
$X \in su(2)$ i.e. X is traceless anti-Hermitian 2×2 matrix

$$X = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$$\pi(X)f \rightarrow X\cdot f = \frac{d}{dt}\Big|_{t=0} \left[e^{tX}\cdot f\right]$$



i.e. $$[X\cdot f]\begin{bmatrix}z\\w\end{bmatrix} = \frac{d}{dt}\Big|_{t=0} f\left(e^{-tX}\begin{bmatrix}z\\w\end{bmatrix}\right)$$



Write $$\begin{bmatrix} z(t)\\ w(t)\end{bmatrix} = \underbrace{e^{-tX}\begin{bmatrix}z\\w\end{bmatrix}}$$

$$\therefore [X\cdot f]\begin{bmatrix}z\\w\end{bmatrix} = \frac{\partial f}{\partial z}\frac{dz}{dt}\Big|_{t=0} + \frac{\partial f}{\partial w}\frac{dw}{dt}\Big|_{t=0}$$

$$= \frac{\partial f}{\partial z}\left(\text{1st component of } -X\cdot\begin{bmatrix}z\\w\end{bmatrix}\right)$$

$$+ \frac{\partial f}{\partial w}\left(\text{2nd component of } -X\cdot\begin{bmatrix}z\\w\end{bmatrix}\right)$$

$$= -\frac{\partial f}{\partial z}\boxed{\left(X_{11}z + X_{12}w\right)} - \frac{\partial f}{\partial w}\boxed{\left(X_{21}z + X_{22}w\right)}$$

i.e. if we think of z and w as linear operators on functions on $\mathbb{C}^2$ ("multiply by z" and "multiply by w") then

$$\boxed{\pi(X) = -\left(X_{11}z + X_{12}w\right)\frac{\partial}{\partial z} - \left(X_{21}z + X_{22}w\right)\frac{\partial}{\partial w}}$$

Get rep of $su(2)_{\mathbb{C}} = \underbrace{sl(2,\mathbb{C})}_{\text{nice basis}}$ = traceless 2×2 complex matrices

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$[X,Y] = H$$
$$[H,X] = 2X$$
$$[H,Y] = -2Y$$

"raising and lowering operators"

basis of $\underbrace{su(2)}$: $\rightarrow$ antiHermitian traceless matrices

So, $su(2)$ is a real vector space with basis

$$E_1 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad E_2 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad E_3 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and Lie brackets

$$[E_1,E_2] = E_3, \quad [E_2,E_3] = E_1, \quad [E_3,E_1] = E_2$$

Following up on loose ends.

Lemma If G is a connected Lie group, then every $A \in G$ can be expressed as a product of exponentials:

$$A = e^{X_n} e^{X_{n-1}} \cdots e^{X_1} \qquad X_1, \dots, X_n \in \mathfrak{g}$$

Proof This follows from the fact that

$$\exp: \quad \mathfrak{g} \longrightarrow G$$

is a local diffeomorphism (see last year's notes). Let $A \in G$. We know there is a path

$$\gamma(t) : [0,1] \longrightarrow G \quad , \gamma(0) = I, \ \gamma(1) = G.$$

For any choice of subdivision of $[0,1]$

$$t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1$$

we can write

$$A = \left(\gamma(t_n)\gamma(t_{n-1})^{-1}\right)\left(\gamma(t_{n-1})\gamma(t_{n-2})^{-1}\right) \cdots \left(\gamma(t_2)\,\gamma(t_1)^{-1}\right)\gamma(t_1)$$

If the subdivision is fine enough we can ensure each of these factors is close enough to $I \in G$, so we can write

$$\gamma(t_i)\gamma(t_{i-1})^{-1} = e^{X_i} \qquad i = 1 \ldots n$$

$\square$

<u>Claim</u> If (V,Π) and (V',Π') are reps of a <u>connected</u> Lie group G, and (V,π), (V',π') the corresponding reps of $\mathfrak{g}$, then

$$\underset{\text{as reps of } G}{(V,\Pi) \cong (V',\Pi')} \iff \underset{\text{as reps of } \mathfrak{g}}{(V,\pi) \cong (V',\pi')}$$

<u>Proof</u> $(\Rightarrow)$ functoriality of $D: \mathrm{Rep}\,G \longrightarrow \mathrm{Rep}\,\mathfrak{g}$

$(\Leftarrow)$ Let $f: V \longrightarrow V'$ be the isomorphism of Lie algebra reps. That is, the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\pi(X)"'] & V' \arrow[d, "\pi'(X)"] \\ V \arrow[r, "f"'] & V' \end{tikzcd} \qquad ⊛ \qquad \text{for all } X \in \mathfrak{g}$$

Then if $A \in G$, we can write

$$A = e^{X_n} \cdots e^{X_1} \qquad X_i \in \mathfrak{g}$$

So

$$\begin{aligned} f \circ \pi(A) &= f \circ \pi\left(e^{X_n}\right) \circ \cdots \circ \pi\left(e^{X_1}\right) \\ &= f \circ e^{\pi(X_n)} \circ \cdots \circ e^{\pi(X_1)} \\ &\qquad ⊛ \qquad ⊛ \qquad ⊛ \\ &= e^{\pi'(X_n)} \circ \cdots \circ e^{\pi'(X_1)} \circ f \\ &= \pi'(A) \circ f. \end{aligned}$$

$\square$

If $\mathfrak{g}$ is the Lie algebra of G, I don't think there is generally a functor

$$\operatorname{Rep} \mathfrak{g} \longrightarrow \operatorname{Rep} G$$

Rather, there is a functor

$$\exp : \operatorname{Rep} \mathfrak{g} \longrightarrow \operatorname{Rep} G_0$$

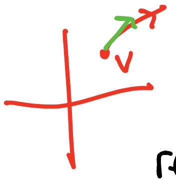
where G_0 is the unique simply connected Lie group whose Lie algebra is $\mathfrak{g}$.

(think eg. $SU(2)$ and $SO(3)$ have the same Lie algebra, but the even-dimensional irreps of $SU(2)$ ("spinor reps") do not have analogues for $\underbrace{SO(3)}_{\text{all irreps are odd-dimensional}}$;)

Lie Algebras
Lecture 3

Last time: From

$$\Pi : G \longrightarrow \mathrm{Aut}(V)$$



rep of G on V

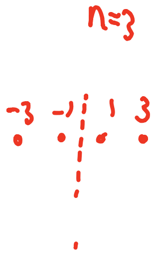
get

$$\pi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$$

rep of $\mathfrak{g}$ on V

$$\underbrace{X(v)}_{} = \frac{d}{dt}\Big|_{t=0} \underbrace{\underbrace{e^{tX}}_{}(v)}_{}$$

shorthand for $\underbrace{\pi(X)(v)}_{}$



We applied this to the irreps V_n of $SU(2)$

$$V_n = \mathrm{span}\{\underbrace{z^n w^0}_{e_0}, \underbrace{z^{n-1}w^1}_{e_1}, \dots, \underbrace{z^0 w^n}_{e_n}\}$$

↑ note: not the "best" normalization

to get rep π of $su(2)$

$$\boxed{\pi(X) = -(X_{11}z + X_{12}w)\frac{\partial}{\partial z} - (X_{21}z + X_{22}w)\frac{\partial}{\partial w}}$$

By complex-linear extension the same formula gives us a rep of $su(2)_{\mathbb{C}} = sl(2,\mathbb{C})$.

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$[H, X] = 2X \qquad [X,Y] = H \qquad [H,Y] = -2Y$$

$$-(n-k)\, z^{n-k} w^k + k\, z^{n-k} w^k$$

We compute:

$$H(e_k) = \left(-z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}\right)\left(\boxed{z^{n-k}w^k}\right) \qquad \frac{\partial}{\partial w}\left(z^{n-k}w^{k+1}\right) = (k+1)z^{n-k}w^k$$

$$= -(n-k+1)z^{n-k}w^k + \underbrace{(k+1)z^{n-k}w^k}$$

$$\overset{n=5 \quad k=5}{=} \underbrace{\underset{5}{(-n+2k)}(e_k)}$$ ← so e_k is an eigenvector of H with eigenvalue $-n+2k$

$$X(e_k) = \left(-w\frac{\partial}{\partial z}\right)\left(z^{n-k}w^k\right)$$

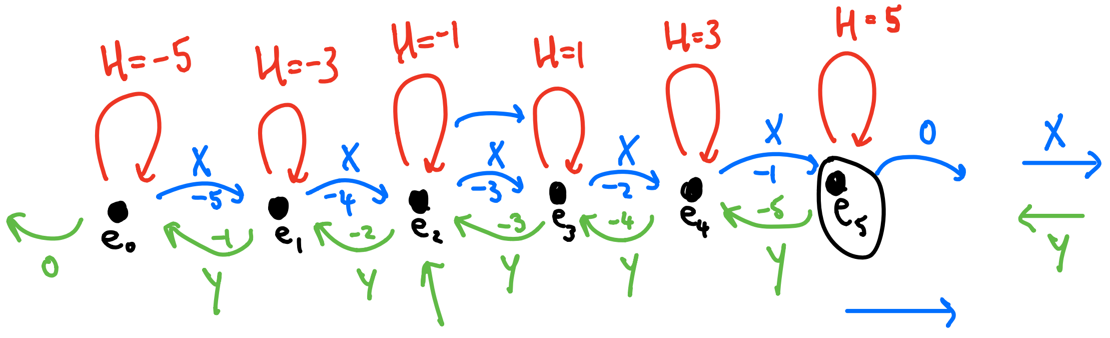
$$= -\overset{5-5}{(n-k)}\,e_{k+1}$$

$$Y(e_k) = \left(-z\frac{\partial}{\partial w}\right)\left(z^{n-k}w^k\right)$$

$$= -k\,e_{k-1}$$

$\therefore\ V_n = V^{(n)}$

eg. V_5:



$[X,Y]=H$

Check commutation relations, eg:

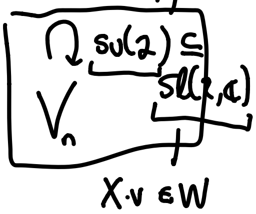
① $XY - YX = H$ on e_2: $\underbrace{\underset{(-4)(-2)}{XY}} - \underbrace{\underset{(-3)(-3)}{YX}} = \boxed{-1 ✓.}$

② $HX - XH = 2X$ on e_4: $\underset{6}{H}\underset{-1}{X} - \underset{-1}{X}\underset{3}{H} = \underset{-2}{2X}$ ✓

Lemma This rep V_n of $sl(2,\mathbb{C})$ is irreducible.

Proof **1st proof** This follows from the fact that V_n is an irreducible rep of $SU(2)$: Let $g_0 = su(2)$, $g = sl(2,\mathbb{C})$, so $g = g_0 \otimes \mathbb{C}. = \{X + iY,\ X, Y \in g_0\}$.

$\boxed{su(2)} \subseteq \boxed{sl(2,\mathbb{C})}$



A rep of g restricts to a rep of g_0.

We've already seen that a g_0-invariant subspace $W \subseteq V_n$ is also a $SU(2)$-invariant subspace, hence trivial, by earlier proof from Lecture 2.

2nd proof Let $W \subseteq V_n$ be a nonzero invariant subspace for $sl(2,\mathbb{C})$, and w a nonzero vector in W.

Then for suitable k, $X^k(w)$ is a nonzero multiple of e_n. $\therefore e_n \in W$

And Once we've established $e_n \in W$, then all e_i are in W (by applying powers of Y). $\therefore W = V_n$ □

$$w = w_0 e_0 + w_1 e_1 + w_2 e_2 + w_3 e_3 + w_4 e_4 + w_5 e_5$$

$$X(w) = \square e_1 + \square e_2 + \square e_3 + \square e_4 + \square e_5$$

$\vdots$

$$X^5(w) = \underset{\neq 0}{\square} e_5$$

$\therefore e_5 \in W$.

Theorem Every irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to V_n for some n.

We'll prove this in stages, in a way which will ultimately generalize to find the irreps of any semisimple Lie algebra.

See:

- Fulton and Harris, Chapter 11, Representations of $\mathfrak{sl}_2\mathbb{C}$.

Firstly, let's recall our basis H, X, Y of $\mathfrak{sl}(2,\mathbb{C})$:

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \qquad H = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \qquad X = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \qquad Y = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

$$[H,X] = 2X, \qquad [H,Y] = -2Y, \qquad [X,Y] = H$$

Let V be a finite-dim irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$.

Fact The action of H on V is diagonalizable ← we'll prove later

So, we have a decomposition

$$V = \bigoplus_{\alpha \in I} V_\alpha$$

where the α run over a finite set I of complex numbers, such that for any $v \in V_\alpha$ we have $\quad v \in V_\alpha$

$$H(v) = \alpha v \qquad H(X(v))$$

How do X and Y behave with respect to this decomposition?

$$AB = \cancel{BA} + \underbrace{[A,B]}_{AB - \cancel{BA}} \qquad [a, a^\dagger$$

<u>Fundamental calculation</u> (1st time)

$$H(X(v)) = (XH + [H,X])(v) \qquad H(v) = \alpha v$$

$$= \alpha X(v) + 2X(v)$$

$$= (\alpha + 2)(X(v)).$$

$$\begin{tikzcd} v \arrow[loop above, "H"] \arrow[r, bend left, "X"] & X(v) \end{tikzcd}$$

$$H(X(v))$$

$$= (\alpha+2)\, X(v)$$

$$\therefore X(v) \in V_{\alpha+2}$$

i.e. $X(v)$ is an eigenvector of H with eigenvalue $\alpha+2$.

So,

$[H,X] = 2X$

$$\boxed{X : V_\alpha \longrightarrow V_{\alpha+2}}$$

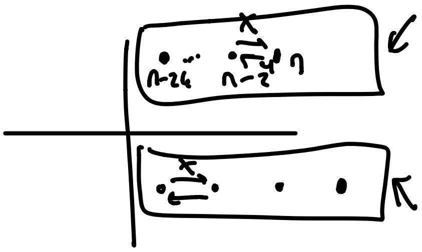
Similarly,

$[H,Y] = -2Y$

$$\boxed{Y : V_\alpha \longrightarrow V_{\alpha-2}}$$

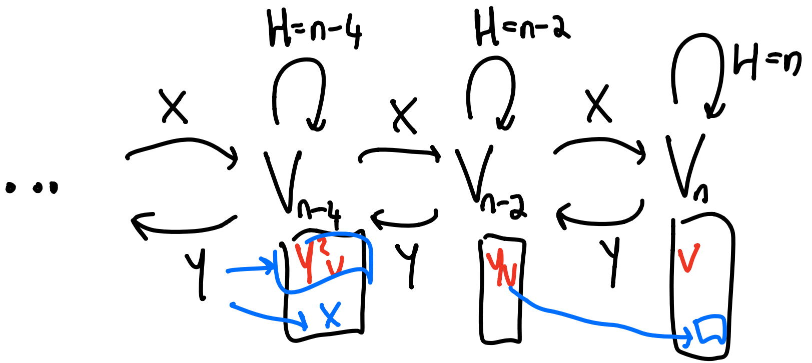
Since V is <u>finite-dimensional</u> and <u>irreducible</u>, the eigenvalues of H must be a finite sequence of the form

$$\begin{tikzcd} \square, & \cdots, & n-4 \arrow[r, bend left, "X"] & n-2 \arrow[l, bend left, "Y"], & n \end{tikzcd}$$



for some $n \in \mathbb{C}$ (we'll shortly see n must be an integer)

The picture is:



Choose any nonzero vector $v \in V_n$. Evidently, we have $X(v) = 0$.

What is $Y(v)$? Firstly: $\circlearrowright^{H}(Y^2(v)) = (n-4)\,Y^2(v)$ $\quad \circlearrowright^{X,Y,H}$

$W :=$

Claim The vectors $\text{span}\{v, Y(v), Y^2(v), \dots\}$ "Verma" span V. $\quad W \subseteq V \quad \therefore W = V$

Proof From the irreducibility of V it is enough to show that the subspace W spanned by these vectors is invariant under the action of H, X and Y. This is clear for the action of H and Y, but needs to be checked for X.

- $X(v) = 0$ (from above picture)

- $X(Y(v)) = (YX + [X,Y])(v)$ ← here is where we use $[X,Y]=H$

  $= 0 + H(v)$

  $= nv$

- $X(Y^2(v)) = (YX + [X,Y])Y(v)$

  $= Y(nv) + HY(v)$

$$= nY(v) + (n-2)Y(v)$$
$$= (n + (n-2))\,Y(v).$$

So, in general

- $$X(\underbrace{y^k(v)}) = \underbrace{(n + (n-2) + \cdots + (n-2k+2))}\,\underbrace{y^{k-1}(v)}$$

$\in W$

and we are done. $\square$

Corollary 1 All the eigenspaces V_α of H are 1-dimensional.

Corollary 2 The representation is determined by a single complex number $n \in \mathbb{C}$.

(It is also true to say that V is determined by the complex numbers α appearing in the decomposition $V = \bigoplus_\alpha V_\alpha$).

$\circlearrowleft sl(2,\mathbb{C})$
$V = \mathbb{C}^2$
$V_n = Sym^n(V)$
$= [\underbrace{V\otimes\cdots\otimes V}_{n}]_{Sym}$

Finally, since V is finite-dimensional, we must have

$$Y^k(v) = 0$$

for sufficiently large k. Let k_0 be the smallest such k. Then:

$$X(Y^{k_0}v) = 0$$

but we've seen

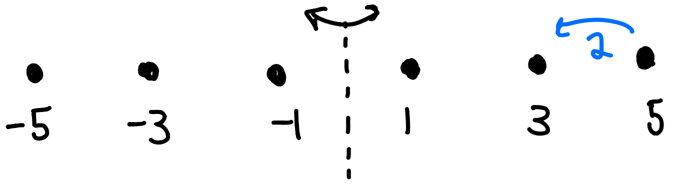
$$X(Y^{k_0}v) = \underbrace{\left(\overbrace{n}^{n} + \overbrace{(n-2)}^{n-2} + \cdots + \overbrace{(n-2k_0+2)}^{n-2(n+1)+2=-n}\right)}_{=\,k_0(n-k_0+1)\;=0}\underbrace{(Y^{k_0-1}v)}_{\neq 0}$$

$\therefore\ = 0$ $\qquad k_0 = n+1$

$\therefore\ n$ is a non-negative integer.

So, the eigenvalues of H form a string of integers differing by 2 and symmetric about the reflection $\alpha \mapsto -\alpha$:

eg $n=5$:



Conclusion For every nonnegative integer n, there ~~exists~~ cheat. a unique ✓ irreducible representation $V^{(n)}$ of $\mathfrak{sl}(2,\mathbb{C})$ whose highest eigenvalue of H is n. It is an $(n+1)$-dimensional representation.

In particular, our irrep V_n earlier is irreducible, and the highest eigenvalue of H was n, so it must be isomorphic to $V^{(n)}$.

What about representations of $sl(2,\mathbb{C})$ which aren't reducible? We're going to need to take a step back for a bit.

Abstract interlude

Definition A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra is called an **ideal** if $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$; i.e. if $\mathfrak{h}$ is a representation of $\mathfrak{g}$ under the adjoint action.

$$\begin{aligned} \mathrm{Ad} : \mathfrak{g} &\longrightarrow \mathrm{End}(\mathfrak{g}) \\ X &\longmapsto [X, -] \end{aligned}$$

Exercise Let G be a connected (matrix) Lie group and $H \leq G$ a connected subgroup. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. Show that

H is a normal subgroup of G $\iff$ $\mathfrak{h}$ is an ideal of $\mathfrak{g}$

Definition A Lie algebra $\mathfrak{g}$ is called **simple** if it contains no nontrivial ideals and if $\dim \mathfrak{g} \geq 2$. It is called **semisimple** if it is a direct sum of simple Lie algebras.

$$\mathfrak{g} \oplus \mathfrak{h} \quad : \quad [(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2])$$

$$X_1, X_2 \in \mathfrak{g}$$
$$Y_1, Y_2 \in \mathfrak{h}.$$

Defn If V and W are representations of $\mathfrak{g}$, their **tensor product** $V\otimes W$ is the representation of $\mathfrak{g}$ given by

$g\cdot(v\otimes w) := (g\cdot v)\otimes(g\cdot w)$

$$X(v\otimes w) = X(v)\otimes w + v\otimes X(w).$$

Note: this definition comes from differentiating the tensor product of two reps π and π' of G on V and W:

$$\begin{aligned} X(v\otimes w) &= \frac{d}{dt}\Big|_{t=0} \underbrace{e^{tX}(v\otimes w)} \\ &= \frac{d}{dt}\Big|_{t=0} \left(e^{tX}v\right)\otimes\left(e^{tX}w\right) \\ &= \frac{d}{dt}\Big|_{t=0} \left(v + tXv + \cdots\right)\otimes\left(w + tXw + \cdots\right) \\ &= \frac{d}{dt}\Big|_{t=0} \left[v\otimes w + t\left(Xv\otimes w + v\otimes Xw\right) + \cdots\right] \\ &= (Xv)\otimes w + v\otimes(Xw) . \end{aligned}$$

Lemma Let (V,π) be a representation of a Lie algebra $\mathfrak{g}$ and define a bilinear form B on $\mathfrak{g}$ by

$$B(X,Y) = \mathrm{Tr}_V\big(\underbrace{\pi(X)\circ\pi(Y)}_{V\to V}\big)$$

Then B is symmetric and $\mathfrak{g}$-invariant.

(ie. B is a morphism $\mathfrak{g}\otimes\mathfrak{g}\longrightarrow\mathbb{C}$ in $\mathrm{Rep}\,\mathfrak{g}$).

trivial $\mathfrak{g}$-rep.

<u>Proof</u> <u>Symmetry</u>

$$\begin{aligned} B(X,Y) &= \mathrm{Tr}_V(\pi(X)\pi(Y)) \\ &= \mathrm{Tr}_V(\pi(Y)\pi(X)) \\ &= B(Y,X). \end{aligned}$$

<u>$\mathfrak{g}$-invariant</u> <u>Exercise</u>

□

<u>Defn</u> The <u>Killing form</u> on a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form

$$B(X,Y) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}_X \, \mathrm{ad}_Y) \nwarrow$$

eg. for $\mathfrak{sl}(2,\mathbb{C})$:

$$[H,X] = 2X \qquad [H,Y] = -2Y \qquad [X,Y] = H$$

$\mathrm{ad}_H : \mathfrak{g} \to \mathfrak{g}$

$$\mathrm{ad}_H = [H,-] = \begin{array}{c|ccc} & H & X & Y \\ \hline H & 0 & 0 & 0 \\ X & 0 & 2 & 0 \\ Y & 0 & 0 & -2 \end{array}$$

$[H,X] = 2X$

$[X,H] = -2X$

$$\mathrm{ad}_X = [X,-] = \begin{array}{c|ccc} & H & X & Y \\ \hline H & 0 & 0 & 0 \\ X & -2 & 0 & 0 \\ Y & 0 & 0 & 2 \end{array}$$

$$\mathrm{ad}_y = [y, -] = \begin{bmatrix} \cdots & & \\ & & \\ & & \end{bmatrix}$$

Exercise Complete this calculation. Compute that

$$B(Z, Z') = 4\,\mathrm{Tr}(ZZ') \qquad Z, Z' \in \mathfrak{sl}(2,\mathbb{C}).$$

Is B non-degenerate?

$$V \qquad B(v,w) \in k$$

$$B \text{ non-degenerate} \iff \begin{bmatrix} B(v,w) = 0 \text{ for all } w \\ \iff v = 0. \end{bmatrix}$$

Lie Algebras 2020

Lecture 4

Last time we ended with the Killing form... but I'm going to pause that and return at a later point.

I want to tie up a loose end from the last lecture:

Conclusion For every nonnegative integer n, there exists a unique irreducible representation $V^{(n)}$ of $\mathfrak{sl}(2,\mathbb{C})$ whose highest eigenvalue of H is n. It is an $(n+1)$-dimensional representation.

cheat.

I said "existence" was a cheat, and that to do it abstractly needed Verma modules (infinite-dim reps). That may be true in general, but for $\mathfrak{sl}(2,\mathbb{C})$, we can write down the rep concretely.

$$\begin{tikzcd}
V_{-n} \arrow[loop above, "H=-n"] \arrow[r, bend left, "X"] & \cdots \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] & V_{n-4} \arrow[loop above, "H=n-4"] \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] & V_{n-2} \arrow[loop above, "H=n-2"] \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] & V_n \arrow[loop above, "H=n"] \arrow[l, bend left, "Y"] \\
 & & & & v \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

If we choose a basis

$$v_k = Y^k(v) \quad , \quad k = 0 \ldots n$$

then we computed:

$[H,X]=2X$

$[H,Y]=2Y$

$[X,Y]=H$

$$H(v_k) = (n-2k)v_k$$

$$Y(v_k) = \begin{cases} v_{k+1} & \text{if } k<n \\ 0 & \text{if } k=n \end{cases}$$

$$X(v_k) = \begin{cases} k(n-(k-1))v_{k-1} & \text{if } k>0 \\ 0 & \text{if } k=0 \end{cases}$$

Conversely, we can <u>define</u> an abstract representation of $\mathfrak{sl}(2,\mathbb{C})$ on

$$V := \mathbb{C}[v_0, \cdots, v_k]$$

by the above formulae (it indeed satisfies the commutation relations). So existence is clear here too.

The other finite-dimensional proof of existence is to use <u>symmetric powers</u>.

Defn The <u>k^{th} symmetric power</u> of a vector space V is

$$\mathrm{Sym}^k(V) = V^{\otimes k} \Big/ \mathrm{span}\left\{ \begin{array}{l} v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_k \\ + \; v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k, \quad i = 1 \ldots k \end{array} \right\}$$

We write the elements of $\mathrm{Sym}^k(V)$ as "homogenous polynomials" eg

$$v_1 v_2 v_3 = [v_1 \otimes v_2 \otimes v_3] = v_2 v_1 v_3 = v_2 v_3 v_1 \quad \text{etc.}$$

For example, if $V = \mathbb{C}[x, y]$, then

$$\mathrm{Sym}^1 V = V$$

$$\mathrm{Sym}^2 V = \mathrm{span}\{ x^2, \underbrace{xy}_{=yx}, y^2 \}$$

$$\mathrm{Sym}^3 V = \mathrm{span}\{ x^3, x^2y, xy^2, y^3 \}$$

etc.

Lemma If V is a representation of a Lie algebra $\mathfrak{g}$, then the associated representation of $\mathfrak{g}$ on $V^{\otimes k}$ descends to $\mathrm{Sym}^k(V)$, via

$$X\Big(\underbrace{[v_1 \otimes \cdots \otimes v_k]}_{\sigma(v_1 \otimes \cdots \otimes v_k)} \Big) := \sum_{p=1}^{k} [v_1 \otimes \cdots \otimes X(v_p) \otimes \cdots \otimes v_k]$$

$$V/W$$

$$P : V^{\otimes k} \longrightarrow V^{\otimes k}$$ ... symmetrizer

$$e_1 \otimes \cdots \otimes e_k \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(k)}$$

eg.

$$V^{\otimes 2} \longrightarrow V^{\otimes 2}$$

$$e_1 \otimes e_1 \longmapsto e_1 \otimes e_1$$

$$e_1 \otimes e_2 \longmapsto \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)$$

$$e_2 \otimes e_2 \longmapsto e_2 \otimes e_2$$

$$P^2 = P$$ (i.e. P is a projector)

$$\mathrm{Sym}^k(V) = \boxed{\mathrm{Im}(P) \subseteq V^{\otimes k}}$$

$$e_1 e_2^2 := P(e_1 \otimes e_2 \otimes e_2)$$

Subspace generated by these vectors.

$$\underset{\mathrm{Sym}^k V}{\Lambda^k V} := V^{\otimes k} \Big/ \overset{\forall i}{\langle v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_k \underset{-}{+} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k \rangle}$$

Proof Must check action of X is well-defined.

