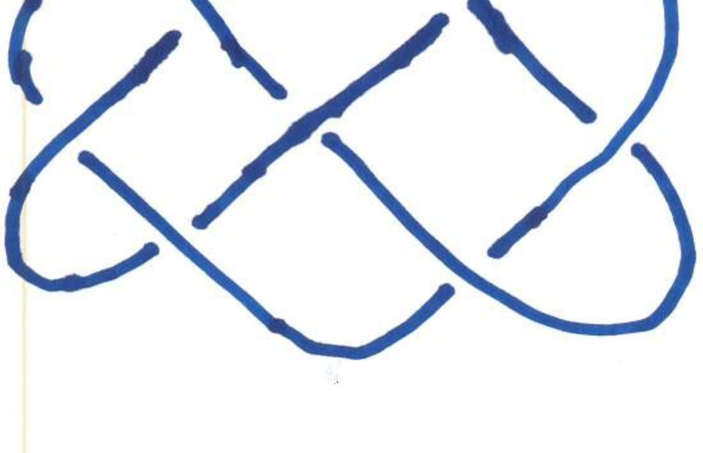
The Jones polynomial of knots

Stellenbosch University Mathematics Society

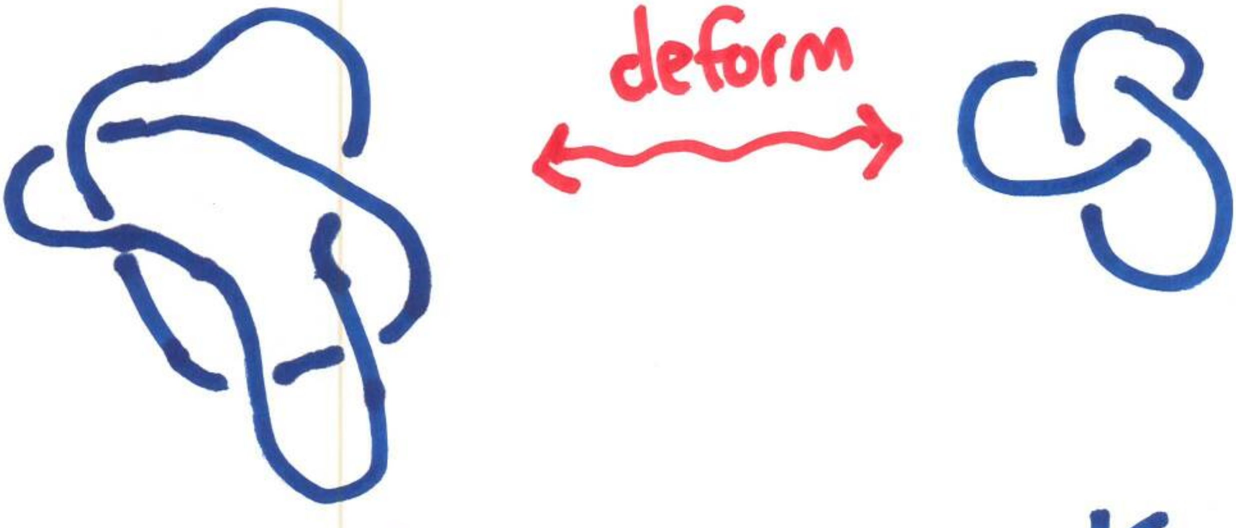
24 April 2019



A **knot** K is a closed loop of string in space.

K

Two knots K, K' are **equivalent** if K can be deformed into K' (without cutting the string).