We have:

$$[v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_k] = [v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k]$$

and indeed

$$X(\text{LHS}) = \sum_{p=1}^{k} [v_1 \otimes \cdots \otimes X(v_p) \otimes \cdots \otimes v_k]$$

$$= X(\text{RHS})$$

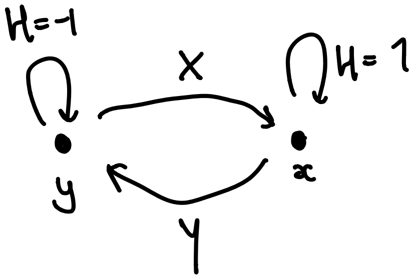
because

$$\underbrace{[v_1 \otimes \cdots \otimes X(v_i) \otimes v_{i+1} \otimes \cdots \otimes v_k]}_{} = \underbrace{[v_1 \otimes \cdots \otimes v_{i+1} \otimes X(v_i) \otimes \cdots \otimes v_k]}_{}. \quad \square$$

Exercise Does it make sense to take symmetric powers of group representations?

For example, let V be the 2-dim irrep of $\mathfrak{sl}(2,\mathbb{C})$:

$$V = \text{span}\{x, y\}$$



$$x^2 = [x \otimes x]$$

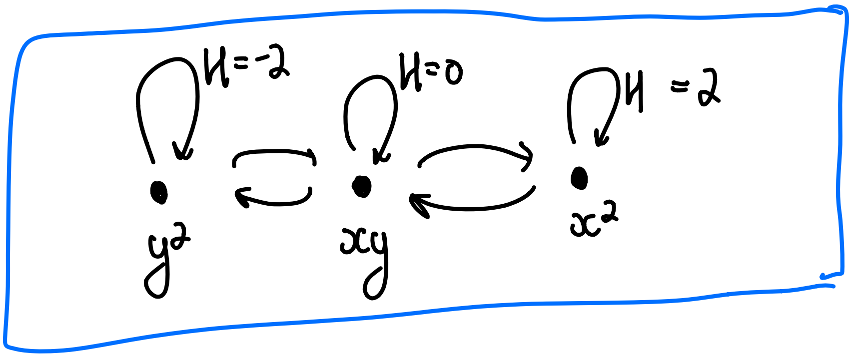
Then $\mathrm{Sym}^2 V = \mathrm{span}\{x^2, xy, y^2\}$ and

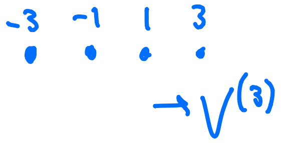
$$H(x^2) = H(x)\cdot x + x\cdot H(x)$$
$$= 2x^2$$

$$H(xy) = H(x)\cdot y + x\cdot H(y)$$
$$= xy \quad - xy$$
$$= 0$$

$$H(y^2) = H(y)\cdot y + y\cdot H(y)$$
$$= -2y^2$$

$V_n :=$ rep of $\mathfrak{sl}(2,\mathbb{C})$ coming from differentiating $SU(2)$-rep V_n.





$V^{(n)}$

So

$$\mathrm{Sym}^2 V \cong \boxed{V^{(2)}} \leftarrow$$ the 3-dim irrep of $\mathfrak{sl}(2,\mathbb{C})$.

Similarly,

$$\mathrm{Sym}^n V \cong V^{(n)}$$

which provides another way to construct the irreps $V^{(n)}$.

Exercise: Decompose $V^{(1)} \otimes V^{(1)} \otimes V^{(1)}$ into a direct sum of irreducible representations.

Representations of $\mathfrak{sl}(3,\mathbb{C})$

We want to classify irreps of $\mathfrak{sl}(3,\mathbb{C})$ like we did for $\mathfrak{sl}(2,\mathbb{C})$. Let V be any representation of $\mathfrak{sl}(3,\mathbb{C})$.

For $\mathfrak{sl}(2,\mathbb{C})$, H played a pivotal role: we decomposed a representation V of $\mathfrak{sl}(2,\mathbb{C})$ into eigenspaces for H:

$$V = \bigoplus_{\gamma} V_{\gamma}$$

For $\mathfrak{sl}(3,\mathbb{C}) = \underbrace{\{ X \in \mathrm{Mat}_{3,3}(\mathbb{C}) : \mathrm{Tr}\, X = 0 \}}_{\text{8-dim}}$, the role of H will be played by the 2-dimensional space $\mathfrak{h} \subseteq \mathfrak{sl}(3,\mathbb{C})$ of diagonal matrices.

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0 \right\}$$

Fact For each $H \in \mathfrak{h}$, the operator $H : V \longrightarrow V$ is diagonalizable.

Note that all the operators $H \in \mathfrak{h}$ commute with each other, so we can find a basis of simultaneous eigenvectors.

By an <u>eigenvector of $\mathfrak{h}$</u>, we mean a vector $v \in V$ that is an eigenvector of each $H \in \mathfrak{h}$. Since the eigenvalue depends linearly on $H \in \mathfrak{h}$, we can write

$$H(v) = \underbrace{\gamma(H)}_{\text{a number}} v$$

for some linear functional $\gamma \in \mathfrak{h}^*$, which we call an <u>eigenvalue</u> for the action of $\mathfrak{h}$ on V. We call these eigenvalues $\gamma \in \mathfrak{h}^*$ for the action of $\mathfrak{h}$ on V the <u>weights</u> of V and the corresponding eigenspaces V_γ the <u>weight spaces</u> of V. So:

Any finite-dimensional representation V of $\mathfrak{sl}(3,\mathbb{C})$ has a decomposition

$$V = \bigoplus_\gamma V_\gamma \qquad \begin{aligned} &v \in V_\gamma \\ &H(v) = \gamma(H)\, v \end{aligned}$$

where V_γ is an eigenspace for $\mathfrak{h}$ and γ ranges over a finite subset of $\mathfrak{h}^*$, the <u>weights</u> of V.

Ok, we know what places the role of "H" for $\mathfrak{sl}(3,\mathbb{C})$. What plays the role of X, Y? Recall for $\mathfrak{sl}(2,\mathbb{C})$,

$$[H, X] = 2X \quad , \quad [H, Y] = -2Y$$

The correct way to understand these commutation relations is that X and Y are <u>eigenvectors</u> for the adjoint action of H on $\mathfrak{sl}(2, \mathbb{C})$:

$$\overset{H(X)}{\mathrm{ad}_H(X)} = 2X \qquad \overset{H(Y)}{\mathrm{ad}_H(Y)} = -2Y$$

In other words, we should consider $V = \mathfrak{g}$, the adjoint representation, and decompose it as above:

$\mathrm{ad}_{H_1}(H_2) = [H_1, H_2] = 0$

$$\mathfrak{g} = \underbrace{\mathfrak{h}}_{} \oplus \left(\bigoplus_{\alpha} \underbrace{\mathfrak{g}_\alpha}_{} \right)$$

H acts as zero

ie. $\alpha = 0$

since $[H, H'] = 0$

roots

For $X \in \mathfrak{g}_\alpha$

$\underbrace{H \cdot X}_{= [H, X]} = \alpha(H)\, X$

note: I'll use γ for general weights, and α for roots.

The weights $\alpha \in \mathfrak{g}^*$ of the adjoint representation are called the <u>roots</u> of $\mathfrak{g}$. Note, we don't usually consider the weight $\alpha = 0$ (with corresponding eigenspace $\mathfrak{h}$) to be a root.

The role of X and Y is thus played by basis vectors for these <u>root spaces</u> $\mathfrak{g}_\alpha$.

Let's calculate the roots for $\mathfrak{sl}(3,\mathbb{C})$.

If we write

$$H = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \qquad a_1 + a_2 + a_3 = 0$$

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \qquad x_{11} + x_{22} + x_{33} = 0$$

then the matrix elements of $\mathrm{ad}_H(X)$ are

$$\boxed{[H,X]_{ij} = (a_i - a_j)\, x_{ij} .}$$

So, if we want X to be an eigenvector for ad_H for all H, we will need x_{ij} to be zero except at a single element, ie. X must be an elementary matrix $\underbrace{E_{ij}}_{\text{matrix elements 0 except at position } (i,j) \text{, where equals 1.}}$:

So,

$$H = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$$

$$\mathrm{ad}_{\boxed{H}}(E_{ij}) = \underbrace{(a_i - a_j)}_{\text{depend on } H} E_{ij}$$

which we can write in a more sophisticated way as

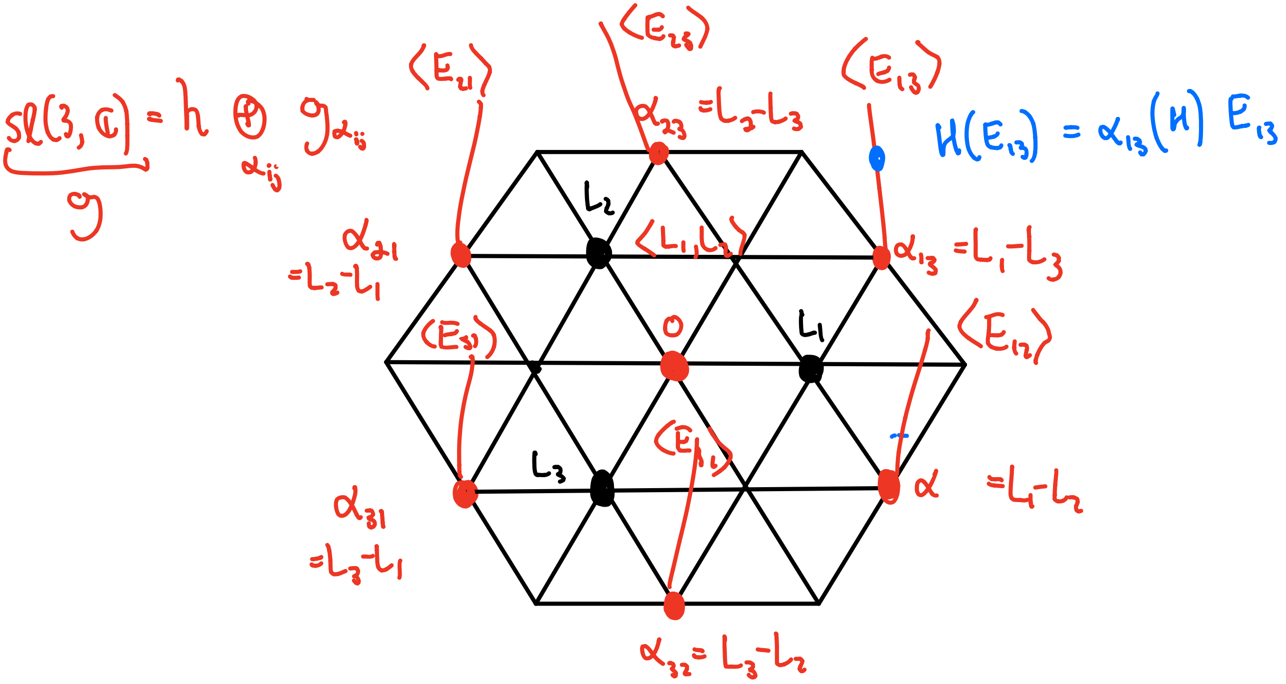
$$\mathrm{ad}_H(E_{ij}) = (L_i - L_j)(H)\; E_{ij}$$

where $L_i \in \mathfrak{h}^*$ are the linear functionals

$$L_i\left(\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}\right) = a_i$$

So,

The 6 roots of $\mathfrak{sl}(3,\mathbb{C})$ are $\alpha_{ij} = L_i - L_j \quad (i \neq j)$, with eigenspace spanned by E_{ij}.



Picture of roots in $\mathfrak{h}^*$. Note: $L_1 + L_2 + L_3 = 0$.

From this picture we can read off basically the entire structure of the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$. We know how each $H \in \mathfrak{h}$ acts on the root spaces $\mathfrak{g}_\alpha$ – it acts by scalar multiplication by $\alpha(H)$.

But, we also know how the rest of the Lie algebra acts.

If $X \in \mathfrak{g}_\alpha$, and $Y \in \mathfrak{g}_\beta$, then where is $[X,Y]$?

Fundamental calculation

$$[H,[X,Y]] = [X,\underbrace{[H,Y]}_{=\beta(H)Y}] + [\underbrace{[H,X]}_{=\alpha(H)X},Y]$$

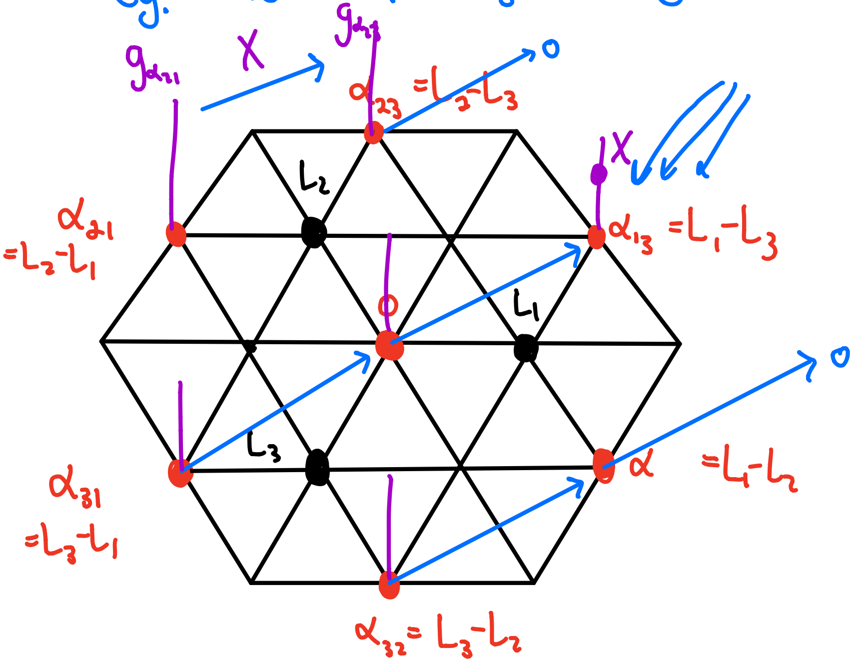
$$= \beta(H)\,[X,Y] + \alpha(H)\,[X,Y]$$

$$= (\beta+\alpha)(H)\,[X,Y]$$

So, $[X,Y]$ is an eigenvector for $\mathfrak{h}$, with eigenvalue $\alpha+\beta$. So:

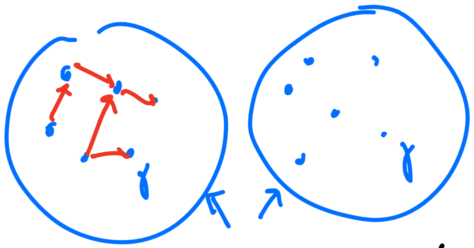
$$\boxed{\mathrm{ad}(\mathfrak{g}_\alpha) : \mathfrak{g}_\beta \longmapsto \mathfrak{g}_{\alpha+\beta}}$$

eg. action of α_{13} on $\mathfrak{g}$:

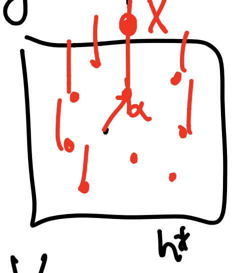


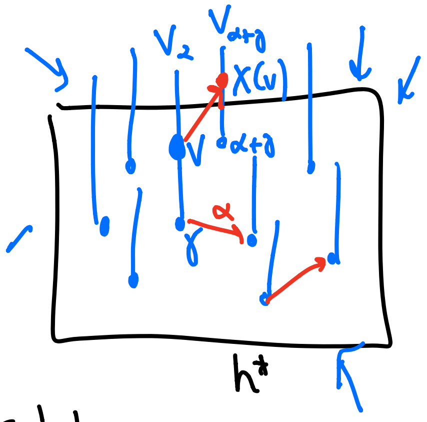
This abstract calculation doesn't yet tell us everything – for instance, it doesn't tell us what the <u>kernel</u> of the action of $X \in \mathfrak{g}_\alpha$ is on some $\mathfrak{g}_\beta$. It just tells us what goes where – but that's a huge help!

The same calculation will apply to <u>any</u> representation V of $\mathfrak{sl}(3,\mathbb{C})$. We decompose V into its weight spaces



$$V = \bigoplus_\gamma V_\gamma$$





and then, if $X \in \mathfrak{g}_\alpha$ and $v \in V_\gamma$, we calculate:

$H(v) = \gamma(H)v$
$H(X) = \alpha(H)X$

<u>Fundamental calculation</u> (for a general representation (V, π))

$(\mathfrak{g}, \mathrm{ad})$

$$H(X(v)) = \left(\left(XH + [H,X]\right)\right)(v)$$
$$= \gamma(H)X(v) + \alpha(H)X(v)$$
$$= (\gamma+\alpha)(H)\ (v)$$

<u>adjoint rep</u>

$\mathfrak{g}$ on $\mathfrak{g}$

$A \cdot B := [A,B]$

So:

$$\boxed{X \in \mathfrak{g}_\alpha : V_\gamma \longrightarrow V_{\alpha+\gamma}}$$

If V is irreducible, we observe:

The weights $\gamma \in \mathfrak{h}^*$ occuring in an irreducible representation V of $\mathfrak{sl}(3,\mathbb{C})$ differ from each other by integral linear combinations of the root vectors $\alpha \in \mathfrak{h}^*$.

Lie Algebras 2020

Lecture 5

Recap: classifying irreps of $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$.

We decompose

$$\mathfrak{g} = \underbrace{\mathfrak{h}}_{=\left\{\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}\right\}} \oplus \left(\bigoplus_{\alpha} \underbrace{\mathfrak{g}_\alpha}_{} \right)$$

$\nwarrow$ $\alpha_{ij} = h_i - h_j$,

$\mathfrak{g}_{\alpha_{ij}} = \text{span}\{E_{ij}\}$

where $\alpha \in \mathfrak{h}^*$ are the <u>roots</u> of $\mathfrak{g}$, satisfying

$$[H, X] = \alpha(H) X \qquad X \in \mathfrak{g}_\alpha.$$

Then we observed that if V is any finite-dim rep of $\mathfrak{sl}(3,\mathbb{C})$, we can decompose

$$V = \bigoplus_{\gamma} V_\gamma$$

where $\gamma \in \mathfrak{h}^*$ are the <u>weights</u> of V, and

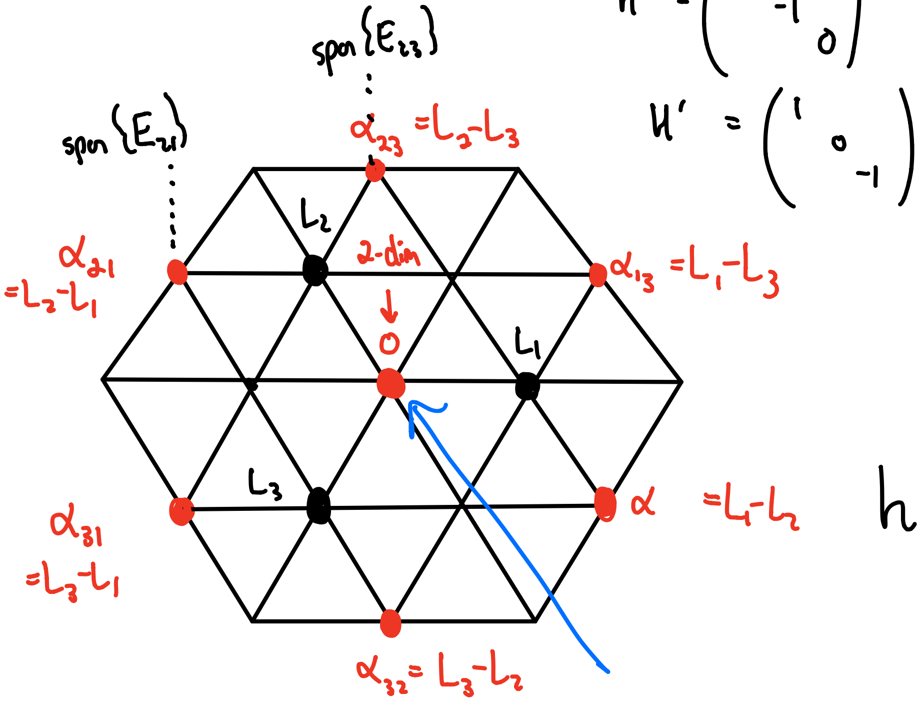
$$H(v) = \gamma(H)\, v \qquad v \in V_\gamma \,.$$

Finally, we found that

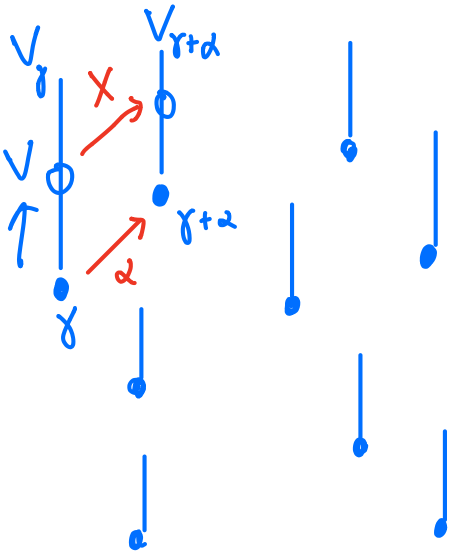
$$\boxed{X \in \mathfrak{g}_\alpha : V_\gamma \longrightarrow V_{\gamma+\alpha}} \leftarrow$$

$$H = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$$H' = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$



roots of $\mathfrak{sl}(3,\mathbb{C}) \subseteq \mathfrak{h}^*$



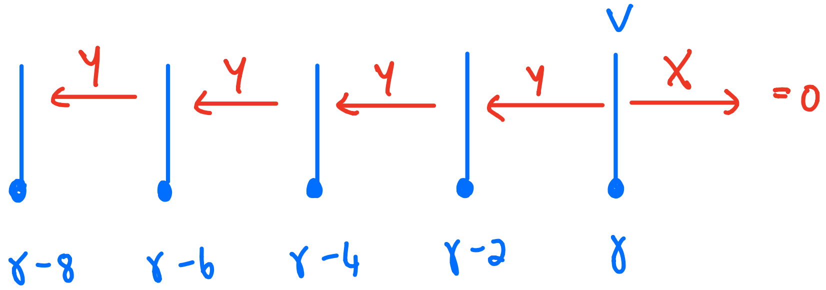
weights of $V \subseteq \mathfrak{h}^*$

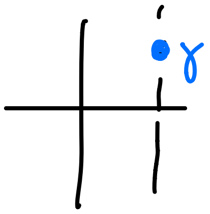
If V is <u>irreducible</u>, we observed:

> The weights $\gamma \in \mathfrak{h}^*$ occuring in an irreducible representation V of $\mathfrak{sl}(3,\mathbb{C})$ differ from each other by integral linear combinations of the root vectors $\alpha \in \mathfrak{h}^*$.

End of recap

For irreps of $\mathfrak{sl}(2,\mathbb{C})$, what we did next was characterize the entire irrep starting with an extremal vector v:

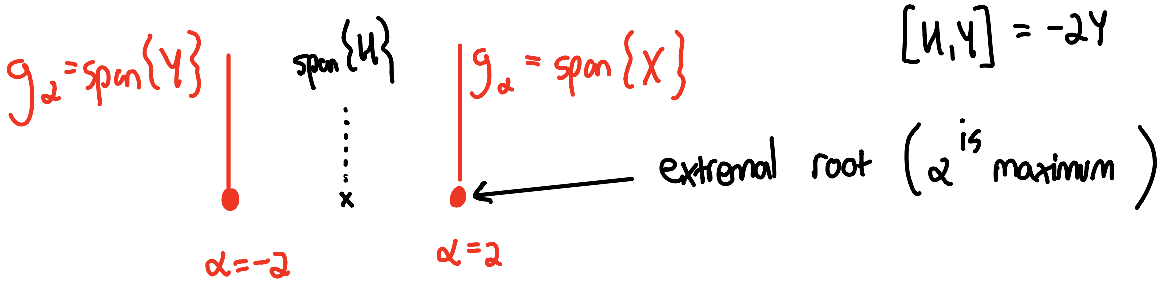




Here, "extremal" meant "v is annihilated by X", i.e. "v is annihilated by the extremal root space".

$[H,X]=2X$

$[H,Y]=-2Y$



For $\mathfrak{sl}(3,\mathbb{C})$, we should thus decide what the "extremal roots" are. Let

$$\Lambda_R = \mathbb{Z}\{\text{roots of } \mathfrak{g}\} \subseteq \mathfrak{h}^*$$

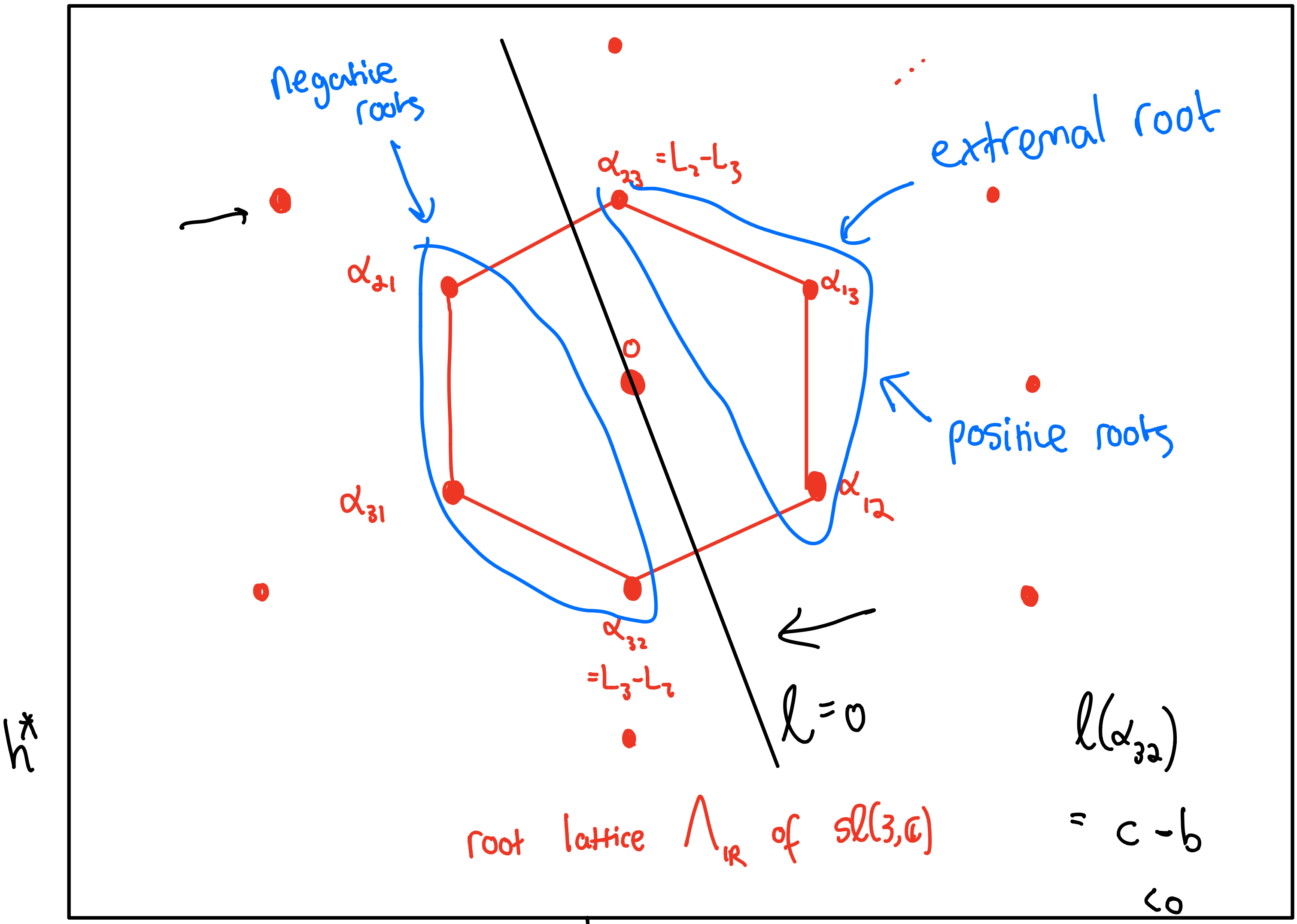
be the root lattice. We fix a linear map

$\Lambda_R \cong \mathbb{Z}^2$

$$\text{Roots}_{\mathbb{R}} := \mathbb{R}\{\text{roots of } \mathfrak{g}\} \subseteq \mathfrak{h}^*$$

$$\ell : \text{Roots}_{\mathbb{R}} \longrightarrow \mathbb{R} \qquad (\text{Roots}_{\mathbb{R}}) \otimes \mathbb{C} \cong \mathfrak{h}$$

which is irrational with respect to the lattice Λ_R. This will divide our roots into the positive roots and the negative roots:



In our case, we can choose

$$\ell(a_1 L_1 + a_2 L_2 + a_3 L_3) = a a_1 + b a_2 + c a_3$$

where $a+b+c = 0$ and $a > b > c$, so that the positive roots (the ones for which $\ell(\alpha) > 0$) and negative roots are:

α	α_{12}	α_{13}	α_{23}	α_{21}	α_{31}	α_{32}
basis for $\mathfrak{g}_\alpha$	E_{12}	E_{13}	E_{23}	E_{21}	E_{31}	E_{32}
	raising operators			lowering operators		

We can extend ℓ to a complex linear functional on $\mathfrak{h}^*$, whose real part then is a real linear functional on Λ_W:

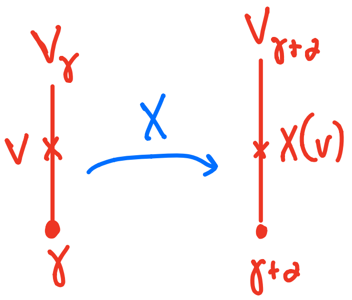
$$\hat{\ell} : \mathfrak{h} \longrightarrow \mathbb{R}$$

$$\tilde{\ell}(v_1 + iv_2) = \ell(v_1) + i\ell(v_2)$$

Now consider again our irreducible representation V.

The moral is:

$$\boxed{\text{Acting with } X \in \mathfrak{g}_\alpha \text{ on } v \in V_\gamma \text{ raises the value of } \hat{\ell} \text{ by } \ell(\alpha).}$$



$\hat{\ell}(X(v)) = \hat{\ell}(\gamma) + \ell(\alpha).$

So, if X comes from a positive root, then acting with X will raise the value of $\hat{\ell}$, and if X comes from a negative root, then acting with X will lower the value of $\hat{\ell}$.

Upshot For any irrep V of $sl(3,\mathbb{C})$, there exists a vector $v \in V$ such that:

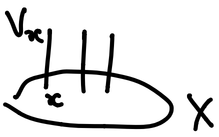
- $v \in V_\gamma$ for some weight γ
- v is killed by the positive roots $E_{1,2}$, $E_{1,3}$, $E_{2,3}$.

We call such a v a highest weight vector for V.

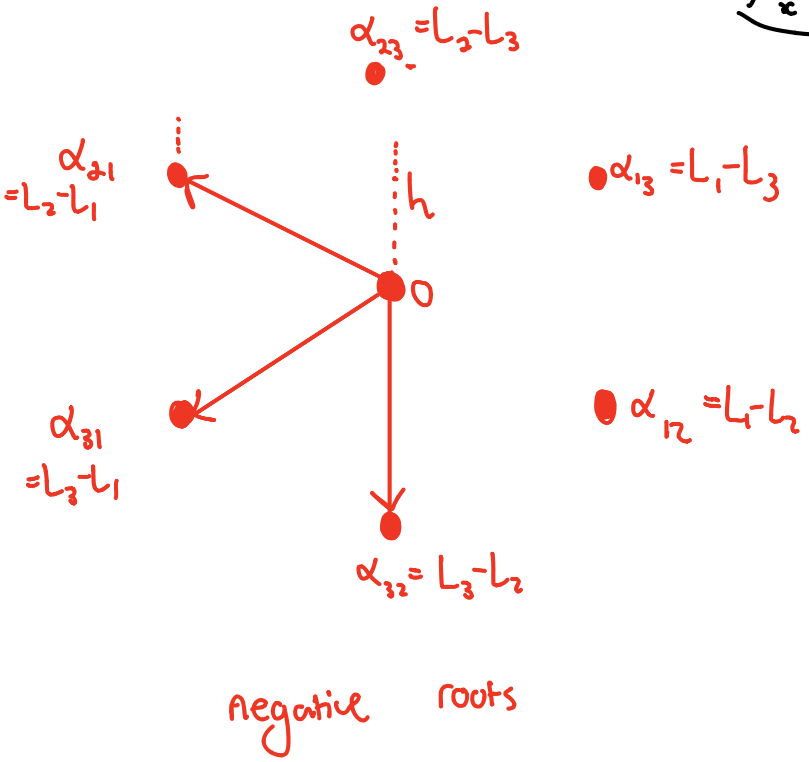
For reps of $sl(2,\mathbb{C})$, we then argued that the representation was spanned by vectors of the form $Y^k(v)$. We have the same here.

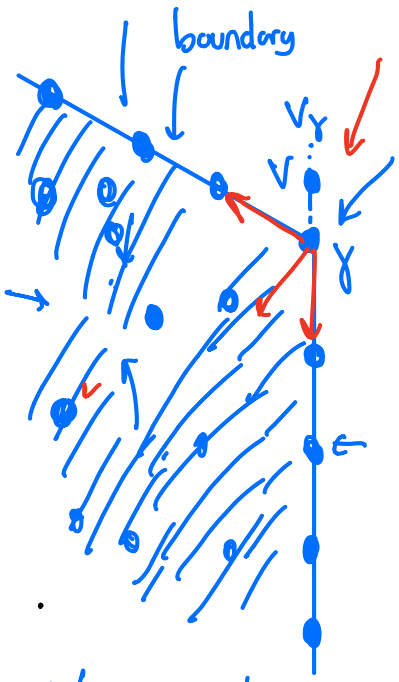
Lemma Let V be an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$, and $v \in V$ a highest weight vector. Then V is spanned by the images of v under successive applications of the lowering operators $E_{2,1}$, $E_{3,1}$, $E_{3,2}$.

As a consequence:



①





$\Rightarrow$ V supported on these weights.

② $\Rightarrow$ $\dim(V_\gamma) = 1$

③ $\Rightarrow$ the "boundary weight spaces" $V_{\gamma+n\alpha_{21}}$, $V_{\gamma+m\alpha_{32}}$ also have dimension 1.

To prove this, we will use the following useful general fact.

Reordering Lemma Suppose $\mathfrak{g}$ is any Lie algebra and that π is a representation of $\mathfrak{g}$. Suppose $X_n, \dots, X_1$ is an ordered basis for $\mathfrak{g}$ as a vector space. Then any expression of the form

$$\underbrace{\pi(X_{j_N})} \cdots \underbrace{\pi(X_{j_2})}\underbrace{\pi(X_{j_1})}$$

can be expressed as a linear combination of terms of the form

$$\underbrace{\pi(X_m)^{k_m} \cdots \pi(X_2)^{k_2}\, \pi(X_1)^{k_1}}$$

where each k_i is a non-negative integer and $\boxed{k_1 + k_2 + \cdots + k_m \leq N.}$

Proof Induction. For $N=1$, it's true, as there is nothing to do. Suppose it's true for products of length N. Then a product of length $N+1$ can by using the induction hypothesis be written as a sum of terms of the form

$$\underbrace{\pi(X_i)}\boxed{\pi(X_m)^{k_m} \cdots \pi(X_1)^{k_1}} \qquad k_1 + \cdots + k_m \leq N$$

repeatedly use $\pi(X_i)\pi(X_j) = \pi(X_j)\pi(X_i) + \underbrace{[\pi(X_i), \pi(X_j)]}_{= \sum_{k=1}^{n} c_{ij}^{k}\pi(X_k)}$

$$= \pi(X_m)^{k_m} \cdots \pi(X_i)^{k_i+1} \cdots \pi(X_1)^{k_1}$$

+ products with $\leq N$ terms (which can be reordered). □

Now we can prove the previous lemma.

Proof As for the proof of $\mathfrak{sl}(2,\mathbb{C})$, let $W \subseteq V$ be the subspace formed by repeated applications of the lowering operators $E_{2,1}, E_{3,1}, E_{3,2}$ to v. We must show W is an invariant subspace, which amounts to checking that the raising operators $E_{1,2}, E_{1,3}, E_{2,3}$ leave W invariant.

Select the following ordered basis for $\mathfrak{g}$:

$$\overset{\longleftarrow}{\underbrace{E_{2,1},\ E_{3,1},\ E_{3,2}}_{\text{lowering operators}}},\ \overset{\longleftarrow}{\underbrace{H_1,\ H_2}_{\text{basis for } \mathfrak{h}}},\ \overset{\longleftarrow}{\underbrace{E_{1,2},\ E_{1,3},\ E_{2,3}}_{\text{raising operators}}}$$

$$\underbrace{H_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix},\ H_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}}$$

Consider applying a raising operator X to a generating vector of W:

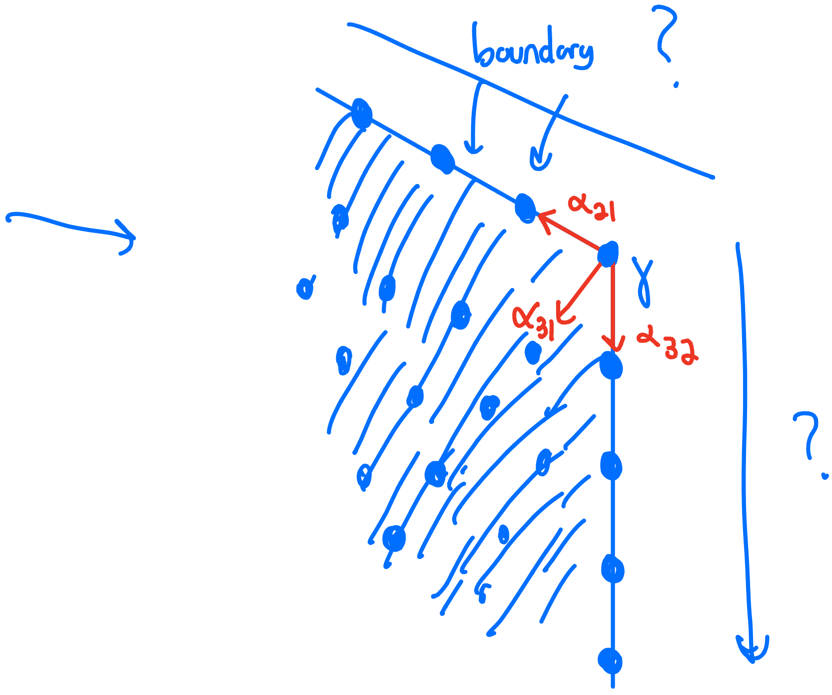
$$\underbrace{\boxed{X\ Y_n \cdots Y_1}(v)}$$

$$\left[\begin{array}{l} X = E_{1,2},\ E_{1,3} \text{ or } E_{2,3} \\ Y_i = E_{2,1},\ E_{3,1} \text{ or } E_{3,2}, \\ \qquad i = 1 \ldots n. \end{array}\right]$$

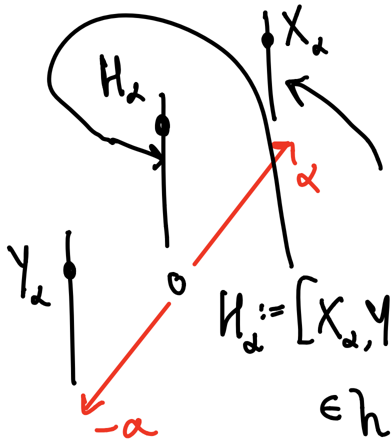
$$\overset{R-L}{=} \sum \underset{\uparrow}{Y \cdots Y} \cdots\ \underbrace{\cancel{H \cdots H}}\ \cancel{X \cdots X}$$

By the Reordering Lemma, we can write this as a sum of terms of the form where first raising operators are applied to v (which kill v), then elements $H \in \mathfrak{h}$ are applied to v (which simply multiply by the scalar factor $\gamma(H)$ as $v \in V_\gamma$), then lowering operators are applied to v. Hence the result is in W. □

Ok. So our picture of the weights are:



V supported on these weights



$H_\alpha := [X_\alpha, Y_\alpha]$

$\in \mathfrak{h}$

$[H_\alpha, X_\alpha] = cX_\alpha$

How long can this boundary extend?

Fact If α is a root of a semisimple Lie algebra $\mathfrak{g}$, then

$$S_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{g}$$

is a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. That is, we can choose $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, $H_\alpha \in \mathfrak{h}$, such that

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, Y_\alpha] = 2Y_\alpha, \quad [X_\alpha, Y_\alpha] = H_\alpha$$

Let's verify this in our example of $\mathfrak{sl}(3,\mathbb{C})$. Take $\alpha = \alpha_{12}$

Write

$$X_\alpha = E_{12}, \quad Y_\alpha = E_{21}, \quad H_\alpha = [X_\alpha, Y_\alpha]$$

$$(= E_{11} - E_{22})$$

Then :

$$[H_\alpha, X_\alpha] = 2X \ , \quad [H_\alpha, Y_\alpha] = -2Y \ , \quad [X_\alpha, Y_\alpha] = H_\alpha .$$

Similarly for the others.

<u>Exercise</u> Check this!

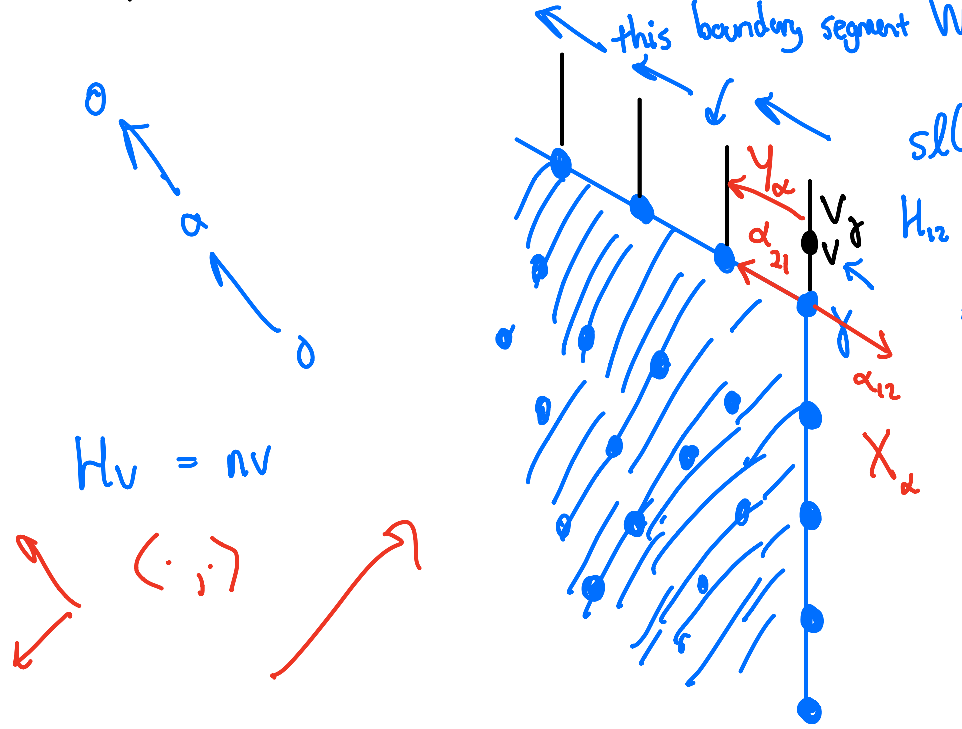
So:

> For any negative root α_{ij}, the subspace $W_{\alpha_{ij}} \subseteq V$ formed by repeatedly applying E_{ij} to v is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$.

In particular, the boundary vectors are irreps of $\mathfrak{sl}(2,\mathbb{C})$:

this boundary segment W_{12} is an irrep of

$$\mathfrak{sl}(2,\mathbb{C}) \cong \operatorname{span}\{\underbrace{E_{12}}_{X}, \underbrace{E_{21}}_{Y}, \underbrace{H_{12}}_{H}\}$$



$\therefore$ eigenvalues of H_{12} are integers, symmetric w.r.t 0.

"n"

$$H_{12}(v) = \underbrace{\gamma(H_{12})}_{\uparrow} v$$

highest eigenvalue, an integer.

From this we see that the roots comprising W_{21} are of the form

$$\overset{-2}{\bullet} \leftarrow \overset{0}{\bullet} \leftarrow \overset{2}{\bullet}$$

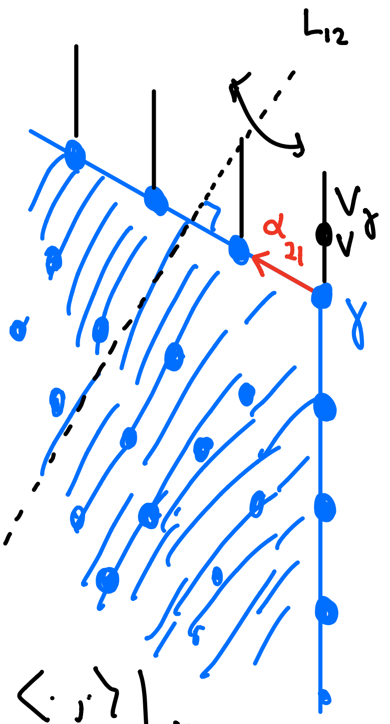
$$\gamma \ , \ \gamma - \alpha_{21} \ , \ \dots \ , \ \gamma - \underbrace{\gamma(H_{12})}_{n} \alpha_{21}$$

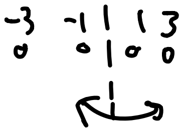
$n=2$

3-dim rep

and they are symmetric with respect to the line

$$L_{12} = \{ \delta \in h^* : \underbrace{\delta(H_{12}) = 0}_{``n=0"} \}$$





$\mathfrak{g} = su(2)$

$\mathfrak{g}_{\mathbb{C}} = sl(2,\mathbb{C}) = su(2)_{\mathbb{C}}$

$\langle \cdot , \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R}$

$\langle \cdot , \cdot \rangle : \mathfrak{g}_{\mathbb{C}} \otimes \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathbb{C}$

$\langle \cdot , \cdot \rangle : \mathfrak{g}_{\mathbb{C}}^* \otimes \mathfrak{g}_{\mathbb{C}}^* \longrightarrow \mathbb{C}$

$\langle f, g \rangle := \langle v, w \rangle \quad f = \langle v, \cdot \rangle \quad g = \langle w, \cdot \rangle$

$\langle \cdot , \cdot \rangle |_{h^*}$

$$\overline{\langle X, Y \rangle = Tr(ad_X\, ad_Y)}$$

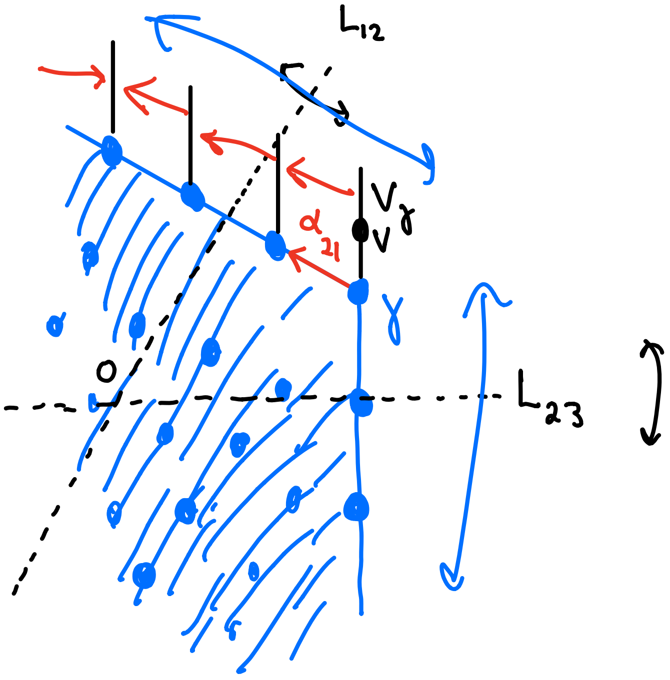
$h \otimes h \longrightarrow \mathbb{C}$

$\langle \cdot , \cdot \rangle |_{\mathfrak{g}^*}$

<u>Exercise</u> Equip the real subspace $Roots_{\mathbb{R}}$ spanned by the roots with the dualized version of the Killing form, thought of as an inner product on $h^*_{\mathbb{R}}$. Show that with respect to this inner product,

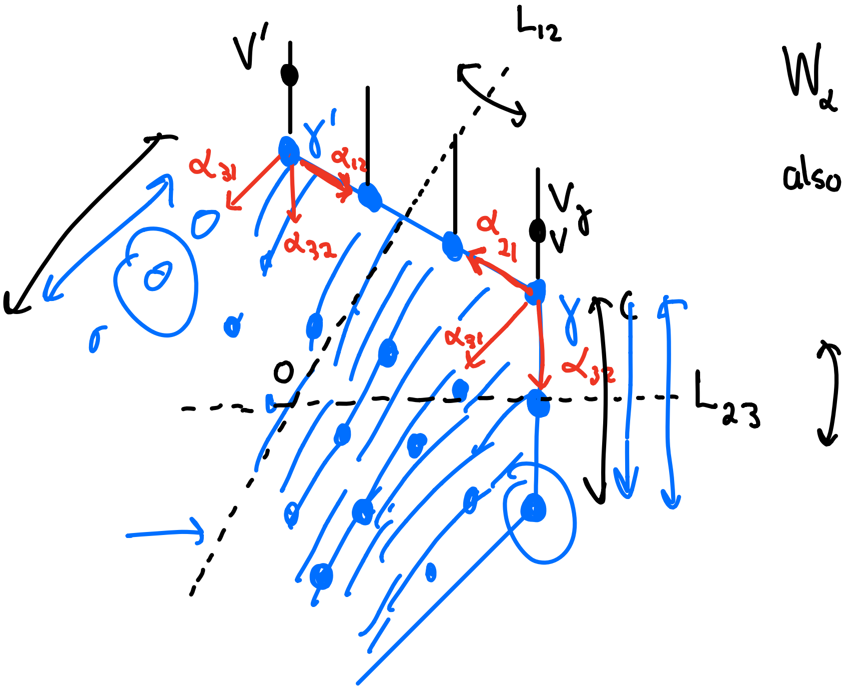
$$L_{12} = \text{line orthogonal to } \alpha_{12}$$

The same analysis applies to the boundary roots $\gamma + k\alpha_{3,2}$. They form an unbroken string, symmetric about the line L_{23}:



Now consider the vector at the end of the top boundary string,

$$v' = E_{2,1}^{n}(v) \quad \in V_{\gamma'} \quad , \quad \gamma' = \gamma + n\alpha_{21}$$



$$W_\alpha(\beta) = \beta - \underbrace{\beta(H_\alpha)\alpha}_{}$$

also a weight? yes.

From the perspective of v', the "positive roots" (the ones whose operators kill v') are

$$\alpha_{13}, \quad \alpha_{23}, \quad \alpha_{21}.$$

These are obtained by swapping $1 \leftrightarrow 2$ (i.e., reflecting in the line L_{12}) in the roots whose operators kill V:

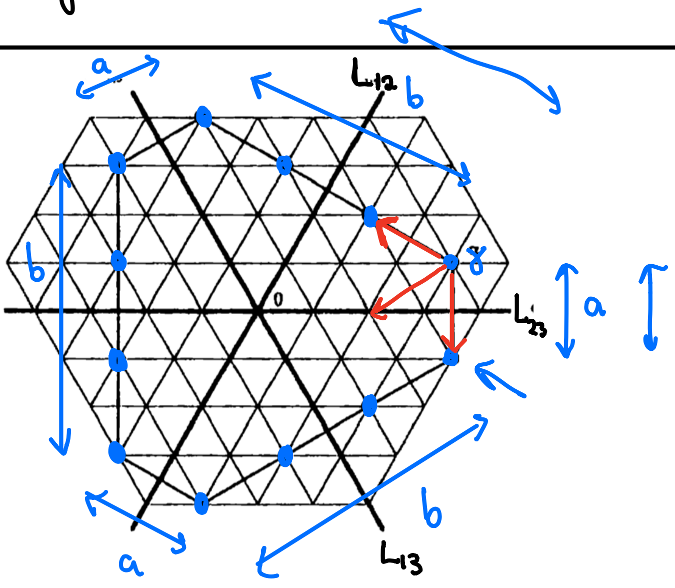
$$\alpha_{23}, \quad \alpha_{13}, \quad \alpha_{12}.$$

In fact, we assert somethin stronger.

Fact Reflection in the lines L_{ij} sends weights to weights

This means that:

The set of weights of the representation V is bounded by a hexagon symmetric with respect to the lines L_{ij} and with one vertex at γ.



- the multiplicity of these border weights is 1

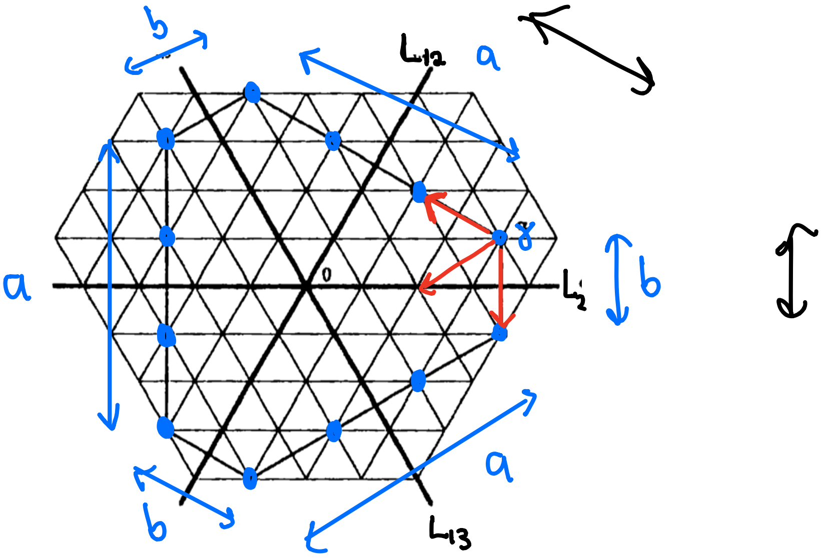
- there are weights inside too, but we don't yet know multiplicities

Lie Algebras

Lecture 6

Last time: we are studying irreducible representations V of $sl(3,\mathbb{C})$. The irrep V has a set of weights, and we arrived at the following picture for the weights:

The set of weights of the representation V is bounded by a hexagon symmetric with respect to the lines L_{ij} and with one vertex at γ. The hexagon is classified by two integers (a,b).



Let's do some examples.