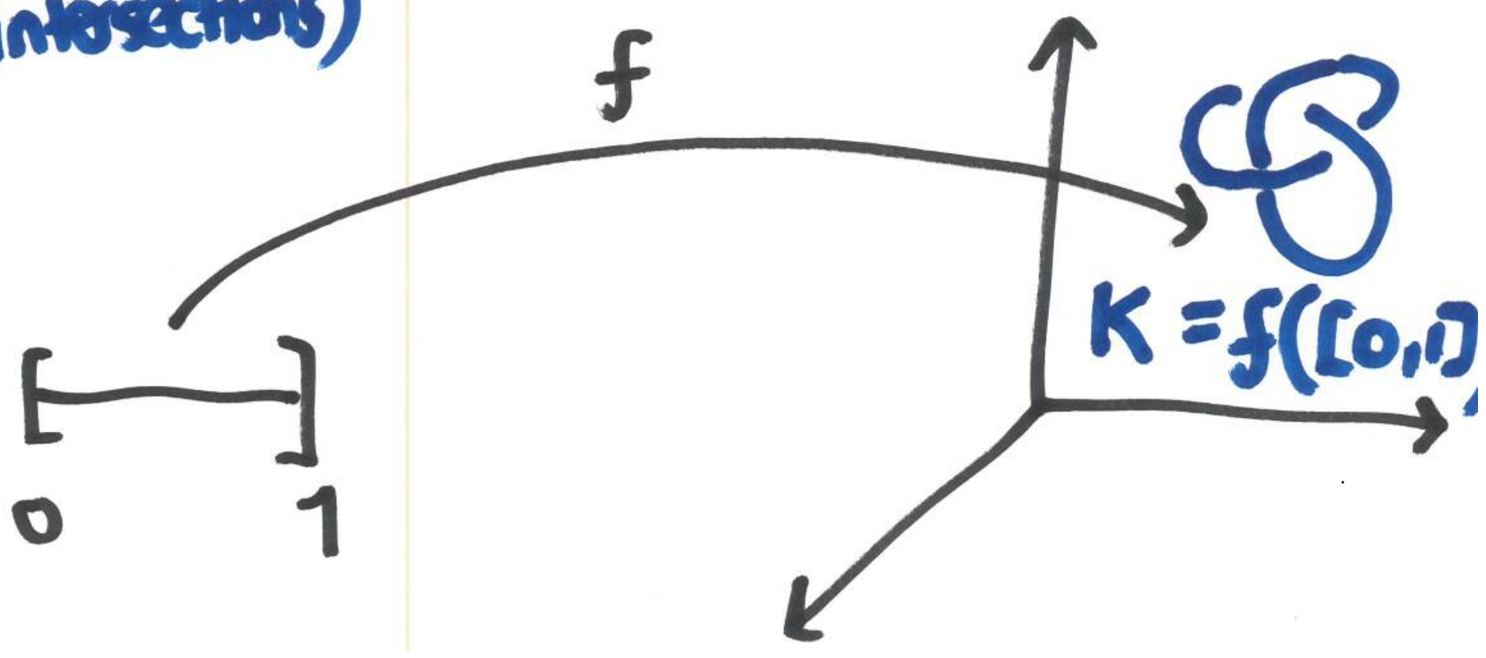
$K' \sim K$

A **knot** $K \subseteq \mathbb{R}^3$ is the image of a map

$$f : [0,1] \longrightarrow \mathbb{R}^3$$

Such that:

- (closed) $f(0) = f(1)$
- (smooth) $\dfrac{d^n f}{dt^n}$ exists for all n
- (smooth) $f'(t) \neq \vec{0}$ for all $t \in [0,1]$
- (no self-intersections) f is injective on $(0,1)$



Two knots K, K' are <u>equivalent</u> if there exists a smooth map ("isotopy")

$$D : [0,1] \times [0,1] \longrightarrow \mathbb{R}^3$$

Such that :

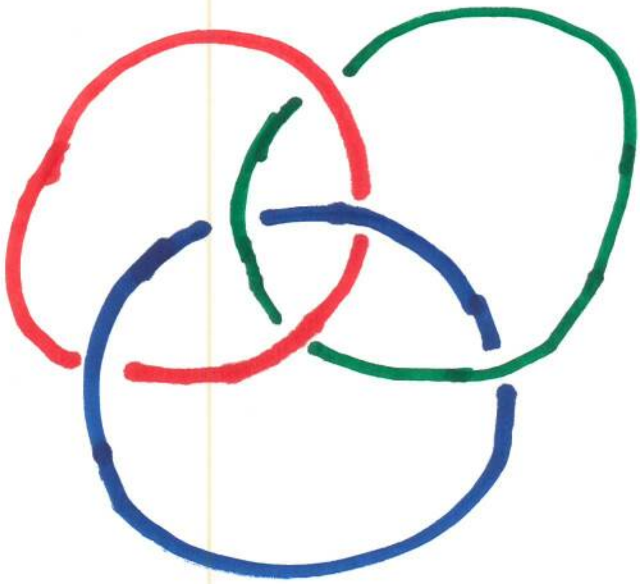
- $D(t,-) : [0,1] \longrightarrow \mathbb{R}^3$ is a knot for each $t \in [0,1]$

- $\mathrm{Im}\big(D(0,-)\big) = K$

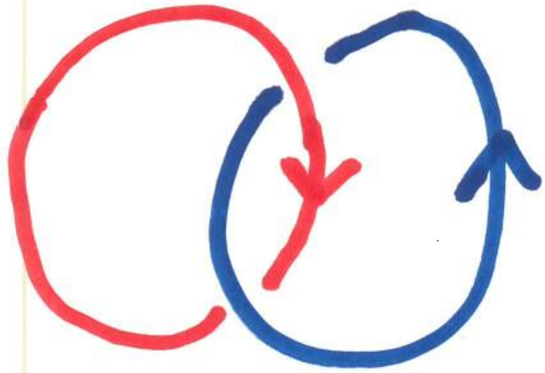
  $\mathrm{Im}\big(D(1,-)\big) = K'$

A <u>link</u> is a disjoint union of knots,

e.g.