1. The standard rep of $\mathfrak{sl}(3,\mathbb{C})$ on $V \cong \mathbb{C}^3$.

$\mathfrak{sl}(3,\mathbb{C})$ acts on $\underbrace{\mathbb{C}^3}_{\text{column vectors}}$ by matrix multiplication,

$$X \cdot v = Xv. \qquad X \in \mathfrak{sl}(3,\mathbb{C}).$$

The eigenvectors for the action of

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} : h_1 + h_2 + h_3 = 0 \right\}$$

, i.e. the weight spaces of V, are the standard basis vectors:

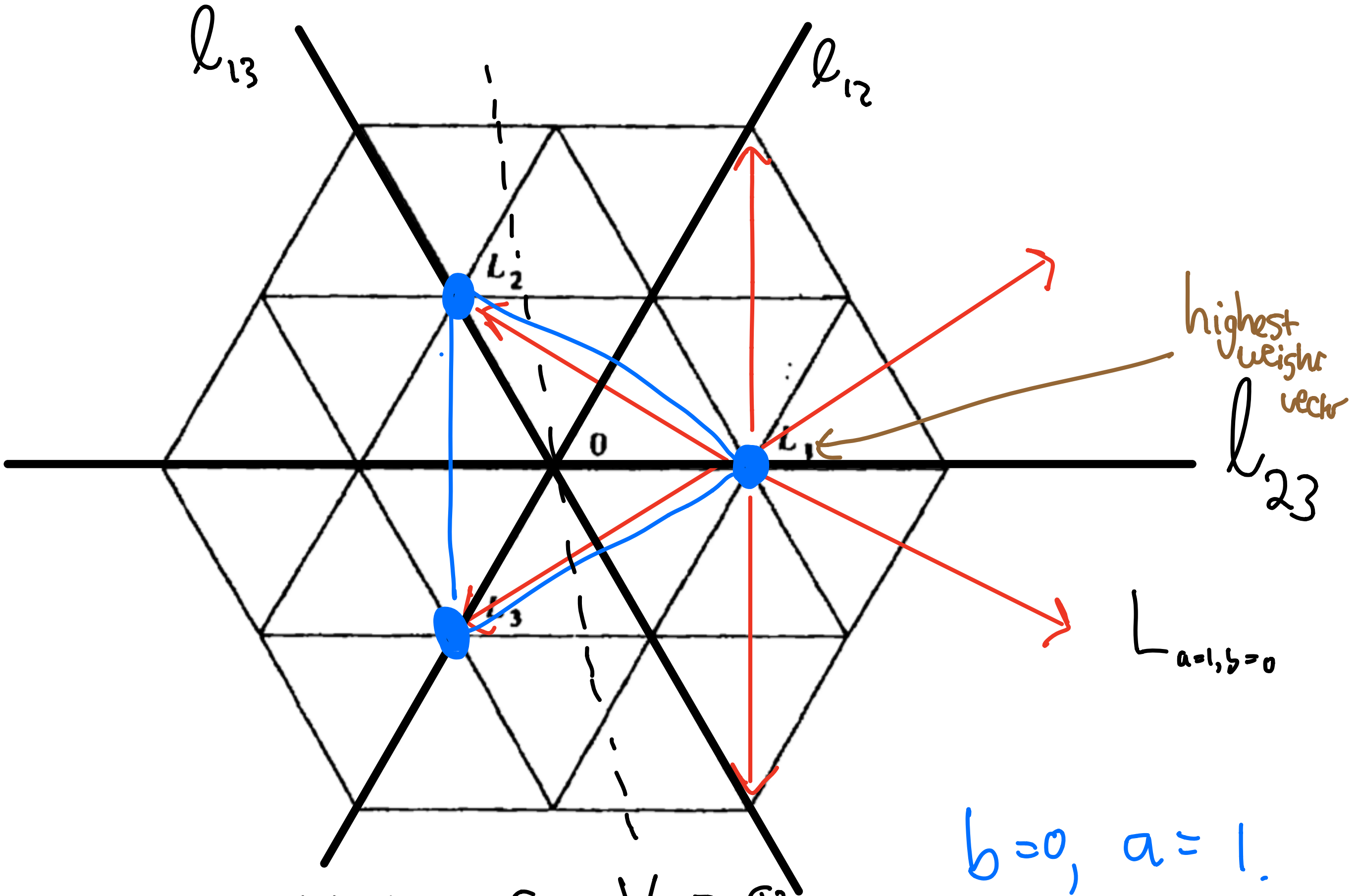
$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

weight: L_1 $\qquad L_2 \qquad L_3$

Recall : $L_i\left(\begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix}\right) = h_i$.

because

$$\begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix} e_i = \underbrace{L_1\left(\begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix}\right)}_{= h_i} e_i .$$



Weights for $V = \mathbb{C}^3$.

How about the dual representation V^*?

Defn ① If V is a representation of a group G, the dual representation of G on V^* is given by

$$(g\cdot f)(v) := f(g^{-1}\cdot v)$$

② If V is a representation of a Lie algebra $\mathfrak{g}$, the dual representation of $\mathfrak{g}$ on V^* is given by:

$$(X\cdot f)(v) := f(\underline{-X\cdot v})$$

Exercise a) Show that these are indeed reps of G and $\mathfrak{g}$ respectively.

b) If $\mathfrak{g}$ is the Lie algebra of G, show that this definition of the dual of the Lie algebra rep is equal to differentiating the dual of the Lie group rep.

Exercise If A is a operator on a f.dim complex vector space V, show that the eigenvalues of A on V are equal to the eigenvalues of $\underbrace{A^*: V^* \longrightarrow V^*}$ on V^*.

$$\underset{X^*\cdot f}{(X\cdot f)}(v) = f(-X\cdot v)$$

the dual linear map, ie.

given $A: V \longrightarrow W$

we get $A^*: W^* \longrightarrow V^*$

defined by $A^*(g)(v) = g(Av)$.

$V \longrightarrow V^{**}$

$$\boxed{X\cdot f = -X^*(f)}$$

V rep of $\mathfrak{g}$

$v \in V$ $\quad X \in \mathfrak{g}$

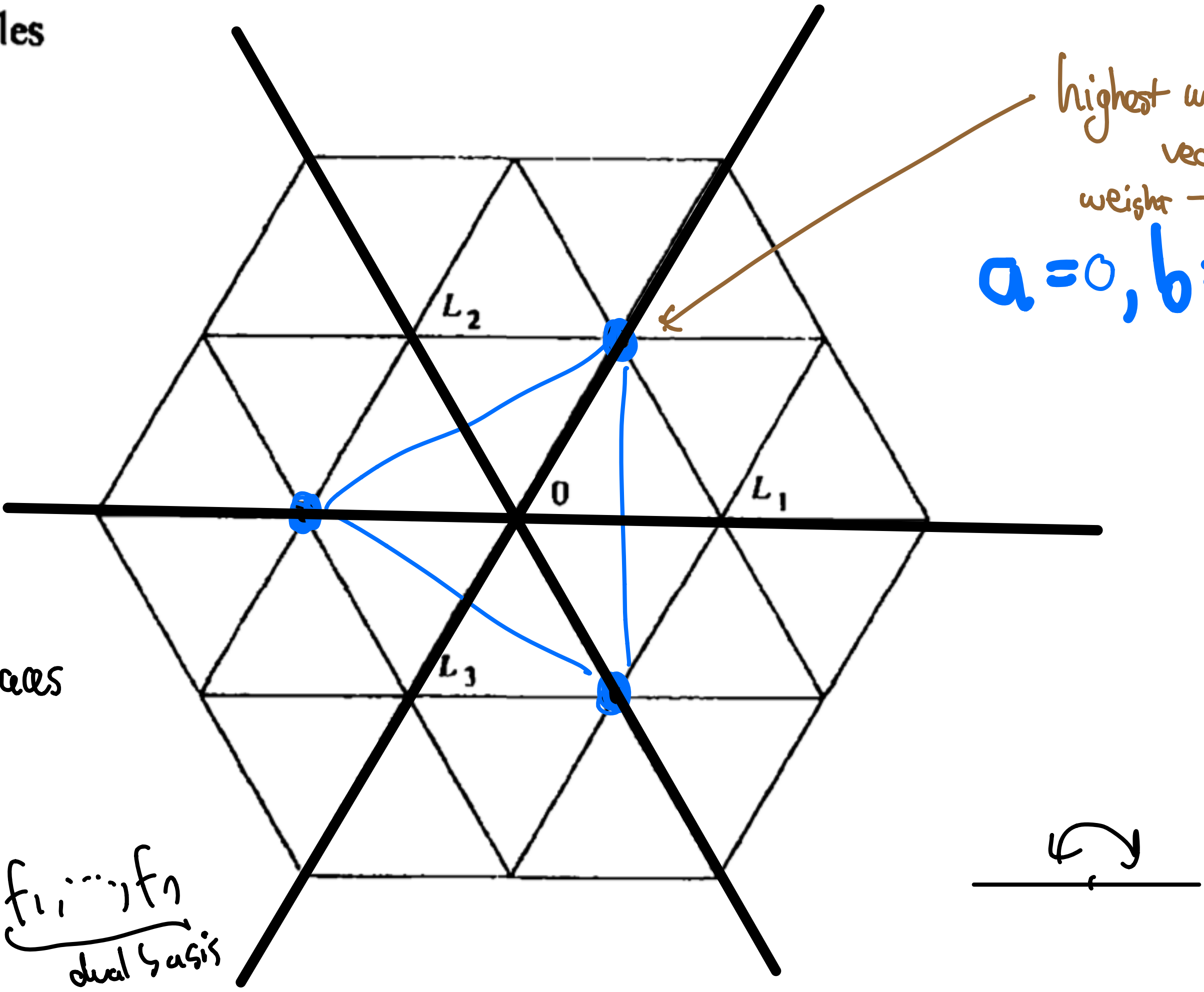
$X(v) = \lambda v$

$\exists\ f \in V^*$ s.t. $X^*(f) = \lambda f$

So, the weights of the dual representation of $\mathfrak{g}$ are the negatives of the weights of the original rep.

amples

2. V^*



highest weight vector weight $-L_3$

$a=0, b=1$

$V \to V^{**}$

f.d vector spaces

$V \cong V^*$

$\underbrace{e_1,\dots,e_n}_{\text{basis}} \rightsquigarrow \underbrace{f_1,\dots,f_n}_{\text{dual basis}}$

$f_i(e_j) = \delta_{ij}$

Weights of $\underbrace{V}_{\text{standard rep}}{}^*$

Observe: the weights of the irreps of $sl(2,\mathbb{C})$ were invariant under $\gamma \longmapsto -\gamma$ (i.e. $V^* \cong V$) but for reps of $sl(3,\mathbb{C})$ this is not true ($V^* \ncong V$ in general).

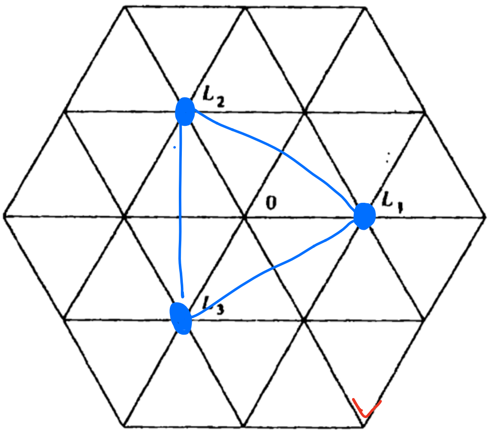
Recall the definitions of the tensor product and symmetric product of Lie algebra reps $V_1 \otimes \cdots \otimes V_n$

$$X(\underbrace{v_1 \otimes \cdots \otimes v_n}_{\in V_1 \otimes \cdots \otimes V_n}) := \underbrace{\sum_{i=1}^{n} v_1 \otimes \cdots \otimes X(v_i) \otimes \cdots \otimes v_n}_{}$$

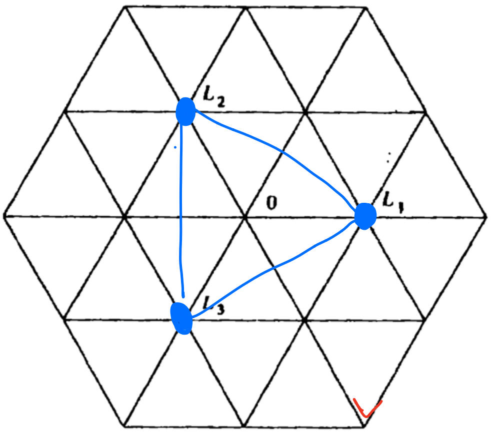
$$X(\underbrace{v_1 v_2 \cdots v_n}_{\in \mathrm{Sym}(V_1 \otimes \cdots \otimes V_n)}) := \sum_{i=1}^{n} v_1 \cdots v_{i-1} X(v_i) \cdots v_n .$$

From this we see that the weights of a tensor product (or a symmetric product) are the sums of the weights of the factors ... it's just that their multiplicities will be different.

eg. 3. $\underline{V \otimes V}$ ... 9-dimensional



$\otimes$



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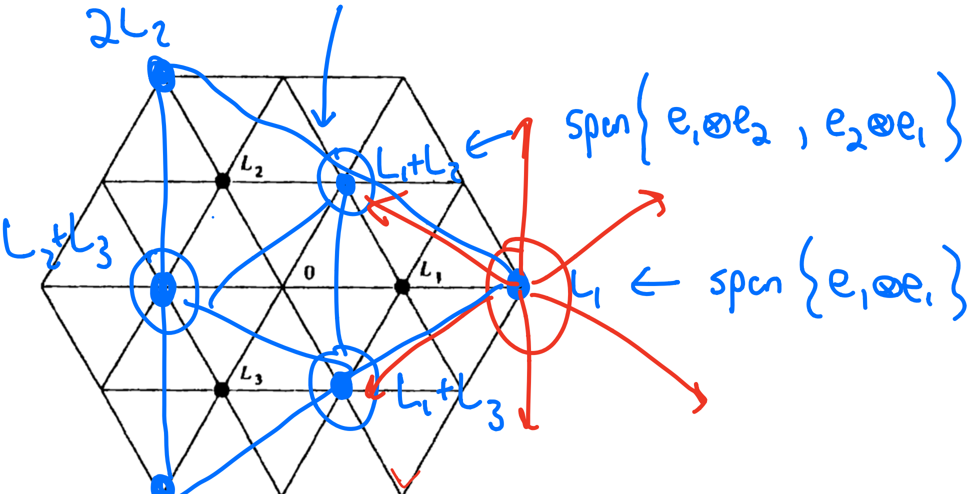
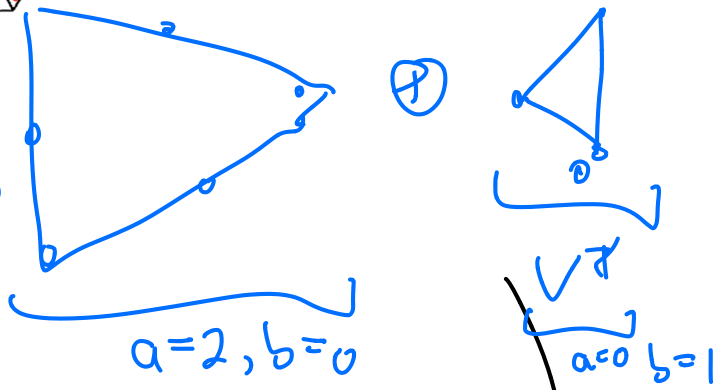
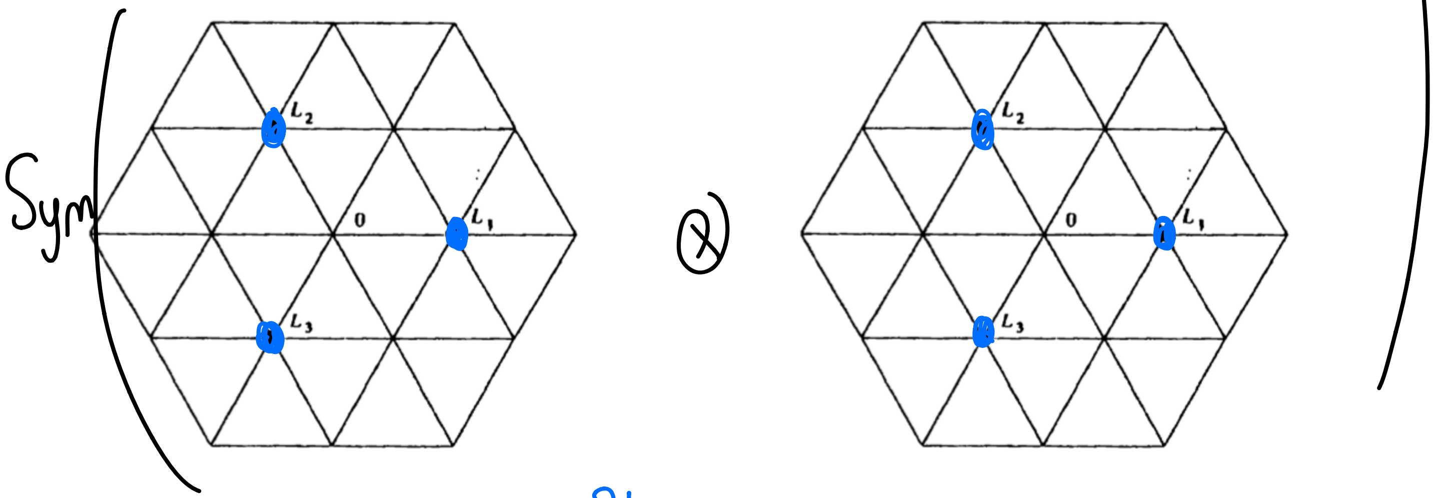




$$V_{(1,0)} \otimes V_{(1,0)} \cong V_{(2,0)} \oplus V_{(0,1)}$$

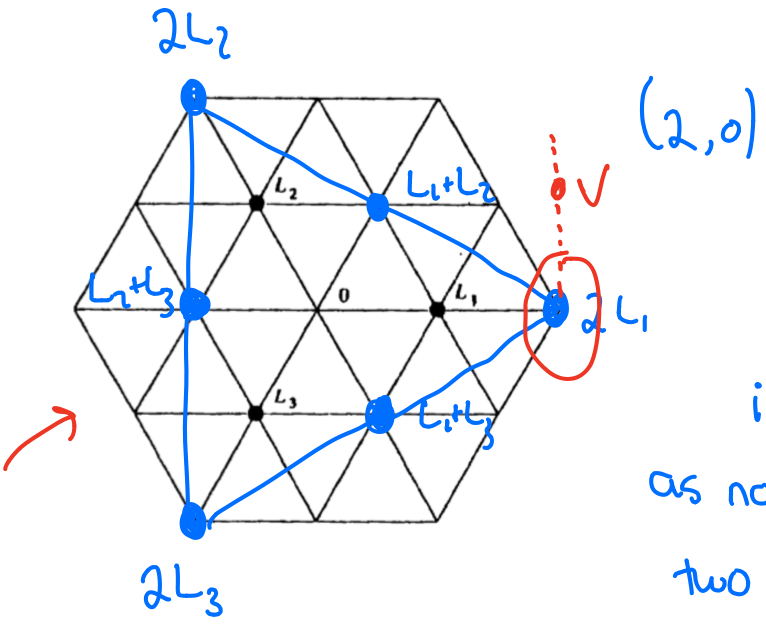
4. $\mathrm{Sym}^2 V$



=

W : Lowering(v)

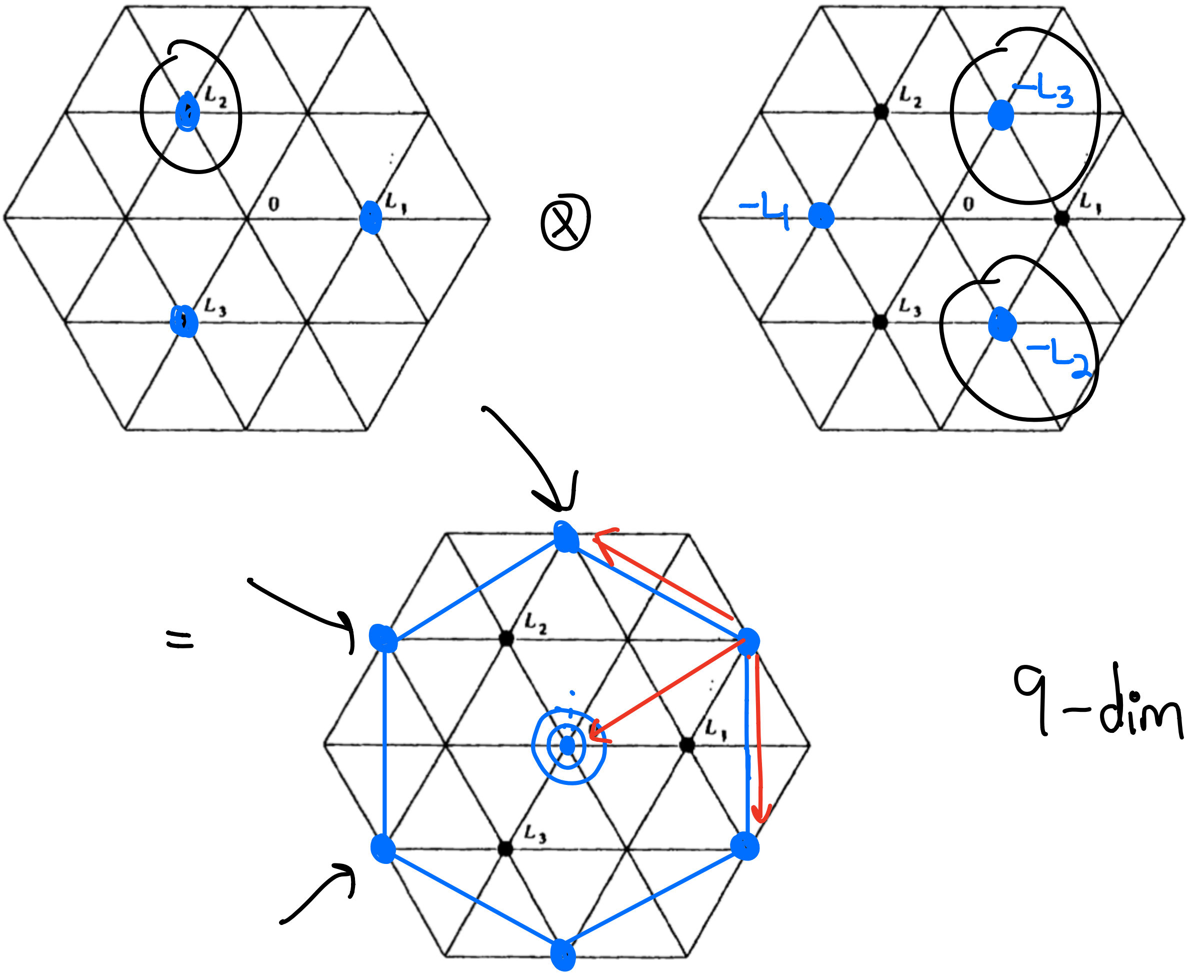
irreducible sub-rep. of



6-dim

irreducible as not union of two hexagons

5. $V \otimes V^*$



9-dim

Now, $V^* \otimes V$ is not irreducible, as there exists a nonzero equivariant map

$\ker(f) \subseteq W \overset{\text{equivariant}}{\xrightarrow{f}} \rho$

$$V^* \otimes V \xrightarrow{ev} \underbrace{\mathbb{C}}_{\text{trivial rep}}$$

$$f \otimes v \longmapsto f(v)$$

Exercise Check that this map is $\mathfrak{g}$-equivariant.

So, the kernel of this map is a non-trivial subrepresentation of $V^* \otimes V$.

Exercise For any f-dim vector space V, we have a canonical isomorphism

$$A : \underbrace{V^* \otimes V}_{\dim(V)^2} \xrightarrow{\cong} \underbrace{\mathrm{End}(V)}_{\dim(V)^2}$$

$$f \otimes v \longmapsto w \longmapsto f(w)v$$

Show that under this identification,

$$W := A(\mathrm{Ker}(f)) = \text{Traceless } 3\times 3 \text{ matrices} \subseteq \mathrm{End}(\overset{\overset{=\mathbb{C}^3}{\downarrow}}{V})$$

and that the resultant action of $\mathfrak{g}$ on W is just the adjoint representation of $\mathfrak{g}$.

In physics, SU(3) "gauge theory" G ⇝ U(1) QED
⇝ SU(2) QCD
⇝ SU(3)

$$\underbrace{V}_{\text{quarks}} \otimes \underbrace{V^*}_{\text{antiquarks}} = \underbrace{\text{adjoint}}_{8 \text{ dim}} \oplus \underbrace{\text{trivial}}_{1\text{-dim}}$$

"eight-fold way".

We can summarize our investigations of irreps of $\mathfrak{sl}(3,\mathbb{C})$ by:

Theorem For any pair of natural numbers $a \geq 0, b \geq 0$ there exists a unique$^{*\text{up to iso}}$ finite-dimensional irreducible representation $\Gamma_{a,b}$ of $\mathfrak{sl}(3,\mathbb{C})$ with highest weight $aL_1 - bL_3$. All irreps are of this form. i.e. V irrep of $\mathfrak{sl}(3,\mathbb{C}) \Rightarrow V \cong \Gamma_{a,b}$

Proof Existence Since $\mathrm{Sym}^a \underbrace{V}$ is generated by a highest weight vector v of weight aL_1, and $\mathrm{Sym}^b V^*$ is generated by a highest weight vector w of weight $-bL_3$,

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$$

will have a highest weight vector $v \otimes w$ of weight $aL_1 - bL_3$. The irrep generated by applying lowering operators to $v \otimes w$ is then $\Gamma_{a,b}$.

Uniqueness Let V and W be two irreducible reps of $\mathfrak{g}$ with the same highest weight α, and let $v \in V$ and $w \in W$ be the corresponding highest weight vectors. Then

$$V \oplus W \qquad X \cdot (v,w) := (Xv, Xw)$$

is a rep of $\mathfrak{g}$ with highest weight vector (v,w) [of weight α]; let

$$U \subseteq V \oplus W$$

be the associated irrep of $\mathfrak{g}$ obtained by applying lowering operators to (v,w). Then the projection maps

$$\pi_V : U \xrightarrow{\cong} V \quad , \quad \pi_W : U \xrightarrow{\cong} W$$

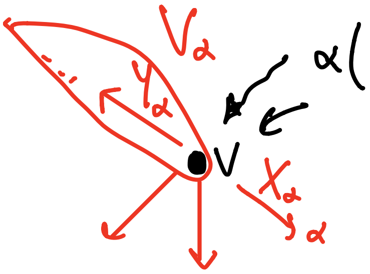
are nonzero maps between irreducible representations, and hence must be isomorphisms (by Schur's Lemma). So $U \cong V$ and $U \cong W$. Hence $V \cong W$.

<u>All irreps are of this form</u> Given any rep V, we know there exists a highest weight vector $v \in V$. This highest weight γ must lie in the <u>weight lattice</u>

$$\Lambda_W = \text{lattice in } \mathfrak{h}^* \text{ generated by the } L_i$$
$$= \{ aL_1 - bL_3 \ , \ a,b \in \mathbb{N}_{\geq 0} \}$$

Since for each positive root α, we have the irrep V_α of $\mathfrak{sl}(2,\mathbb{C})$ sitting inside V,

$$V_\alpha := Y_\alpha^k (v)$$



nonnegative

and the eigenvalues of H_α on V_α must be integers, i.e.

$\mathfrak{sl}(2,\mathbb{C})$

$: \{X_\alpha, Y_\alpha, \underset{\uparrow}{H_\alpha}\}$

$$H_\alpha(v) = \underbrace{\gamma(H_\alpha)}_{\in\mathbb{Z}} v$$

α positive root

$\gamma(H_1) = a$

$\gamma(H_3) = -b$ $\therefore \gamma = \begin{pmatrix} aL_1 \\ -bL_3 \end{pmatrix}$

In our case, by calculation,

$\alpha = \alpha_{12}$ $$H_\alpha := [\underbrace{X_\alpha}_{E_{12}}, \underbrace{Y_\alpha}_{E_{21}}] = \gamma\left(\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}\right) \in \mathfrak{h}$$

$\alpha = \alpha_{13}$ $$H_\alpha = \ldots = \gamma\left(\begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix}\right) \in \mathfrak{h}$$

$\alpha = \alpha_{23}$ $$H_\alpha = \ldots = \gamma\left(\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}\right) \in \mathfrak{h}$$

So, $$\gamma(H_{12}),\ \gamma(H_{13}),\ \gamma(H_{23}) \in \mathbb{N}_{\geq 0}$$

i.e. $$\underbrace{\gamma = aL_1 - bL_3.}$$

So, for any irrep V, the highest weight γ of V must lie in the weight lattice. So if $v \in V_\gamma$, then the irrep obtained by applying lowering operators to v will be a copy of $\Gamma_{a,b}$.

$\square$

Lie Algebras Lecture 7

It is time to generalize our study of irreps of $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(3,\mathbb{C})$ to arbitrary <u>semisimple Lie algebras</u>.

However, I'm going to take a <u>geometric</u> approach (same as Hall). That means, I perceive a <u>compact Lie group</u> G as the fundamental object, and I am studying irreps of its Lie algebra $\mathfrak{g}$ and its complexification $\mathfrak{g}_{\mathbb{C}}$ not for its own sake but because I'm interested in G.

This simplifies things a lot, as it enables us to avoid a lot of algebra, which is good, since I don't really like that.

The thing that a <u>compact Lie group</u> G immediately gives us (which a purely algebraic approach must work very hard to do) is an <u>inner product</u> on representations of G and of $\mathfrak{g}$.

... (Explain Haar integral from Haar integral notes)...

<u>Lemma</u> Let G be a compact Lie group. Then any finite-dim rep (V,Π) of G admits an inner product making Π a <u>unitary</u> representation.

<u>Proof</u> Let $\langle\cdot,\cdot\rangle_0$ be <u>any</u> inner product on V. Then we can use the normalized Haar measure ω on G to "average it over G" in order to make a <u>new</u> inner product $\langle\cdot,\cdot\rangle$ with respect to which π is unitary.

$$\langle v, w\rangle := \int_{a\in G} \langle \pi(a)v, \pi(a)w\rangle_0 \quad \omega_a$$

This is clearly an inner product. Let's check if its unitary:

$$\langle \pi(g)v, \pi(g)w\rangle = \int_{a\in G} \langle \pi(a)\pi(g)v, \pi(a)\pi(g)w\rangle_0 \, \omega_a$$

$b = ag$

$$= \int_G \langle \pi(ag)v, \pi(ag)w\rangle_0 \quad \omega_a$$

$$= \int_G f\,\omega$$

where $f(a) = \langle \pi(ag)v, \pi(ag)w\rangle_0$.

$$= \int_G R_{g^{-1}}^{*}(f\omega) \quad \left[\begin{array}{l}\text{invariance of integral}\\ \text{under}\\ \text{diffeomorphism } R_{g^{-1}}: G \to G\end{array}\right]$$

$$= \int_G L^*_{g^{-1}}(f)\ \underbrace{L^*_{g^{-1}}(\omega)}_{=\,\omega} \quad \text{as } \omega \text{ is } G\text{-invariant}$$

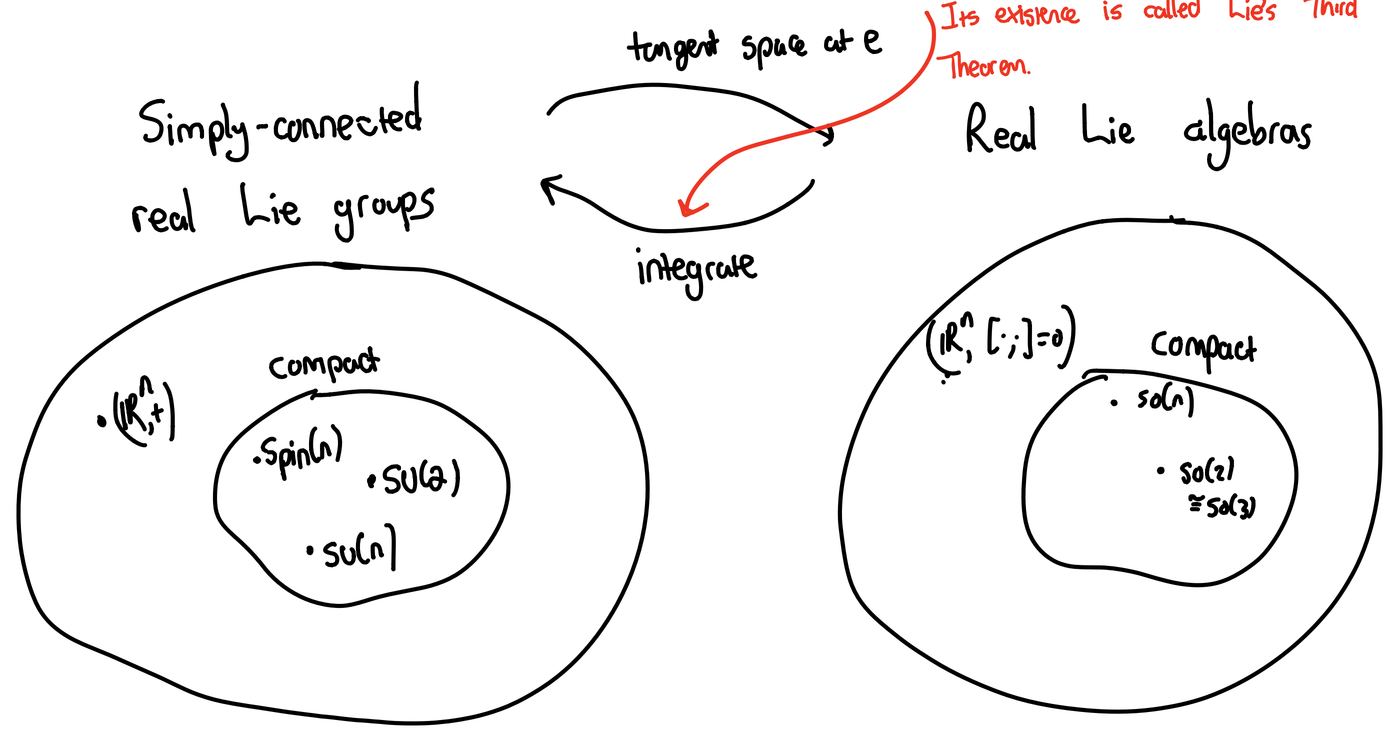
$$= \int_{a\in G} \langle \pi(a)v, \pi(a)w\rangle_0 \ \omega_a$$

$$= \langle v, w\rangle.$$

□

Let's call a real Lie algebra which is isomorphic to the Lie algebra of a compact Lie group, a compact Lie algebra.

Bigger picture: we have an equivalence of categories (Cartan's theorem)



didn't do this functor last year! Its existence is called Lie's Third Theorem.

Corollary The Lie algebra $\mathfrak{g}$ of a compact Lie group G admits a $\mathfrak{g}$-invariant inner product, i.e.

$$\langle \mathrm{ad}_X Y, Z \rangle = - \langle Y, \mathrm{ad}_X Z \rangle.$$

Proof From the above theorem, there exists an inner product on $\mathfrak{g}$ making the Adjoint rep of G unitary:

$$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$$
$$\mathrm{Ad}(g)(X) = gXg^{-1}$$

(at least, for a matrix Lie group)

Now differentiate.

$\square$

Corollary Every f-dim representation (V, π) of a compact Lie algebra $\mathfrak{g}$ admits a $\mathfrak{g}$-invariant inner product (ie. the operators $\pi(X)$ are skew self-adjoint)

$$\langle \pi(X) v, w \rangle = - \langle v, \pi(X) w \rangle$$

Proof $\mathfrak{g}$ integrates to a compact Lie group G, and V becomes a representation of G. So it admits a G-invariant inner product. Then we differentiate.

$\square$

Exercise Work out explicitly the inner product on su(2) using this technology. i.e. choose an arbitrary inner product

$$X = i\sigma_x \, , Y = i\sigma_y \quad , Z = i\sigma_z$$

and then average it out! And compare to Killing form,

$$K(A,B) = Tr(ad_A \, ad_B)$$

and

$$(A,B) = Tr(A^\dagger B)$$

Haar measure on compact Lie groups

"short version"

The "longer version" of these notes is also available.

Every compact Lie group G admits a unique normalized Haar measure, i.e. a measure μ on G such that

- (G-invariant) $\mu(gU) = \mu(U)$ for every Borel set $U \subseteq G$, $g \in G$
- (normalized) $\mu(G) = 1$

That's the "measure theory" definition of Haar measure, but its not constructive and not very geometric.

Here is a geometric description, which uses the language of <u>manifolds</u> and <u>differential forms</u>.

A <u>volume form</u> ω on a manifold M is a section of the <u>top exterior power</u> of the <u>cotangent bundle</u>:

$$\begin{tikzcd} \text{total space} \arrow[d, shift right] & \Lambda^n(T^*M) \arrow[d, shift right, "\pi"'] \\ \text{base space} \arrow[u, shift right] & M \arrow[u, shift right, "\omega"'] \end{tikzcd}$$

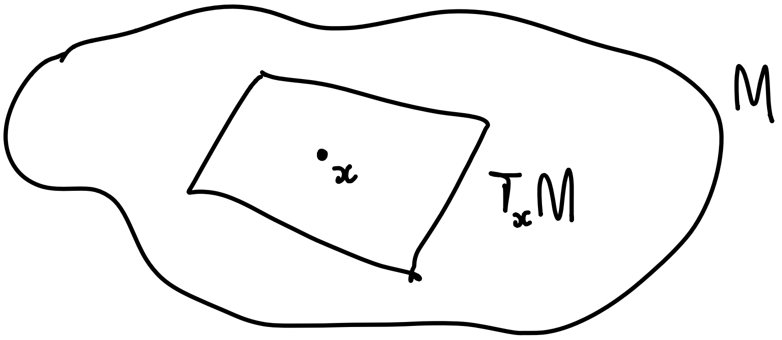
if $\dim M = n$

$$\boxed{\pi \circ \omega = id_M}$$

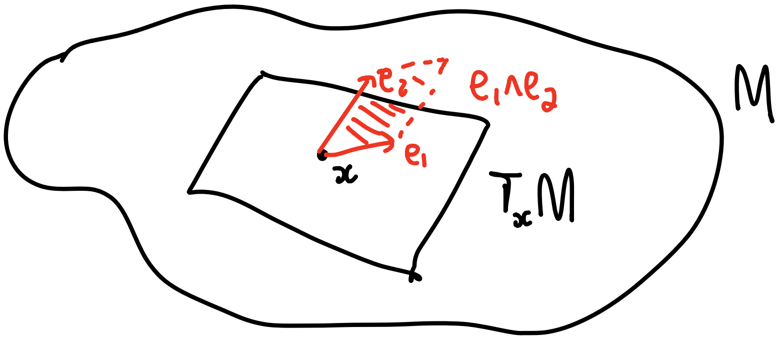
$$\boxed{\text{i.e. for each } x \in M, \ \omega_x \in \Lambda^n(T_x^* M)}$$

That is, for each $x \in M$, we have

$$\omega_x \in \Lambda^n(T_x^* M) \cong (\Lambda^n T_x M)^*$$



That is, since $\Lambda^n T_x M$ is the space of volume elements in the tangent space at x,



$$\omega_x : \Lambda^n T_x M \longrightarrow \mathbb{R}$$

for each x,

is thus, a linear functional on the volume elements at $x \in M$.

(and it depends smoothly on x)

This is just to motivate why it makes sense to integrate a volume form ω over an oriented manifold M:

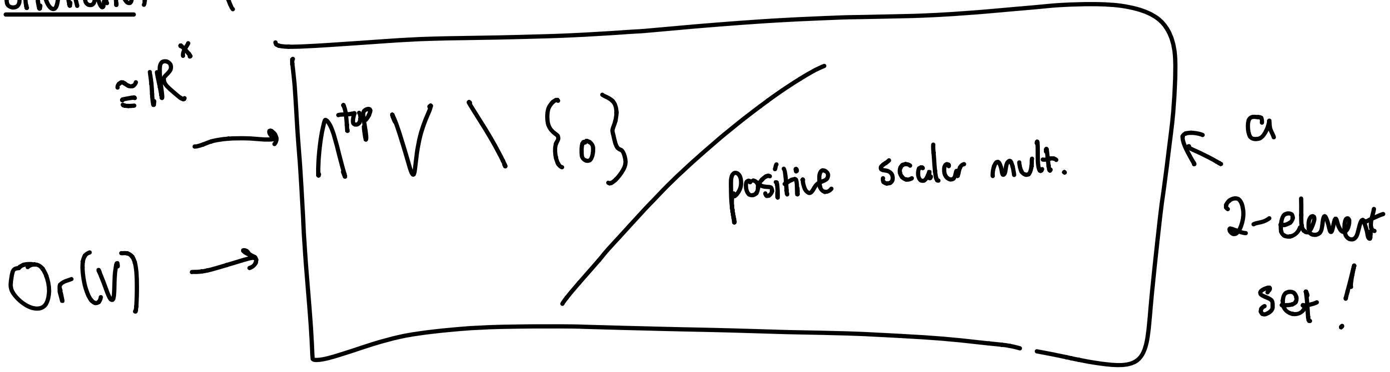
$$\omega \longmapsto \int_M \omega$$

So, a volume form ω is a smooth way of constructing a measure on M:

$$\mu(U) := \int_U \omega. \qquad U \overset{\text{open}}{\subseteq} G$$

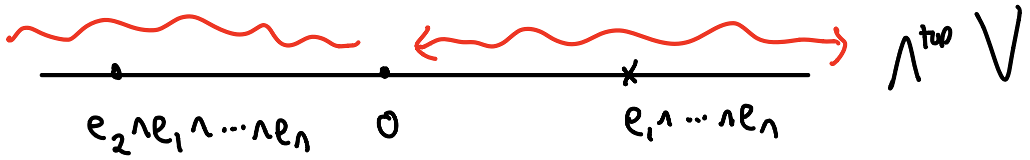
How do we define the integral of an n-form on an n-dim oriented manifold M?

An <u>orientation</u> of a vector space V is an element of



If V has basis $e_1, \dots, e_n$

Basis for $\Lambda^{top} V$ is $e_1 \wedge \dots \wedge e_n = - e_2 \wedge e_1 \wedge e_3 \wedge \dots \wedge e_n$



So, any manifold M has an <u>orientation bundle</u>

$$\begin{tikzcd} Or(M) \arrow[d, shift right, "\pi"'] \\ M \arrow[u, shift right, bend right, "s"'] \end{tikzcd}$$

An <u>orientation</u> of M is a section of the orientation bundle.
An <u>oriented manifold</u> is one equipped with an orientation.

To integrate an n-form ω over an n-dimensional oriented compact manifold M, we choose a finite cover

$$\Big(U_i \subseteq M, \quad \phi_i : \underbrace{V_i}_{\subseteq \mathbb{R}^n} \xrightarrow{\cong} U_i \Big)$$

of M by oriented coordinate charts, and a partition of unity

- $\lambda_i : M \xrightarrow{\text{smooth}} [0,1] \quad , i \in I$

- At each $x \in M$,

  $$\sum_i \lambda_i(x) = 1$$

- support of $\lambda_i \subseteq U_i$

Then we set:

$$\int_M \omega := \sum_i \underbrace{\int_{V_i} \lambda_i(x) \, \phi_i^* \omega \; dx_1 \cdots dx_n}_{\substack{\text{ordinary Riemann integral} \\ \text{over } V_i \text{. The orientation} \\ \text{on } M \text{ is necessary to make this} \\ \text{well-defined.}}}$$

So, in geometric terms, we are interested in constructing a canonical volume form ω on a compact Lie group. Also, as far as G-invariance goes,

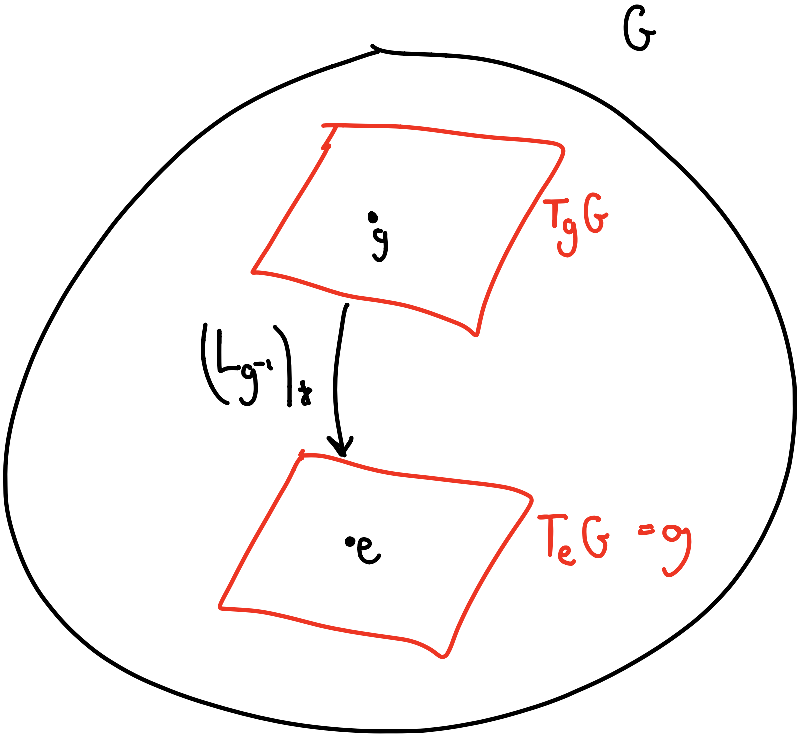
$$\mu \text{ is } G\text{-invariant} \iff \omega \text{ is a } G\text{-invariant } n\text{-form,}$$

$$\left(\text{i.e. } \mu(gK) = \mu(K) \ \forall g \in G,\ K \overset{\text{compact}}{\subseteq} G\right) \qquad \left(\text{i.e. } L_g^* \omega = \omega \ \ \forall g \in G\right)$$

That's easy: choose an arbitrary nonzero element

$$\omega_e \in \underbrace{\Lambda^n T_e^* G}_{1\text{-dim}}$$

and then translate it to each tangent space:



$$L_g : G \longrightarrow G$$
$$a \longmapsto ga$$

$$(L_{g^{-1}})_* : T_gG \longrightarrow T_eG$$

$$\Lambda^n (L_{g^{-1}})_* : \Lambda^n T_gG \longrightarrow \Lambda^n T_eG$$

Pulling back forms :

$$\begin{tikzcd} M \arrow[r, bend left, "f"] & N \end{tikzcd}$$

smooth map

$$\underbrace{\Omega^k(M)}_{k\text{-forms on } M} := \text{smooth sections of}$$

$$\begin{tikzcd} \Lambda^k T^* M \arrow[d, shift right, "\pi"'] \\ M \arrow[u, shift right, "\omega"'] \end{tikzcd}$$

For each $x \in M$,

$$\omega_x \in \Lambda^k T^*_x M \hat{=} \left(\Lambda^k T_x M\right)^*$$

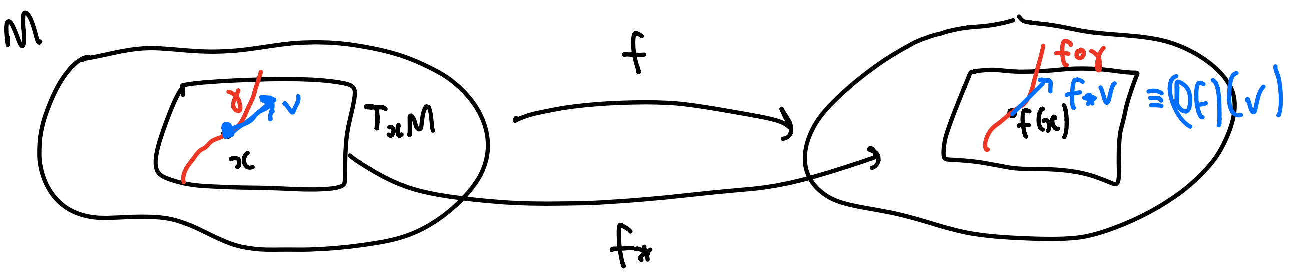
$$\text{i.e.} \quad \omega_x : \Lambda^k T_x M \longrightarrow \mathbb{R}$$

$$v_1 \wedge \cdots \wedge v_k \longmapsto \omega_x(v_1 \wedge \cdots \wedge v_k)$$

$$\begin{tikzcd} M \arrow[r, bend left, "f"] & N \end{tikzcd}$$

$$f_* \equiv Df$$

$$(f_*)_x : T_x M \longrightarrow T_{f(x)} N$$



$$\omega \in \Omega^k(N)$$

$$f^*\omega \in \Omega^k(M)$$

$x \in M$

$$\begin{tikzcd} (f^*\omega)_x : \Lambda^k T_x M \arrow[rr] \arrow[dr, "\Lambda^k f_*"'] & & \mathbb{R} \\ & \Lambda^k T_{f(x)} N \arrow[ur, "\omega_{f(x)}"'] & \end{tikzcd}$$

$$A : V \longrightarrow W$$

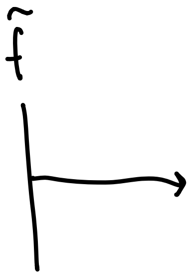
$$\Lambda^k A : \Lambda^k V \longrightarrow \Lambda^k W$$

$$v_1 \wedge \cdots \wedge v_k \longmapsto Av_1 \wedge \cdots \wedge Av_k$$

$f : S^1 \longrightarrow S^1$ $\qquad \tilde{f}(2\pi) - \tilde{f}(0) \in \mathbb{Z}.$

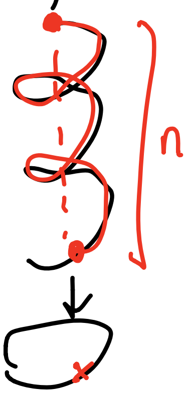
$\parallel$

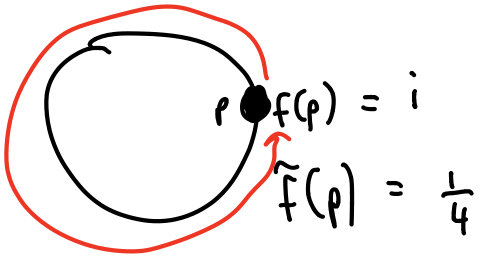
$\operatorname{wind}(f) = \int_0^{2\pi} \tilde{f}'(t)\,dt$ (and why is it an integer?)



$$\begin{tikzcd} t \arrow[d, mapsto] & \mathbb{R} \arrow[d, "\pi"] & \\ e^{2\pi i t} & S^1 \arrow[r, "f"'] & S^1 \arrow[ul, dashed, "\tilde{f}"'] \end{tikzcd}$$

$\tilde{f} \leftarrow$ choose a lift $\hat{f}$, ie. a branch of the logarithm.

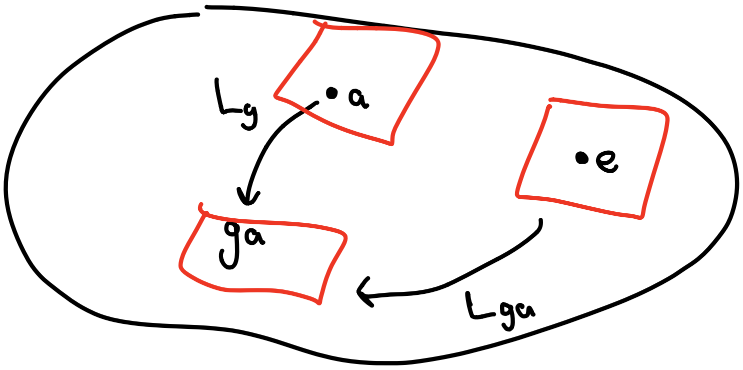




$$\begin{tikzcd}
\omega_g : & \Lambda^n T_g G \arrow[rr] \arrow[dr, "\Lambda^n(L_{g^{-1}})_*"'] & & \mathbb{R} \\
& & \Lambda^n T_e G \arrow[ur, "\omega_0"'] &
\end{tikzcd}$$

Since it is <u>defined</u> by taking something at the identity and translating it over all of G, it will be G-invariant <u>by defn</u>:

$$\begin{aligned}
(L_g^* \omega)_a &= \omega_{ga} \circ \Lambda^n(L_g)_* \\
&= \omega_e \circ \Lambda^n(L_{(ga)^{-1}})_* \circ \Lambda^n(L_g)_* \\
&= \omega_e \circ \Lambda^n(L_{a^{-1}g^{-1}} \circ L_g)_* \\
&= \omega_e \circ \Lambda^n(L_{a^{-1}})_* \\
&= \omega_a .
\end{aligned}$$



$$\begin{tikzcd}
\Lambda^n T_a G \arrow[rr] \arrow[dr, "\Lambda^n(L_g)_*"'] & & \mathbb{R} \\
& \Lambda^n T_{ga} G \arrow[ur, "\omega_{ga}"'] &
\end{tikzcd}$$

By multiplying our initial choice of $\omega_e \in \Lambda^n T_e^* G$ (an element of a 1-dim space) by an appropriate scalar, we can ensure that

$$\int_G \omega = 1$$

Moral There is a canonical way to construct a G-invariant volume form ω on a compact Lie group, normalized s.t. $\int_G \omega = 1$.

Ok... can we make this more explicit?

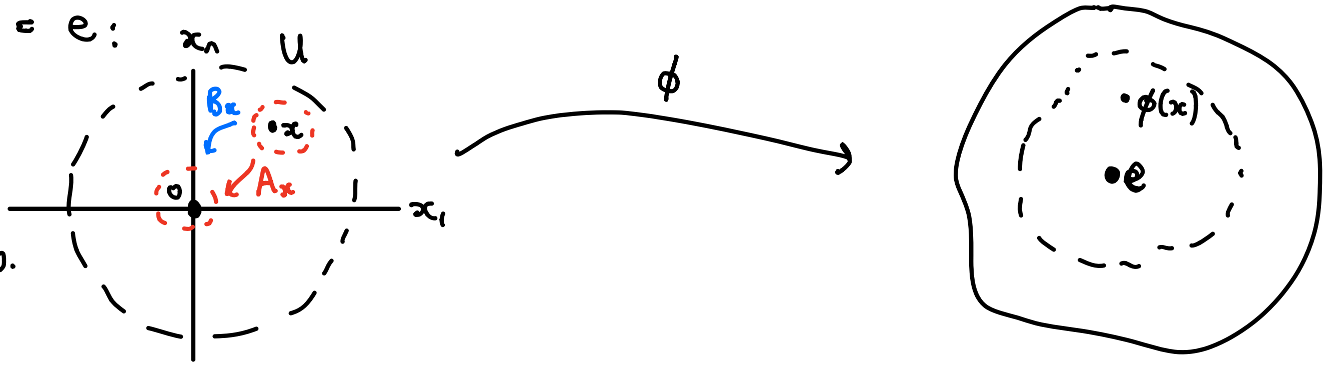
In the accompanying "Haar measure on Lie groups" notes which go into this in a bit more depth, you will find:

1. How ω looks like in an arbitrary coordinate chart for G

Let $(U \subseteq \mathbb{R}^n, \phi : U \longrightarrow G)$ be a chart for G, such that

$\phi(0) = e$:

not actually necessary.



Then, pulled back to the local coordinates $x_1, \cdots, x_n$ on U,

$$\boxed{\phi^*(\omega)_x = \det\left(DA_x\right) dx_1 \wedge \cdots \wedge dx_n}$$

where

$$A_x = \phi^{-1} \circ L_{\phi(x)^{-1}} \circ \phi$$

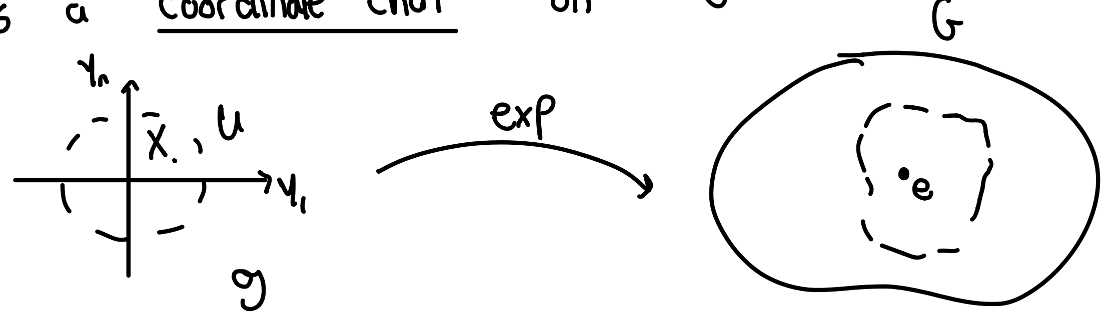
is the smooth map of $\mathbb{R}^n$, defined in a neighborhood of $x \in U$, which sends $x \longmapsto 0$, and DA_x is its derivative, at x.

2. How ω looks like in the exponential coordinate chart

Recall that the exponential map is a local diffeomorphism, i.e. there exists an open ball $U \subseteq \underbrace{T_e G}_{\mathfrak{g}}$ such that restricting

$$\exp : \mathfrak{g} \longrightarrow G$$

to U is a diffeomorphism onto its image. Now, T_eG is just a vecter space (isomorphic to $\mathbb{R}^n$), so we can regard exp as a coordinate chart on G



If we pick a basis $Y_1, \cdots, Y_n$ for $\mathfrak{g}$, then

$$\boxed{\left[\exp^*(\omega)\right]_X = \det(\Phi)\, dY_1 \wedge \cdots \wedge dY_n}$$

where

$$\begin{aligned} \Phi_X : \mathrm{Lie}(G) &\longrightarrow \mathrm{Lie}(G) \\ Y &\longmapsto \left(\frac{I - e^{-ad_X}}{ad_X}\right)(Y) \end{aligned}$$

is the "derivative of the exponential" which you learnt about last year when proving the Baker-Campbell-Hausdorff formula!

Here, ad_X is the linear map

$$\begin{aligned} ad_X : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ Y &\longmapsto [X, Y] \end{aligned}$$

and

$$\frac{I - e^{-ad_X}}{ad_X}$$

is just a shorthand for the power series of

$$g(z) = \frac{1-e^{-z}}{z} = \frac{1}{z}\left(1 - \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right)\right)$$

$$= 1 - \frac{z}{2!} + \frac{z^2}{3!} - \frac{z^3}{4!} + \cdots$$

applied to ad_X :

$$\frac{I - e^{-ad_X}}{ad_X} := I - \frac{ad_X}{2!} + \frac{ad_X^2}{3!} - \frac{ad_X^3}{4!} + \cdots$$

3. How ω looks like for $SU(2)$

It turns out that for $SU(2)$, which is S^3 as a manifold, ω is just $\frac{1}{2\pi^2}$ × canonical volume form on S^3 arising from the fact that S^3 is a submanifold of $\mathbb{R}^4$.

In terms of "spherical coordinates" on S^3,

$\theta \in [0,\pi]$ $\qquad$ $\cos^2\theta + \sin^2\theta \underbrace{\|\hat{n}\|^2}_{=1} = 1$

$$\phi : (\theta, \underbrace{\hat{n}}_{\in S^2}) \longmapsto (\underbrace{\cos\theta}_{\in\mathbb{R}}, \underbrace{\sin\theta\,\hat{n}}_{\in\mathbb{R}^3}) \qquad \hat{n}(\alpha,\beta) = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$$

i.e. $$(\theta, \alpha, \beta) \longmapsto (\cos\theta, \sin\theta\sin\alpha\cos\beta, \sin\theta\sin\alpha\sin\beta, \sin\theta\cos\alpha)$$

we have:

$$\boxed{\omega = \frac{1}{2\pi^2} \sin^2\theta \sin\alpha \; d\theta \wedge d\alpha \wedge d\beta}$$

← the volume of S^3

$$\int_{\theta=0}^{\pi}\int_{\alpha=0}^{\pi}\int_{\beta=0}^{2\pi} \sin^2\theta \sin\alpha \; d\theta \, d\alpha \, d\beta = 2\pi^2$$

$$\therefore \; \mathrm{vol}(S^3) = 2\pi^2$$

Lie Algebras Lecture 8

Every (real or complex) f-dim representation (V,π) of a compact Lie algebra k admits a k-invariant inner product (ie. the operators $\pi(X)$ are skew self-adjoint)

$$\langle \pi(X)v,\ w\rangle = -\langle v,\ \pi(X)w\rangle \quad , X\in k$$

<u>Corollary</u> If $W\leq V$ is a sub-representation of k, then $W^\perp$ is also a sub-representation, so

$$V = W \oplus W^\perp$$

as representations of k.

<u>Proof</u> Let $v\in W^\perp$. We must show $\overset{X\in k}{X(v)}\in W^\perp$. Indeed, for all $w\in W$,

$$\langle w,\ X(v)\rangle = \langle X^*(w),\ v\rangle \qquad \text{[defn of } X^*\text{]}$$

$$= -\underbrace{\langle X(w),\ v\rangle}_{=0} .$$

□

<u>Corollary</u> Every f-dim representation of a compact Lie algebra splits as a direct sum of irreducible representations.

<u>Proof</u> Induction on dimension.

□

Note: this isn't true for non-compact Lie algebras.

eg. $\left(\mathbb{R}, [\cdot,\cdot] = 0 \right)$ is not a compact Lie algebra (= Lie algebra of $(\mathbb{R}, +)$).

Have rep

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathrm{End}(\mathbb{C}^2) \\ t &\longmapsto \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Clearly $\mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a sub-representation, but it doesn't admit a complementary sub-representation. This is simply the fact that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not diagonalizable.

<u>Definition</u> A Lie algebra $\mathfrak{g}$ is called <u>simple</u> if it contains no nontrivial ideals and $\underbrace{\text{if } \dim \mathfrak{g} \geq 2}$; It is called <u>semisimple</u> if it is a direct sum of simple Lie algebras. i.e. is not abelian

<u>Theorem</u> A complex Lie algebra $\mathfrak{g}$ is semisimple $\Leftrightarrow$ $\mathfrak{g} \cong \mathfrak{k}_{\mathbb{C}}$ where K is a compact Liegroup, and $\underbrace{\text{the center of } \mathfrak{k} \text{ is trivial.}}_{\text{i.e. center}(K) \text{ is discrete group}}$

Alternatively:

$$\underbrace{\text{A complex Lie algebra } \mathfrak{g} \text{ is simple}}_{\text{i.e. } \mathfrak{g} \text{ is non-abelian and has no non-trivial ideals}} \iff \underbrace{\mathfrak{g} = \mathfrak{k}_{\mathbb{C}} \text{ where } K \text{ is a compact simple Lie group}}_{\text{i.e. } K \text{ is compact and has no nontrivial connected normal subgroups}}$$

Proof of theorem ($\Leftarrow$) If $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$, then we can decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$$

where each $\mathfrak{g}_i$ contains no nontrivial ideals. We must just show that

$$\underbrace{\text{center}(\mathfrak{g})}_{= \bigoplus_i \text{center}(\mathfrak{g}_i)} = 0 \implies \dim \mathfrak{g}_i \geqslant 2.$$

Indeed, if $\text{center}(\mathfrak{g}_i) = 0$, then clearly $\dim \mathfrak{g}_i \geqslant 2$, because the only 1-dim Lie algebra is $\mathbb{C}$, which is abelian so its center is $\mathbb{C}$.

$\implies$ Omitted.

$\square$

Examples

Amongst the matrix Lie groups from last year: (and one from this year)

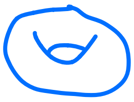
		center	Lie algebra k	Complexified Lie algebra	ADE classification
simple ✓	$SU(n)$, $n \geq 2$	$\pm I$	$su(n)$	$s\ell(n, \mathbb{C})$	A_{n-1}
	$SO(2n+1)$, $n \geq 1$	I	$so(2n+1)$	$so(2n+1, \mathbb{C})$	B_n
	$SO(2n)$, $n \geq 2$	I	$so(2n)$	$so(2n, \mathbb{C})$	D_n
	$Sp(2n)$, $n \geq 1$	$\pm I$	$sp(n)$	$sp(n, \mathbb{C})$	C_n
×	$U(n)$	$U(1)$	$\mathbb{R}$		
	$SO(2) = U(1)$	$U(1)$	$\mathbb{R}$		
not matrix Lie group →	$Spin(n)$, $n \geq 3$	$\pm e$	$so(n)$	$so(n, \mathbb{C})$	B_n or D_n (odd or even)

Classification of finite-dim representations of semisimple Lie algebras

Let g be a semisimple complex Lie algebra. Write $g = k_{\mathbb{C}}$.

Fix a <u>maximal torus</u> $t \subseteq k$, i.e. a commutative subalgebra which is <u>maximal</u> (i.e. there doesn't exist a commutative subalgebra of k which strictly contains t). torus $= U(1)^n = (S^1)^n$  $(S^1)^2$

Lie algebra $= \boxed{\mathbb{R}^n}$



Clearly such a maximal torus <u>exists</u>. We write $h = t_{\mathbb{C}}$.

Now let V be any representation of g. Equip V with the K-invariant inner product. So,

$$X(\cdot) : V \longrightarrow V$$

is skew self-adjoint for all $X \in k$.

Hence it is dia onalizable! With purely imaginary eigenvalues! Hence:

V decomposes as an orthogonal direct sum of its weight spaces.

$h = t_{\mathbb{C}}$

$\gamma \in h^* = (t_{\mathbb{C}})^*$

$t^* \subset (t_{\mathbb{C}})^*$

$$\bigoplus_{\gamma \in \underbrace{N(V)}_{\text{weights of } V} \subseteq h^*} V_\gamma \qquad ①$$

$H \in t$

$\gamma(H)$ is purely imag.

Moreover, each weight $\gamma \in i t^* \subseteq h^*$.

Recall:

Lemma Let $A : V \longrightarrow V$ be a skew-self adjoint map. Then eigenvalues of A are purely imaginary.

Proof Let $Av = \lambda v$ where v is normalized. So,

$$\begin{aligned} \lambda &= \langle v, Av \rangle \\ &= \langle A^* v, v \rangle \\ &= \langle -Av, v \rangle \\ &= -\overline{\langle v, Av \rangle} \\ &= -\overline{\lambda} \end{aligned}$$

$\therefore$ λ is purely imaginary □

Applying this to the adjoint representation gives:

$$g = \underbrace{t_{\mathbb{C}}}_{h} \oplus \bigoplus_{\alpha \in R} g_\alpha \qquad ②$$

where the set of roots R is purely imaginary, ie.

$$R \subseteq it^* \subseteq h^*.$$

Also, by the same proof as before:

$$\boxed{X_\alpha : V_\gamma \longrightarrow V_{\gamma+\alpha} \qquad \alpha \in R}$$

Lemma $\bigcap_{\alpha \in R} \ker(\alpha) = \{0\}$. (We're regarding $\alpha : t \xrightarrow{\text{real linear}} i\mathbb{R}$)

Proof Suppose $H \in \mathfrak{t}$, and $\alpha(H) = 0$ for all roots α.

Then, for $X \in \mathfrak{g}_\alpha$,

$$[H, X] = \underbrace{\alpha(H)}_{=0} X$$

By the decomposition ② we conclude H commutes with everything in $\mathfrak{g}$. So H is in the center, so it is zero. □

$\alpha : \mathfrak{t} \longrightarrow i\mathbb{R}$ $\alpha : i\mathfrak{t} \longrightarrow \mathbb{R}$

Corollary R spans the real linear space $i\mathfrak{t}^* \subseteq \mathfrak{g}^*$. ③

Proof If this was false, then there would be a nonzero $H \in \mathfrak{t}$ such that $\alpha(H) = 0$ for all $\alpha \in R$. But by the Lemma, this implies $H = 0$, so it's a contradiction. □

Now, we have a complex conjugation map on $\mathfrak{g} = \mathfrak{k}_\mathbb{C}$:

$$\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$$
$$X_1 + iX_2 \longmapsto X_1 - iX_2$$

Lemma τ is a real linear automorphism of $\mathfrak{g}$.

Proof $[\tau(X_1 + iX_2), \tau(Y_1 + iY_2)]$ $X_1, X_2, Y_1, Y_2 \in \mathfrak{k}$

$$= [X_1 - iX_2,\ Y_1 - iY_2]$$

$$= [X_1, Y_1] - [X_2, Y_2] - i\left([X_1, Y_2] + [X_2, Y_1]\right)$$

$$= \tau\left([X_1 + iX_2,\ Y_1 + iY_2]\right).$$

□

Lemma If $\alpha \in R$, then $-\alpha \in R$, and

$$\tau : \mathfrak{g}_\alpha \xrightarrow{\cong} \mathfrak{g}_{-\alpha}$$

④

Proof Suppose $H \in \mathfrak{t}$, and $X \in \mathfrak{g}_\alpha$. Then

$$[H, \tau(X)] = [\tau(H), \tau(X)] \quad (\tau(H) = H)$$

$$= \tau([H, X]) \quad (\tau \text{ is automorphism})$$

$$= \tau(\alpha(H) X)$$

$$= \overline{\alpha(H)}\, \tau(X)$$

$$= -\alpha(H)\, \tau(X).$$

So $\tau(X)$ is an eigenvector of ad_H with eigenvalue $-\alpha(H)$.

□

If V is a rep of $\mathfrak{g}=k_{\mathbb{C}}$ with invariant inner product $\langle \cdot, \cdot \rangle$, then we know

$$X(\cdot) : V \longrightarrow V \qquad X \in k$$

$$\mathfrak{g} = \{X_1 + iX_2 \; ; X_1, X_2 \in k\}$$

is skew self-adjoint:

$$\langle X(v), w \rangle = - \langle v, X(w) \rangle. \qquad v, w \in V. \quad X \in k.$$

But what about $X \in \mathfrak{g} = k_{\mathbb{C}}$?

<u>Lemma</u> For $X \in \mathfrak{g}$,

$$\langle \tau(X)(v), w \rangle = - \langle v, X(w) \rangle$$

<u>Proof</u> Write $X = X_1 + iX_2$, $\quad X_1, X_2 \in k$.

$$\begin{aligned}\langle \tau(X)(v), w \rangle &= \langle (X_1 - iX_2)(v), w \rangle \\ &= \langle X_1(v), w \rangle + i \langle X_2(v), w \rangle \\ &= - \langle v, X_1(w) \rangle - i \langle v, X_2(w) \rangle \\ &= - \langle v, X(w) \rangle.\end{aligned}$$

□

Definition For each root $\alpha \in R$, we define its coroot

$$H_\alpha \in it \subseteq h$$

as the unique element satisfying

$$H_\alpha \perp \ker\alpha \quad , \quad \alpha(H_\alpha) = 2$$

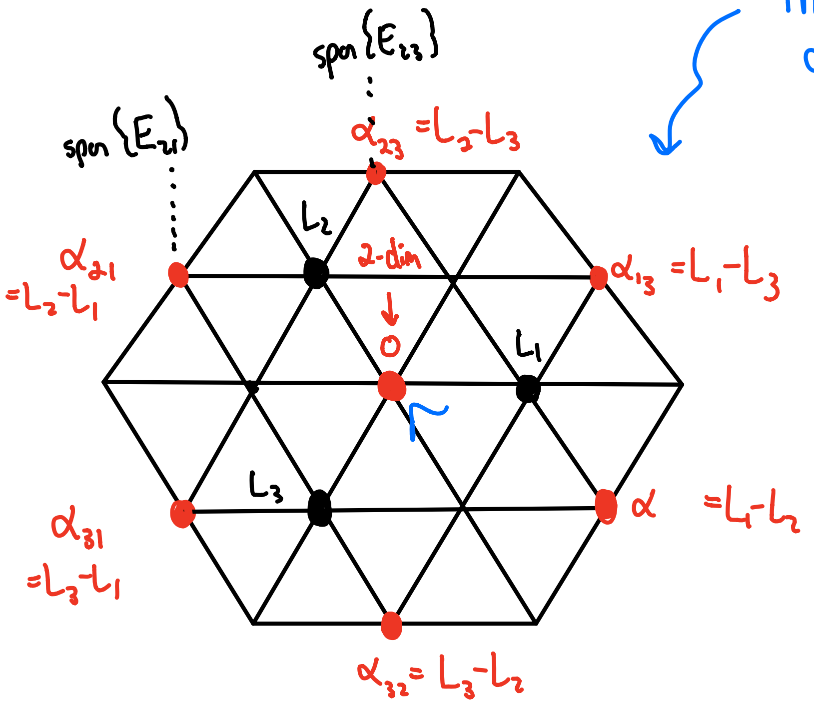
Recall:

it^*

$= (it)^*$

the roots lie in a $\dim(h)$ real subspace of h^*, namely it^*.

And $it^* = (it)^*$.



h

roots of $sl(3,\mathbb{C}) \subseteq h^*$

k

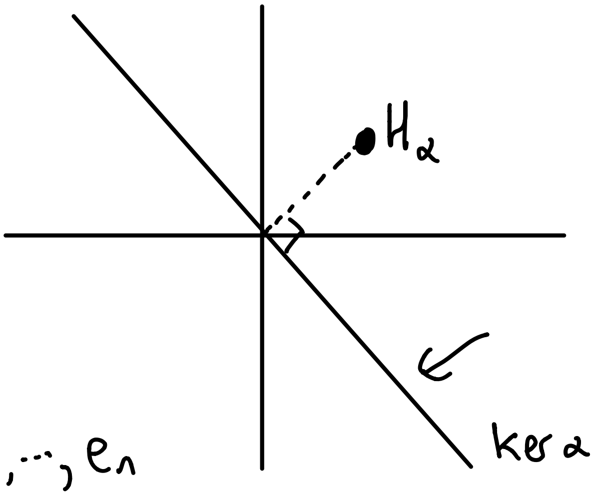
$g := k_{\mathbb{C}}$

$\alpha \in R$:

$\alpha : t \longrightarrow i\mathbb{R}$

or $\alpha : it \longrightarrow \mathbb{R}$

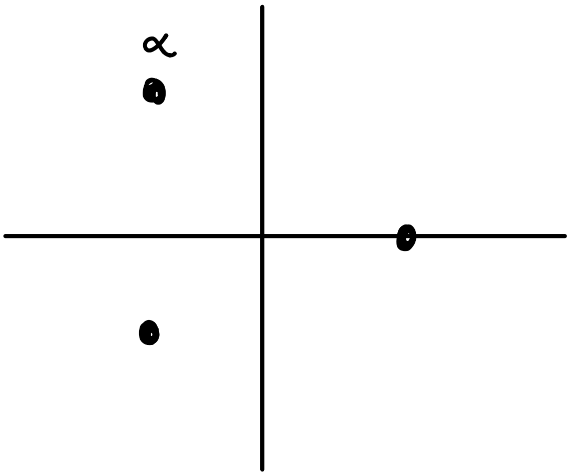
Real spaces:



$e_1, \cdots, e_n$

basis

$\left(it \,,\, \langle \cdot , \cdot \rangle \right)$



it^* = real linear dual of it

$\langle \cdot , \cdot \rangle$

Exercise Since $\alpha \in (it)^*$, it can be written as

$$\alpha = \langle S_\alpha , - \rangle$$

for some vector $S_\alpha \in it$. Determine S_α and compare it to H_α.

Theorem Let $\alpha \in R$. Then there exists $X_\alpha \in \mathfrak{g}_\alpha$ such that

$$X_\alpha, \quad Y_\alpha := -\tau(X_\alpha), \quad H_\alpha$$

$H_\alpha \perp \ker\alpha$, $\alpha(H_\alpha) = 2$

satisfy the $\mathfrak{sl}(2,\mathbb{C})$ commutation relations

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, Y_\alpha] = -2Y_\alpha, \quad [X_\alpha, Y_\alpha] = H_\alpha$$

⑤

To prove, we will need:

Lemma Suppose $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, and $H \in \mathfrak{h}$. Then $[X,Y] \in \mathfrak{h}$ and

$$\boxed{\langle H, [X,Y]\rangle = -\alpha(H)\,\langle \tau(X), Y\rangle}$$

Proof Clearly $[X,Y] \in \mathfrak{h}$. We compute:

$$\begin{aligned}
\langle H, [X,Y]\rangle &= \langle H, \mathrm{ad}_X(Y)\rangle \\
&= -\langle \mathrm{ad}_{\tau(X)}(H), Y\rangle \qquad \text{(earlier Lemma)} \\
&= -\langle [\tau(X), H], Y\rangle \\
&= \langle [H, \tau(X)], Y\rangle \\
&= \langle -\alpha(H)\tau(X), Y\rangle \\
&= -\alpha(H)\,\langle \tau(X, Y)\rangle
\end{aligned}$$

because $\tau : \mathfrak{g}_\alpha \to \mathfrak{g}_{-\alpha}$ so $[H, \tau(X)] = -\alpha(H)\tau(X)$.

□

Proof of theorem: Choose any nonzero $X_\alpha \in \mathfrak{g}_\alpha$. We automatically have:

$$[H_\alpha, X_\alpha] = \alpha(H_\alpha) X_\alpha = 2X_\alpha \quad , \quad [H_\alpha, Y_\alpha] = [H_\alpha, -\tau(X_\alpha)] = -2Y_\alpha$$

Also, we have $[X_\alpha, Y_\alpha] \in \ker\alpha^{\perp}$ as if $H' \in \ker\alpha$,

$$\langle H', [X_\alpha, Y_\alpha] \rangle = \underbrace{\alpha(H')}_{=0} \langle \tau(X_\alpha), Y_\alpha \rangle$$

And,

$$\langle H_\alpha, [X_\alpha, Y_\alpha] \rangle = -\underbrace{\alpha(H_\alpha)}_{=2} \langle \tau(X_\alpha), Y_\alpha \rangle \quad \text{(Lemma)}$$
$$= -2 \langle -Y_\alpha, Y_\alpha \rangle$$
$$= 2 \|Y_\alpha\|^2$$
$$> 0$$

So $[X_\alpha, Y_\alpha]$ differs from H_α by a positive real factor.

If we scale $X_\alpha \longmapsto \lambda X_\alpha$, $\lambda \in \mathbb{C}^*$, then

$$Y_\alpha \longmapsto \bar{\lambda} Y_\alpha$$
$$[X_\alpha, Y_\alpha] \longmapsto \underbrace{|\lambda|^2}_{\text{positive}} [X_\alpha, Y_\alpha]$$

So by choosing λ appropriately we can arrange

$$[X_\alpha, Y_\alpha] = H_\alpha.$$

(Note: there was a $U(1)$'s worth of choices of λ:

$$\underbrace{X_\alpha, Y_\alpha,}_{U(1)\text{-degree of freedom}} \; \underbrace{H_\alpha}_{\text{fixed}}$$

)

□

Lie Algebras Lecture 9

How does this all work for $su(3)$?

$$k = su(3) = \text{Lie algebra of } SU(3)$$

($\therefore$ compact Lie algebra)

$= 3\times3$ antiHermitean traceless matrices

It turns out the invariant inner product can be taken to be:

$$\langle X, Y \rangle = Tr(X^* Y) \qquad X^* = \text{conjugate transpose}$$

Check: need $\langle ad_X(Y), Z \rangle = -\langle Y, ad_X(Z) \rangle$?

$$\Leftrightarrow \langle XY - YX, Z \rangle = -\langle Y, XZ - ZX \rangle$$

$$\Leftrightarrow Tr(Y^*X^*Z - X^*Y^*Z) = -Tr(Y^*XZ - Y^*ZX)$$

use $X^* = -X$

$$\Leftrightarrow Tr(YXZ - XYZ) = Tr(YXZ - YZX) \checkmark$$

The maximal torus $t \subseteq k$ is the diagonal matrices (2-dim)

$$t = \left\{ \begin{pmatrix} ia_1 & & \\ & ia_2 & \\ & & ia_3 \end{pmatrix}, a_i \in \mathbb{R}, a_1 + a_2 + a_3 = 0 \right\}$$

$\nwarrow$ write (a_1, a_2, a_3) for this matrix

The inner product on k, restricted to t, is:

$$\langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle = (-i) i [a_1 b_1 + a_2 b_2 + a_3 b_3]$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

= Euclidean inner product on $\mathbb{R}^3$,
restricted to subspace $a_1+a_2+a_3=0$

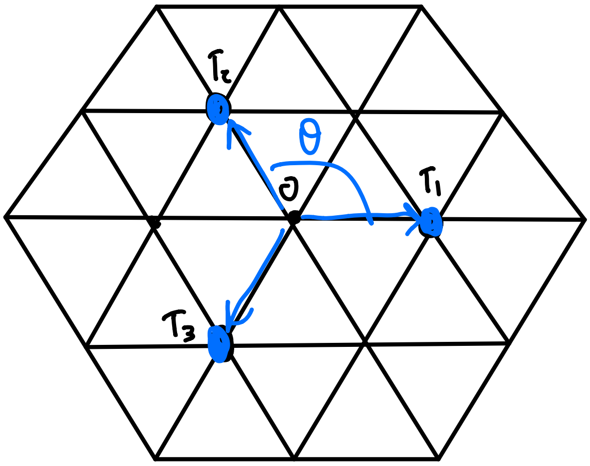
The vectors $T_1=(1,-1,0)$, $T_2=(-1,0,+1)$, $T_3\,(0,1,-1)$ span $\mathfrak{t}$:

Note: $\cos\begin{pmatrix}\text{angle between}\\ T_1 \text{ and } T_2\end{pmatrix} = \dfrac{\langle T_1,T_2\rangle}{\|T_1\|\|T_2\|}$

$= \dfrac{-1}{\sqrt{2}\cdot\sqrt{2}}$

$= \dfrac{-1}{2}$

$\therefore \theta = 120^\circ$



Complexification $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}} = \mathfrak{sl}(3,\mathbb{C})$

= traceless 3x3 complex matrices

Cartan subalgebra $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \left\{ \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{pmatrix} \;\; \begin{matrix} h_i \in \mathbb{C}, \\ h_1+h_2+h_3=0 \end{matrix} \right\}$

Roots $\alpha_{ij} \in \mathfrak{h}^*$, $\alpha_{ij} = L_i - L_j$, $(i\neq j)$

$$L_i\begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{pmatrix} = h_i$$

root space of α_{ij} = $\text{span}\{E_{ij}\} \subseteq \mathfrak{g}$.

Note:

$$\underbrace{R}_{\text{roots}} \subseteq i\mathfrak{t}^*$$

because the roots take on imaginary values on $\mathfrak{t}$:

$$\alpha_{ij}\begin{pmatrix} ia_1 & & \\ & ia_2 & \\ & & \underset{\uparrow\, a_i \in \mathbb{R}}{ia_3} \end{pmatrix} = i(a_i - a_j) \;.$$

Recall:

Theorem Let $\alpha \in R$. Then there exists $X_\alpha \in \mathfrak{g}_\alpha$ such that

$$X_\alpha, \quad Y_\alpha := -\tau(X_\alpha), \quad H_\alpha$$

$H_\alpha \perp \ker\alpha$, $\alpha(H_\alpha) = 2$

Satisfy the $\mathfrak{sl}(2,\mathbb{C})$ commutation relations

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, Y_\alpha] = -2Y_\alpha, \quad [X_\alpha, Y_\alpha] = H_\alpha$$

Also, we showed that for any root α,

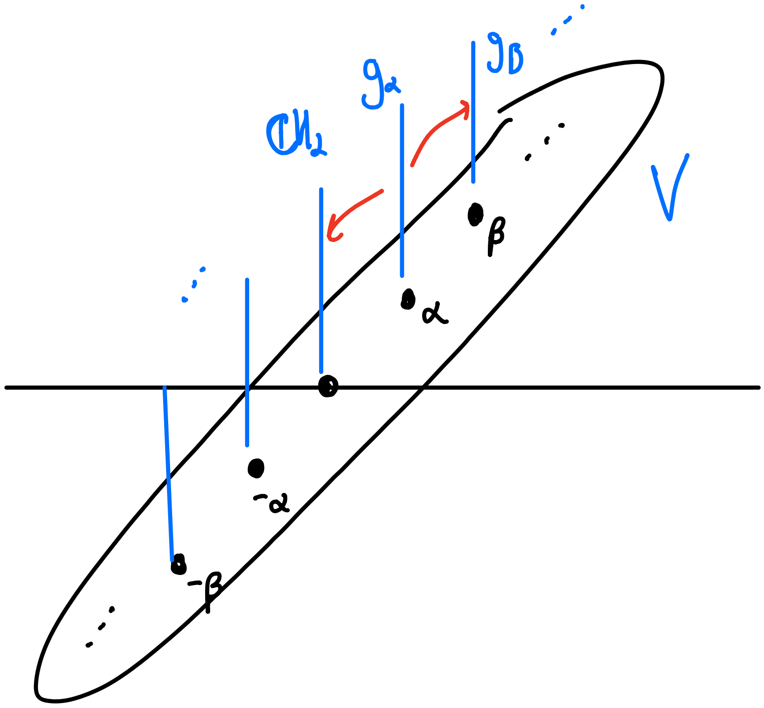
$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}H_\alpha$$

which hints that $\mathfrak{g}_\alpha$ is 1-dimensional.

Lemma Let $\alpha \in R$. Then $\dim \mathfrak{g}_\alpha = 1$. Moreover, $R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$.

Proof Let

$$V = \mathbb{C}H_\alpha \oplus \bigoplus_{\beta \in R \cap \mathbb{R}\alpha} \mathfrak{g}_\beta$$



Then V is a representation of

$$S_\alpha = \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \cong \mathfrak{sl}(2,\mathbb{C}).$$

So, it decomposes into weight spaces. These weight spaces are the eigenspaces of H_α, which are simply the $\mathfrak{g}_\beta$, as for $X \in \mathfrak{g}_\beta$,

$$[H_\alpha, X_\beta] = \beta(H_\alpha)\, X_\beta$$

So the weights of V are just $\gamma = \beta|_{\mathbb{C}H_\alpha}$, with weight space $\mathfrak{g}_\beta$.

Let V_{even} be the part of V with <u>even</u> weights, ie.

$$V_{even} = \underbrace{\mathbb{C}H_\alpha}_{\text{weight}=0} \oplus \bigoplus_{\left.\begin{subarray}{l}\beta\in R\cap\mathbb{R}\alpha \\ \beta(H_\alpha)\in 2\mathbb{Z}\end{subarray}\right] = \{n\alpha \,,\, n\in\mathbb{Z}_{\neq 0}\}} g_\beta$$

Then V_{even} splits up into $\underbrace{\dim(\mathbb{C}H_\alpha)}_{=1}$ irreducible reps. So, V_{even} is irreducible. But,

$$\mathbb{C}H_\alpha \oplus g_\alpha \oplus g_{-\alpha} \overset{\text{sub-representation}}{\subseteq} V_{even}$$

$$\therefore \quad V_{even} = \mathbb{C}H_\alpha \oplus g_\alpha \oplus g_{-\alpha}$$

$$\therefore \quad \dim g_\alpha = 1 \quad \text{and} \quad R\cap\mathbb{Z}\alpha = \{-\alpha,\alpha\}.$$

There remains the possibility that <u>odd</u> weights exist, ie. that

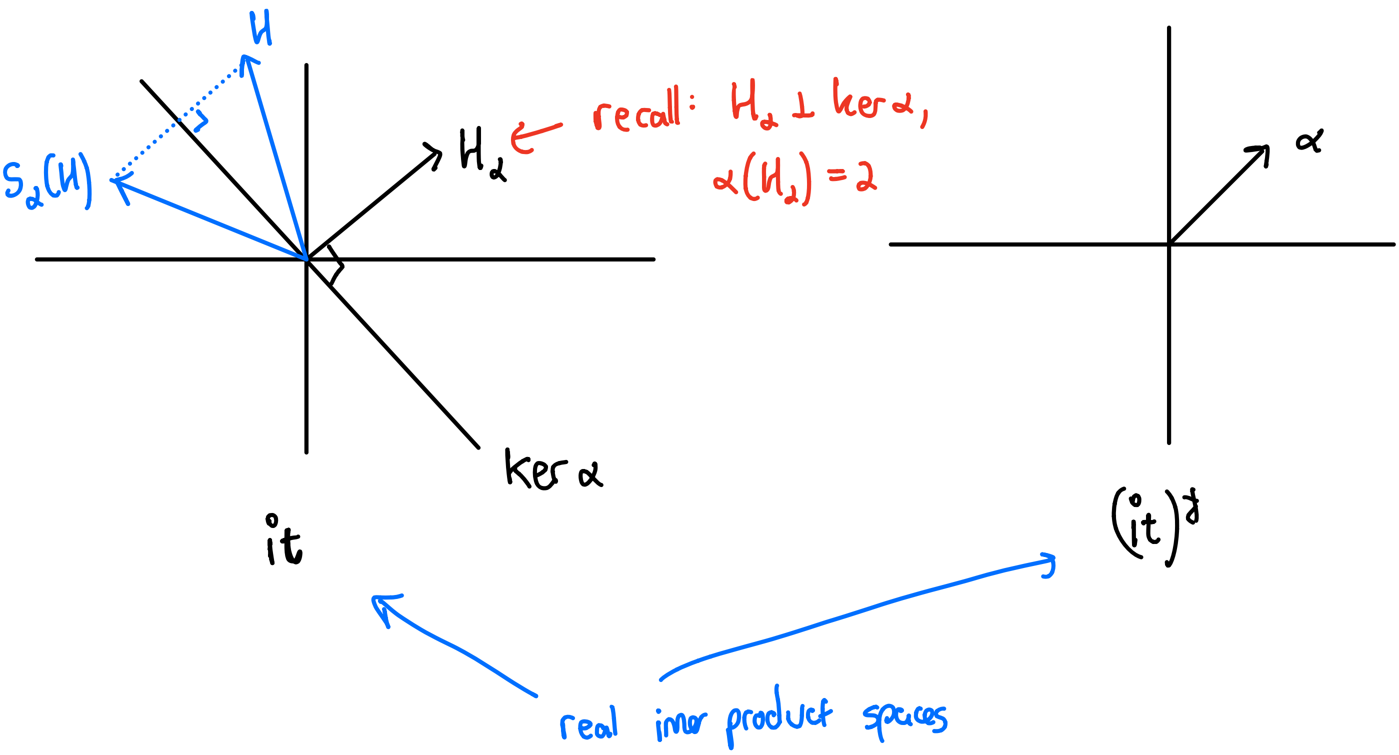
$$V_{odd} = \bigoplus_{\left.\begin{subarray}{l}\beta\in R\cap\mathbb{R}\alpha \\ \beta(H_\alpha)\text{ is odd}\end{subarray}\right] = (\mathbb{Z}+1/2)\,\alpha} g_\beta$$

is nonzero. But V_{odd} splits into $\dim(g_{\frac{\alpha}{2}})$ irreducible reps. And if $\frac{\alpha}{2}$ is a root of g, then we could run the entire analysis again, starting with $\alpha' = \frac{\alpha}{2}$. We would conclude that $2\alpha' = \alpha$ is not a root, which is a contradiction. So $V_{odd} = 0$. □

Let $\alpha \in R$. We write

$$S_\alpha : i\mathfrak{t} \longrightarrow i\mathfrak{t}$$

for $\underline{\text{orthogonal reflection in } \ker\alpha}$ map.



In a formula,

$$S_\alpha(H) = H - \alpha(H)\, H_\alpha, \qquad ①$$

as the reader will verify.

Exercise Verify this.

The dual map

$$W_\alpha := S_\alpha^* : (it)^* \longrightarrow (it)^*$$

sends

$$\lambda \longmapsto \lambda - \lambda(H_\alpha)\alpha \qquad ②$$

and can be interpreted as orthogonal reflection in the hyperplane $\alpha^\perp$, as the reader will verify

Exercise Verify that S_α^* is given by ② and that it can be interpreted as orthogonal reflection in the hyperplane $\alpha^\perp$.

We also write

$$S_\alpha : h \longrightarrow h$$
$$r_\alpha \equiv S_\alpha^* : h^* \longrightarrow h^*$$

for their complex-linear extensions (given by ① and ② resp.).

Lemma For every $\alpha \in R$, there exists a Lie algebra automorphism

$$\phi_\alpha : g \longrightarrow g$$

such that $\phi_\alpha\Big|_h = S_\alpha$.

Proof Fix X_α, Y_α such that $(H_\alpha, X_\alpha, Y_\alpha)$ is a standard $\mathfrak{sl}(2,\mathbb{C})$ triple. Set

$$U_\alpha = \frac{\pi}{2}(X_\alpha - Y_\alpha)$$

Then $\phi_\alpha := e^{ad_{U_\alpha}}$ is an automorphism of $\mathfrak{g}$.

> Recall: For $Z \in \mathfrak{g}$, $e^{ad_Z}(\cdot) = e^{Z}(\cdot)e^{-Z}$ is an automorphism of $\mathfrak{g}$.

Also, for $H \in \mathfrak{h}$,

$$\begin{aligned} ad_{U_\alpha}(H) &= \tfrac{\pi}{2}[U_\alpha, H] \\ &= \tfrac{\pi}{2}\left([H, Y_\alpha] - [H, X_\alpha]\right) \\ &= \tfrac{\pi}{2}\left(\underbrace{-\alpha(H)}_{=0}Y_\alpha - \underbrace{\alpha(H)}_{=0}X_\alpha\right) \end{aligned}$$

So $\phi_\alpha = 0$ on $\ker\alpha$. It remains to prove that

$$\phi_\alpha(H_\alpha) = -H_\alpha.$$

But this is an equation formulated completely in terms of the structure of $\mathfrak{sl}(2,\mathbb{C})$. So we can check it there:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \frac{\pi}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$\therefore \quad e^{U} = \begin{pmatrix} \cos\pi/2 & \sin\pi/2 \\ -\sin\pi/2 & \cos\pi/2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\therefore \quad e^{U} H e^{-U} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= -H. \quad \checkmark$$

$\square$

<u>Corollary</u> For every root $\alpha \in R$, orthogonal reflection in $\alpha^{\perp}$

$$r_\alpha \equiv S_\alpha^{*} : (i\mathfrak{t})^{*} \longrightarrow (i\mathfrak{t})^{*}$$

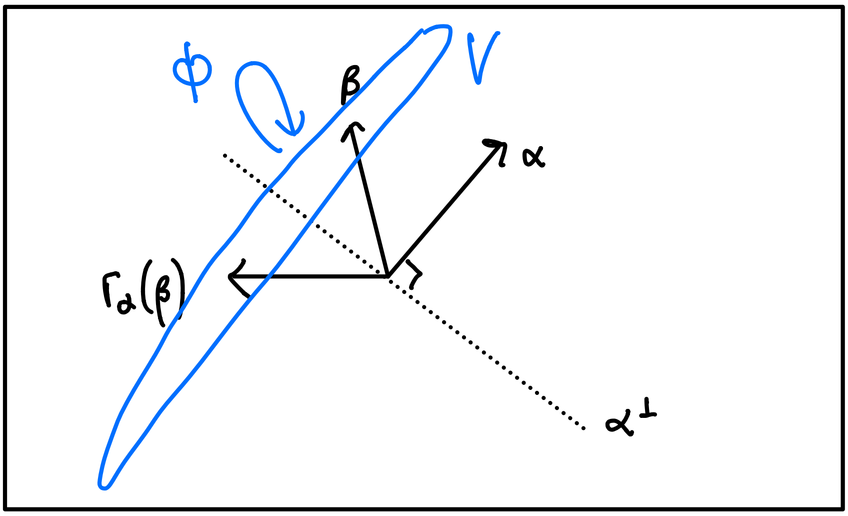
sends R to R. Moreover, for $\alpha, \beta \in R$, $r_\alpha(\beta) \in \beta + \mathbb{Z}\alpha$.

<u>Proof</u> Let ϕ be an automorphism of $\mathfrak{g}$ which leaves $\mathfrak{h}$ invariant. Let $\alpha \in R$ and $X \in \mathfrak{g}_\alpha$. Then for $H \in \mathfrak{h}$,

$$[H, \phi(X)] = \phi\left([\phi^{-1}(H), X]\right)$$
$$= \phi\left(\alpha(\phi^{-1}(H))\, X\right)$$
$$= \alpha(\phi^{-1}(H))\ \ \phi(X).$$

So $(\phi^{-1})^*(\alpha) := \alpha \circ \phi^{-1}$ is a root. This applies to our case $\phi = \phi_\alpha$. (Note that, restricted to it, ϕ is just orthogonal reflection in $\ker\alpha$, so $\phi^{-1} = \phi = S_\alpha$ on $\mathfrak{h}$).

Must still prove that $r_\alpha(\beta) \in \beta + \mathbb{Z}\alpha$:



$(it)^*$

Let

$$V = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$$

Then clearly V is a representation of $sl(2,\mathbb{C})_\alpha$. And, since $U_\alpha \in sl(2,\mathbb{C})_\alpha$,

$$\phi_\alpha = e^{ad_{U_\alpha}} \quad \text{sends} \quad V \longrightarrow V.$$

But that means

$$\mathfrak{g}_{r_\alpha(\beta)} = \mathfrak{g}_{\phi_\alpha^{-1}(\beta)} = \phi_\alpha\left(\mathfrak{g}_\beta\right) \subseteq V.$$

□

<u>Exercise</u> Choose one of the roots α for $sl(3,\mathbb{C})$, and work out

$$\phi_\alpha : sl(3,\mathbb{C}) \longrightarrow sl(3,\mathbb{C}).$$

Does it restrict to the real subspace $\underbrace{i\,su(2)}_{\cong \mathbb{R}^8} \subseteq sl(3,\mathbb{C})$?

Does it have a geometric interpretation?

Lie Algebras

Lecture 10

Definition The Weyl group W is the group generated by the reflections

$$r_\alpha : i\mathfrak{t}^* \longrightarrow i\mathfrak{t}^* \quad , \quad \alpha \in R.$$

Lemma The Weyl group is finite.

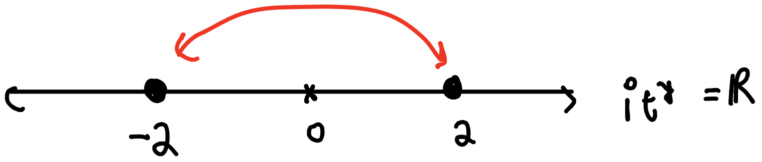
Homomorphism?
$f(r_\alpha r_\beta) = f(r_\alpha) f(r_\beta)$

Proof We have an injective homomorphism

$$f : W \longrightarrow \underbrace{\text{permutations of } R.}_{\text{finite}}$$

□

Example For $\mathfrak{sl}(2, \mathbb{C})$, $W = \mathbb{Z}_2$:



Exercise Work out the Weyl group for $\mathfrak{sl}(3,\mathbb{C})$.

Review

We can now classify the irreducible representations of any semisimple Lie algebra $\mathfrak{g} = K_{\mathbb{C}}$.

① Let $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ where $\mathfrak{t}$ is a maximal torus in $\mathfrak{k}$. We call $\mathfrak{h}$ the <u>Cartan subalgebra</u> of $\mathfrak{g}$. We can <u>simultaneously diagonalize</u> the actions of the $H \in \mathfrak{h}$ on any rep.

That is, if V is a rep of $\mathfrak{g}$, we can decompose

$$V = \bigoplus_{\gamma \in \Lambda(V)} V_\gamma$$

② In particular,

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right)$$

and each $\mathfrak{g}_\alpha$ is 1-dimensional.

③ We have

$$X_\alpha : V_\gamma \longrightarrow V_{\gamma+\alpha} \qquad X_\alpha \in \mathfrak{g}_\alpha.$$

④ Each $\alpha \in R$ has an associated copy of $\mathfrak{sl}(2,\mathbb{C})$:

$$\mathfrak{sl}(2,\mathbb{C})_\alpha = \underbrace{\mathbb{C}H_\alpha}_{\text{coroot of } \alpha} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

$v \in V_\gamma$

$Hv = \gamma(H)v$

and V is thus also a rep of $\mathfrak{sl}(2,\mathbb{C})_\alpha$.

⑤ In particular, the eigenvalues of H_α must be integers. So, the weights $\Lambda(V)$ live in the <u>weight lattice</u>

$Hv = nv$

$H_\alpha v = \underbrace{\gamma(H_\alpha)}v$

$$\Lambda := \left\{ \gamma \in (i\mathfrak{t})^* : \gamma(H_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in R \right\}$$

⑥ We fix a linear functional

$$\ell : (i\mathfrak{t})^* \longrightarrow \mathbb{R}$$

which decomposes the roots as

$$R = R^+ \cup R^- . \quad \text{Note: } R^- = -R^+ .$$

The <u>raising operators</u> are the $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in R^+$
<u>lowering operators</u> are the $Y_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in R^-$.

⑦ Any rep V must contain a <u>highest weight</u> γ_0 (maximizing ℓ) (if V is irreducible then $\dim V_{\gamma_0} = 1$.) and a

highest weight vector $v_0 \in V_{\gamma}$. Clearly

V is generated by successively applying lowering operators to v_0.

⑧ The set of weights $\Lambda(V) \subseteq i\mathfrak{t}^*$ is invariant under the Weyl group reflections $r_\alpha : i\mathfrak{t}^* \longrightarrow i\mathfrak{t}^*$.

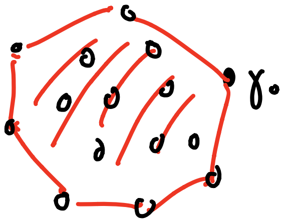
Exercise Check this. More precisely, define

$$\psi_\alpha : V \longrightarrow V$$

to be the linear map $\psi_\alpha = e^{\pi(U_\alpha)}$, where π is the representation of $\mathfrak{g}$ on V. Show that

$$\boxed{\begin{array}{l}\gamma \text{ a weight of } V \\ \text{and } v \in V_\gamma\end{array} \Longrightarrow \begin{array}{l} r_\alpha(\gamma) \text{ is also a weight} \\ \text{and } \psi_\alpha^{-1}(v) \in V_{r_\alpha(\gamma)}\end{array}}$$

In fact,



$$\Lambda(V) = \text{Convex-Hull}\left(W(\gamma_0)\right) \cap \Lambda$$

⑨ The possible highest weights γ_0 are those satisfying

$$\gamma_0(H_\alpha) \geq 0 \qquad \forall \alpha \in R^+$$

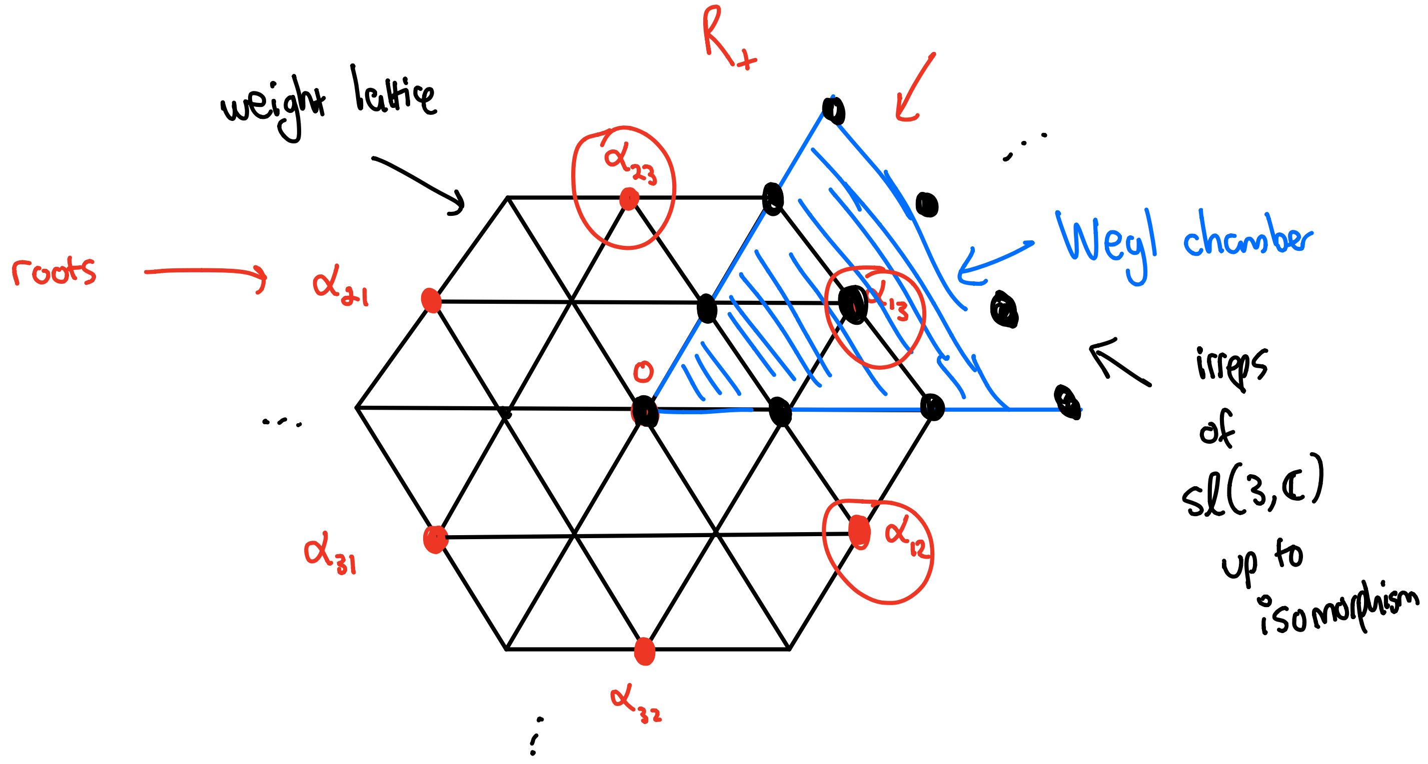
The set of weights satisfying $\overset{\text{all}}{\wedge}$ these inequalities is called the <u>positive Weyl chamber</u>. In terms of inner products,

$$\langle \gamma_0, \alpha \rangle \geqslant 0 \qquad \alpha \in R^+$$

In summary, we have:

<u>Theorem</u> The irreducible representations of $\mathfrak{g}$ are classified by their highest weight. The set of possible highest weights is

$$\Lambda \cap \text{positive Weyl chamber.}$$



Root systems

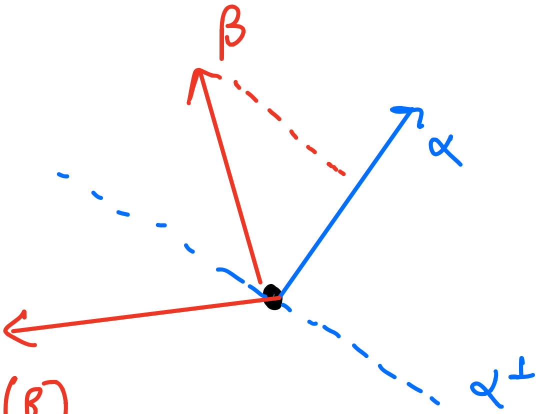
From a semisimple Lie algebra, we have extracted a collection of vectors in a real inner-product space (the root vectors $\alpha \in R$ living in it*) having certain properties. Let's axiomatize this.

$\dim(E)$ is called the rank of the root system.

Definition A root system is a pair (E, R) where E is a finite dim real inner product space and a finite set $R \subset E \setminus \{0\}$, satisfying:

1. R spans E.
2. If $\alpha \in R$, then $R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$.
3. If $\alpha, \beta \in R$, then so is $r_\alpha(\beta)$, where r_α is reflection in the hyperplane $\alpha^\perp$.
4. For $\alpha, \beta \in R$, $r_\alpha(\beta) \in \underbrace{\beta + \mathbb{Z}\alpha}$.

↑ i.e. $\mathrm{proj}_\alpha(\beta)$ is an integer or half-integer multiple of α.



$$r_\alpha(\beta) = \beta - 2\,\mathrm{proj}_\alpha(\beta) = \beta - \boxed{2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}}^{\in \mathbb{Z}} \alpha$$

Two root systems (E, R) and (E', R') are isomorphic if there exists a linear isomorphism $T : E \longrightarrow E'$ with $T(R) = R'$.

Let us review. Given a (necessarily real) compact Lie algebra k with trivial center ($\Leftrightarrow$ the Lie group K has at most discrete, hence finite, center), and a choice of maximal abelian subalgebra $t \subseteq k$, we constructed a root system $(E = it^*, R \subseteq it^*)$

Theorem 1. The isomorphism class of $(E = it^*, R \subseteq it^*)$ does not depend on the choice of maximal torus $t \subseteq k$.

2. The resulting map

$$\left\{ \begin{matrix} \text{real compact Lie algebras} \\ \text{with trivial center} \end{matrix} \right\} \Big/ \text{iso} \longrightarrow \left\{ \text{root systems} \right\} \Big/ \text{iso}$$

is a bijection.

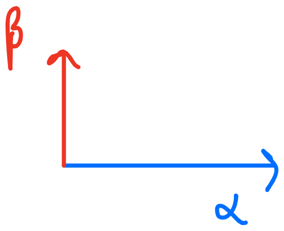
Serre wrote down an explicit inverse map.

For now, let's look at some more properties of root systems and some examples.

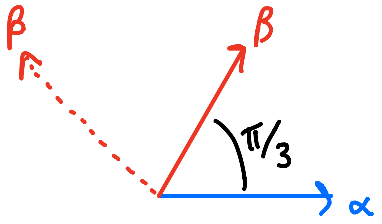
Lemma In a root system, suppose α, β are roots, $\alpha \neq \pm\beta$, and $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$. Then one of the following holds:

$0 \leq \theta \leq \pi.$

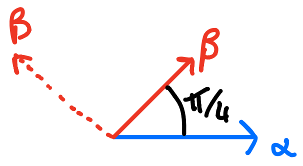
1. $\langle \alpha, \beta \rangle = 0$
2. $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$ and $\text{Angle}(\alpha, \beta) = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.
3. $\langle \alpha, \alpha \rangle = 2\langle \beta, \beta \rangle$ and $\text{Angle}(\alpha, \beta) = \frac{\pi}{4}$ or $\frac{3\pi}{4}$
4. $\langle \alpha, \alpha \rangle = 3\langle \beta, \beta \rangle$ and $\text{Angle}(\alpha, \beta) = \frac{\pi}{6}$ or $\frac{5\pi}{6}$



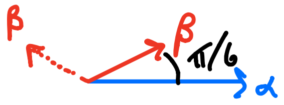
1.



2.



3.



4.

Note: θ acute $\Rightarrow \text{proj}_\alpha(\beta) = \frac{\alpha}{2}$

θ obtuse $\Rightarrow \text{proj}_\alpha(\beta) = -\frac{\alpha}{2}$

Proof Let $m_1 = \dfrac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$, $m_2 = \dfrac{2\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$. Then

- $m_1 m_2 = \dfrac{4\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} = 4\cos^2\theta$, $\theta = \text{Ang}(\alpha, \beta)$

$m \quad = \langle \alpha, \alpha \rangle \quad \geq 1$ if $\langle \alpha \beta \rangle \neq 0.$

- $\frac{m_2}{m_1} = \frac{\quad}{\langle \beta, \beta \rangle}$

Only possibilities:

$4\cos^2\theta$	θ	m_1	m_2
0	$\pi/2$	•	•
1	$\pi/3$ or $\frac{2\pi}{3}$	1	1
2	$\frac{\pi}{4}$ or $\frac{5\pi}{4}$	1	2
3	$\frac{\pi}{6}$ or $\frac{5\pi}{6}$	1	3

□

<u>Corollary</u> Suppose α and β are roots. Let $\theta = \mathrm{Ang}(\alpha,\beta)$.

- θ is <u>strictly acute</u> $(0 < \theta < \pi/2) \Rightarrow \alpha - \beta$ is a root.
- θ is <u>strictly obtuse</u> $(\frac{\pi}{2} < \theta < \pi) \Rightarrow \alpha + \beta$ is a root

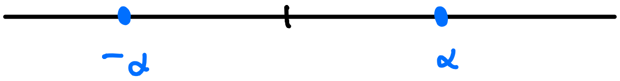
<u>Proof</u> We know $r_\alpha(\beta) = \beta - \underbrace{2\,\mathrm{proj}_\alpha(\beta)}$ is always a root.

$= \alpha$ if θ is acute

$= -\alpha$ if θ is obtuse. □

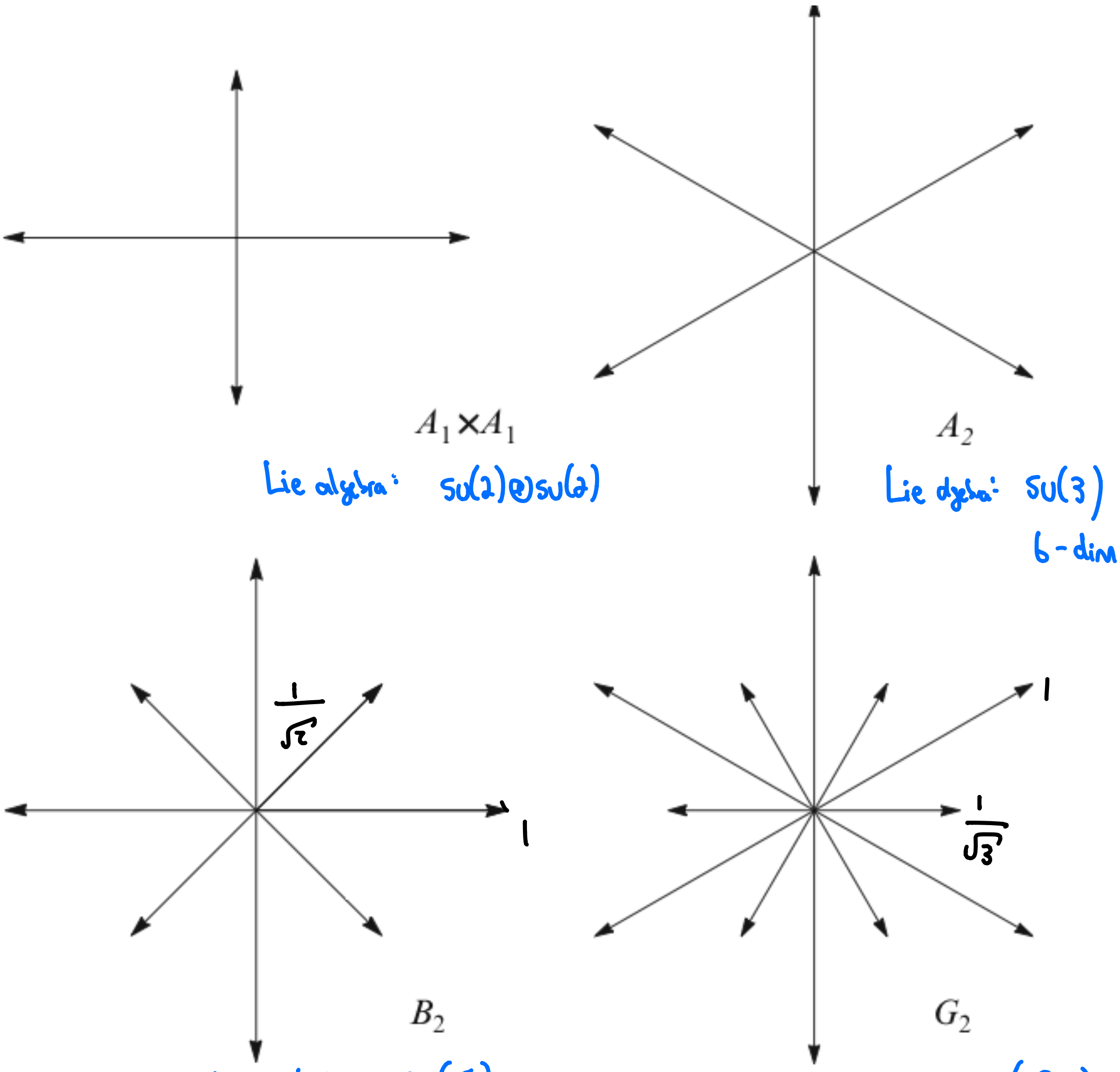
Examples of root systems

- Rank 1 : only $\{\alpha, -\alpha\}$ allowed (overall scale irrelevant).



= root system of $su(2)$.

- Rank 2 :



$A_1 \times A_1$

Lie algebra: $su(2) \oplus su(2)$

A_2

Lie algebra: $su(3)$
6-dim

B_2

Lie algebra: $so(5)$
dim 10

G_2

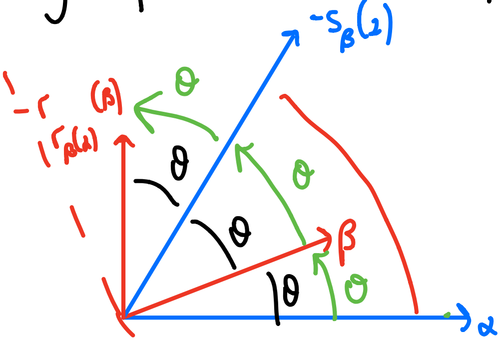
$\mathrm{Lie}(G_2)$.
(dim 14 Lie algebra)

Fig. 8.3 The rank-two root systems

Proposition Every rank 2 root system is isomorphic to one of these.

Proof Let $R \subset \mathbb{R}^2$. Let θ be smallest angle between roots. Let α, β be lin. ind. roots. If $\text{Ang}(\alpha,\beta) > \frac{\pi}{2}$, then $\text{Ang}(\alpha,-\beta) < \frac{\pi}{2}$ so $\theta \leq \pi/2$. So $\theta \in \{\pi/2, \pi/3, \pi/4, \pi/6\}$.

Let α, β have $\text{Ang}(\alpha,\beta) = \theta$. Then $-r_\beta(\alpha)$ is at angle 2θ from α.



Similarly, $-s_{s_\beta(\alpha)}(\beta)$ is at angle 3θ from α. We will eventually come back to α. These must be all the roots (else there's an angle smaller than θ).

$\theta = \pi/2 \rightsquigarrow A_1 \times A_1$

$\theta = \pi/3 \rightsquigarrow A_2$

$\theta = \pi/4 \rightsquigarrow B_2$

$\theta = \pi/3 \rightsquigarrow G_2$.

$\square$

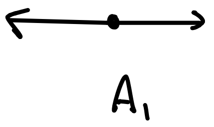
Lie algebras

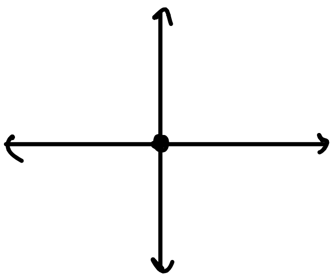
Lecture 11

We want to classify root systems (E, R)

Note that if (E_1, R_1) and (E_2, R_2) are root systems, then so is the orthogonal direct sum $(E_1 \oplus E_2, R_1 \oplus R_2)$. We say a root system is <u>irreducible</u> if it cannot be written as a nontrivial orthogonal direct sum. So, we want to classify <u>irreducible</u> root systems.

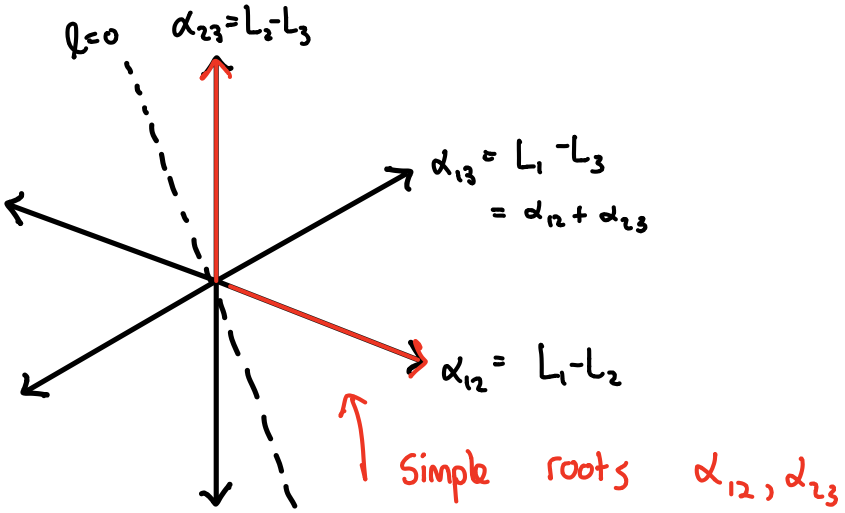
eg





$A_1 \times A_1$
(not irreducible)

We call a positive root <u>simple</u> if it cannot be written as a sum of other positive roots.



Note: the angle between two simple roots α, β cannot be acute, else $\left(\|\alpha\| > \|\beta\| \right)$ $\alpha - \beta$ is a positive root and $\beta = \alpha + (\alpha - \beta)$, contradiction.

The simple roots must be <u>linearly independent</u>, by the following exercise.

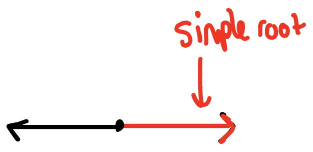
Exercise Show that if a collection of vectors $v_1, \cdots, v_n$ in a Euclidean space all have pairwise obtuse angles and if they all lie on one side of a hyperplane, then they are linearly independent.

It follows that the simple roots form a basis for E.

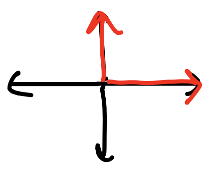
The <u>Dynkin diagram</u> of the root system is obtained by drawing one node for each simple root and joining two nodes by a certain number of edges, depending on the angle between the simple roots:

Note: in the case $\theta = \frac{3\pi}{4}$, $\theta = \frac{5\pi}{6}$, the arrow goes from <u>longer</u> to <u>shorter</u> root.

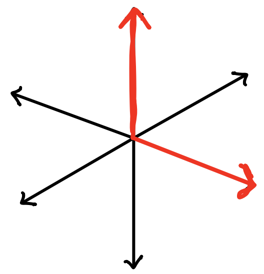
eg.: root system Dynkin diagram



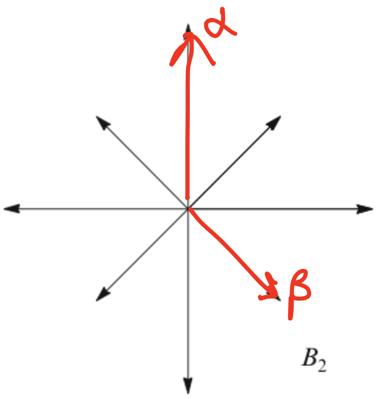
A_1



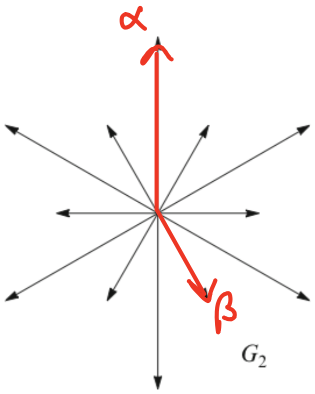
$A_1 \times A_1$



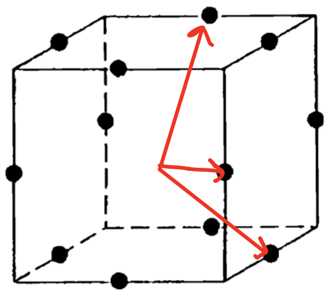
A_2



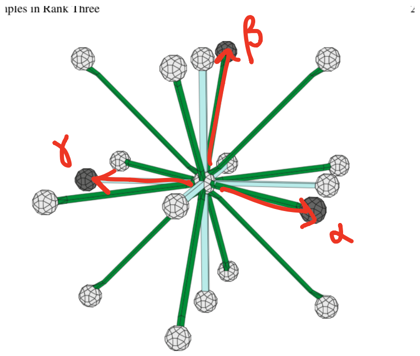
B_2

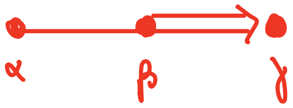


G_2



A_3



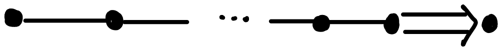


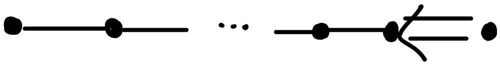
B_3

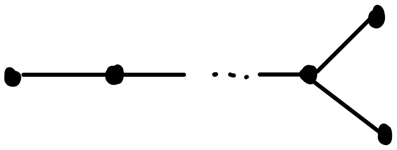
Theorem The Dynkin diagrams of irreducible root systems are precisely:
(The subscripts indicate the number of nodes

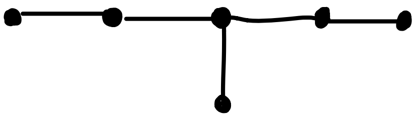
A_n

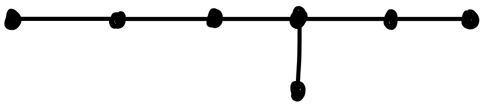
$(n \geq 1)$ $su(n+1)$

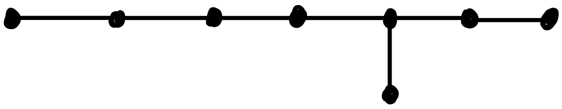
B_n

$(n \geq 2)$ $so(2n+1), n \geq 2$

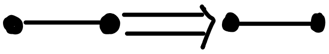
C_n

$(n \geq 3 \ldots$ since $B_2 = C_2)$ $sp(2n)$

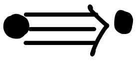
D_n

$(n \geq 4$, else $= A_n)$ $so(2n)$

E_6

e_6

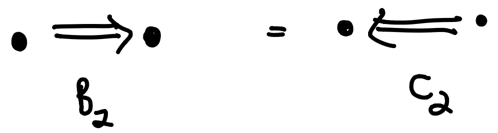
E_7

e_7

E_8

e_8

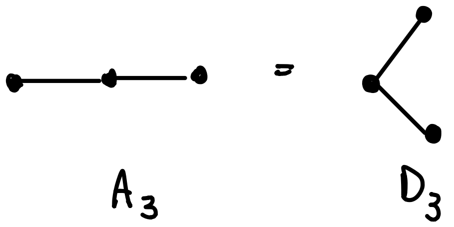
F_4

f_4

G_2

g_2

Note some "coincedences" for low n:



reflects $so(5) \cong sp(4)$



reflects $su(4) \cong so(6)$

from Griffiths + Harris

Proof The angles alone (no arrows to indicate relative lengths and no restrictions coming from the axioms of a root system) determine the possible diagrams. Let's say that a diagram of n nodes, with each pair of nodes separated by 0, 1, 2, or 3 edges, is <u>admissible</u> if there exists a configuration of n unit vectors $e_1, \cdots, e_n$ in Euclidean space such that

$$\mathrm{Ang}(e_i, e_j) = \begin{cases} \pi/2 & \text{if } \bullet\qquad\bullet \\ \frac{2\pi}{3} & \text{if } \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \\ 3\pi/4 & \text{if } \bullet\!=\!\!=\!\bullet \\ \frac{5\pi}{6} & \text{if } \bullet\!\equiv\!\!\equiv\!\bullet \end{cases}$$

The claim is that the above diagrams are the only connected admissible diagrams. Note that

$$\boxed{4(e_i, e_j)^2 = \text{number of edges between } e_i \text{ and } e_j.}$$

Then:

1. Any subdiagram of an admissible diagram, obtained by removing a node and all edges connected to it, is admissible.

Clear.

2. There are at most $(n-1)$ pairs of nodes that are connected by edges. Also, the diagram contains no cycles (i.e. a connected admissible diagram is a <u>tree</u>).

If e_i and e_j are connected by an edge, then $2(e_i, e_j) \leq -1$.

So,

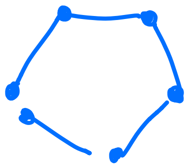
$$\underbrace{\left(\sum_i e_i, \sum_i e_i\right)}_{>0} = n + \sum_{i<j} 2(e_i, e_j)$$

since the e_i are lin. ind., hence their sum is not zero

$$= n + \sum_{\substack{\text{pairs} \\ \text{of nodes connected} \\ \text{by edges}}} \underbrace{2(e_i, e_j)}_{\leq -1}$$

$$\therefore \# \left\{ \begin{matrix} \text{pairs of nodes} \\ \text{connected by edges} \end{matrix} \right\} \leq n-1$$

This also proves there are no cycles, else

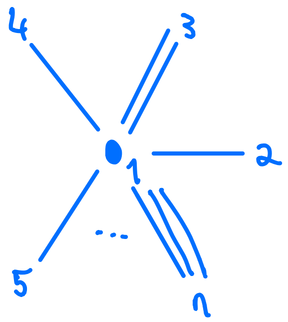


n nodes

n pairs of nodes connected by edges

3. No node has more than 3 edges coming out of it.

By (1), we can restrict ourselves to diagrams of this form:



Note: by (2), no edges between i,j for i,j ≥ 2.

We must show

$$\sum_{j=2}^{n} 4(e_1,e_j)^2 < 4$$

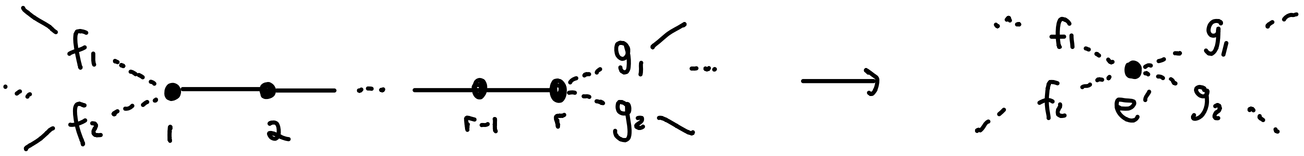
But, since $e_2, \cdots, e_n$ are all mutually orthogonal, and e_1 is not in their span, we must have

$$\underbrace{(e_1,e_1)^2}_{=1} < \sum_{j=2}^{n} (e_1,e_j)^2$$

$$\left[v = \sum \langle e_i, v\rangle e_i + v_\perp \right]$$

which is what we wanted to show.

4. In an admissible diagram, any string of single edges can be collapsed, and the resulting diagram is still admissible:



If $e_1, \cdots, e_r$ are the unit vectors corresponding to the string of nodes, then $e' = e_1 + e_2 + \cdots + e_r$ is a unit vector, as

$$(e',e') = \sum_{i=1}^{r} (e_i,e_i) + 2\sum_{i=1}^{r-1} \underbrace{(e_i,e_{i+1})}_{=-1/2}$$

$$= r - (r-1)$$

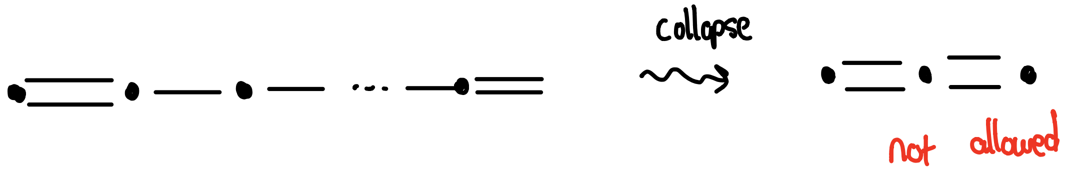
$$= 1$$

Also, e' has the same inner products with the border nodes as e_1 and e_r used to have, e.g.

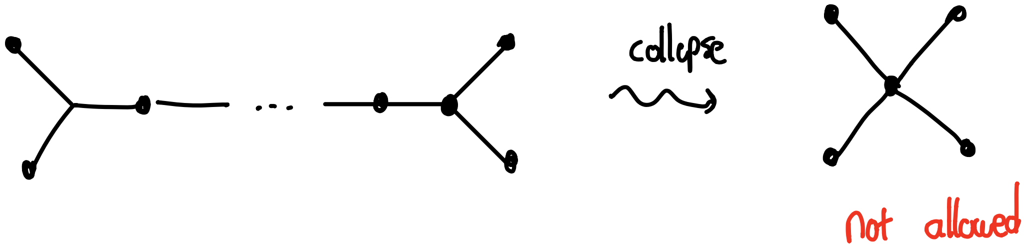
$$\langle f_1, e' \rangle = \langle f_1, e_1 \rangle \ \checkmark$$

$$\langle g_1, e' \rangle = \langle g_1, e_r \rangle \ \checkmark$$

Ok, now we can begin. Clearly G_3 is the only diagram with a triple edge. Next, there can only be <u>one</u> double edge in a diagram, else we would have a subdiagram of the form

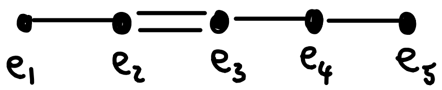


By the same reasoning, there can be at most one triple node (a node with single-edges to three other nodes):



Similarly, you can't have a double edge and a triple edge in the same diagram, or a double edge and a triple node. (ie. only one "feature" allowed!)

To finish off double edges, it remains only to rule out the following diagram:



It turns out such a configuration will violate the Cauchy-Schwarz inequality!

Set

$$v = a_1e_1 + a_2e_2 \quad , \qquad w = a_3e_3 + a_4e_4 + a_5e_5$$

Cauchy-Schwarz:

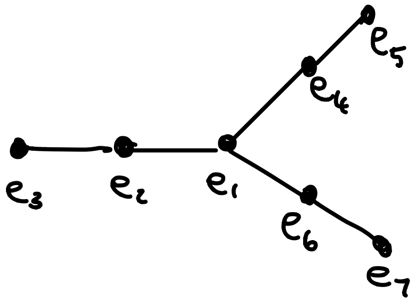
$$\underbrace{\langle v, w\rangle^2}_{= \frac{a_2^2 a_3^2}{2}} < \underbrace{\langle v,v\rangle}_{\substack{=a_1^2+a_2^2 \\ 2a_1a_c}} \underbrace{\langle w,w\rangle}_{= a_3^2+a_4^2+a_5^2 - a_3a_4 - a_4a_5}$$

$(=18) \qquad (=3) \qquad (=6)$

To violate, we want to make a_2 and a_3 as large as possible, while keeping $\|v\|$ and $\|w\|$ fixed. The following maximize this, and violate C-S:

$$v = e_1 + 2e_2 \qquad w = 3e_3 + 2e_4 + e_5$$

It remains to classify the single-edge diagrams, in particular those with a triple node. We must rule out the following cases:



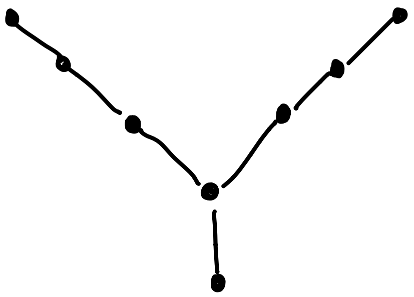
Consider the three perpendicular unit vectors:

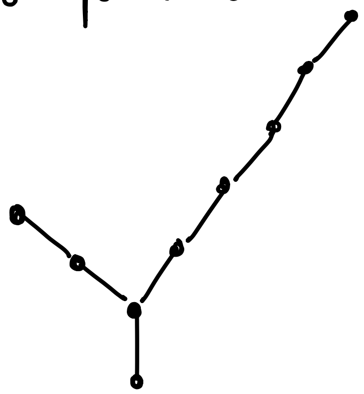
$$u = \frac{2e_2 + e_3}{\sqrt{3}}, \quad v = \frac{2e_4 + e_5}{\sqrt{3}}, \quad w = \frac{2e_6 + e_7}{\sqrt{3}}$$

Then, as in (3), since e_1 is not in the span of them, we must have

$$\underbrace{\langle e_1, e_1 \rangle^2}_{=1} > \underbrace{\langle u, e_1 \rangle^2}_{1/3} + \underbrace{\langle v, e_1 \rangle^2}_{1/3} + \underbrace{\langle w, e_1 \rangle^2}_{1/3},$$

a contradiction. This, and the other two possibilities

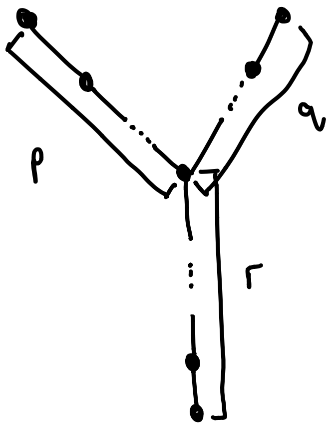




can be ruled out by the following exercise.

Exercise Show that if the legs emanating from a triple node have p, q, and r nodes respectively, then

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$



QED.

$\square$

Since we can concretely construct a Lie algebra whose root system corresponds to each of these diagrams, we have completed the classification of semisimple complex Lie algebras ($=$ compact real Lie algebras with trivial center).

Finally, I want to write down Serre's inverse functor:

$$\begin{tikzcd}\text{Semisimple Lie algebras} \arrow[r, bend left] & \text{Dynkin diagram} \arrow[l, bend left]\end{tikzcd}$$

note: this is better than root systems, more functorial.

Theorem (Serre) Given a Dynkin diagram with n nodes, form the free Lie algebra on the following generators,

$$H_1, \dots, H_n, \quad X_1, \dots, X_n, \quad Y_1, \dots, Y_n$$

and quotient by the following relations:

- $[H_i, H_j] = 0 \qquad \forall\, i,j$
- $[H_i, X_j] = n_{ji} X_j \qquad \forall\, i,j$
- $[H_i, Y_j] = -n_{ji} Y_j \qquad \forall\, i,j$
- $[X_i, Y_i] = H_i \qquad \forall\, i$

- $[X_i, Y_j] = 0 \qquad \forall\, i \neq j$

and

- $\mathrm{ad}_{X_i}^{n_{ij}}(X_j) = 0$
- $\mathrm{ad}_{Y_i}^{n_{ij}}(Y_j) = 0$

Then the result is a semisimple Lie algebra whose Dynkin diagram is the one you started with.