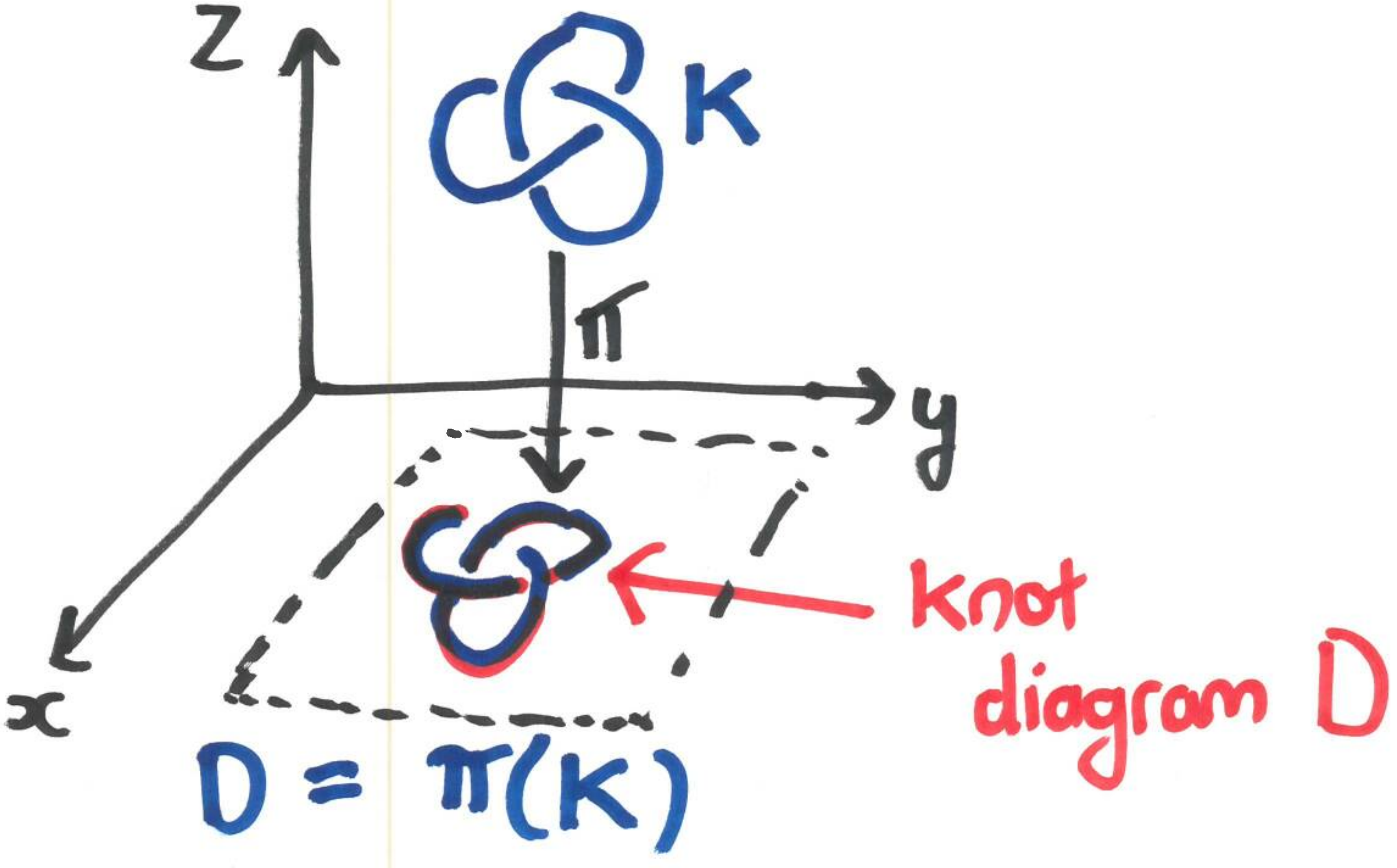


A link is <u>oriented</u> if each component is oriented, eg.



A **knot diagram** is a closed loop in the plane with over/under crossing info indicated.

By projecting a knot onto the plane, we get a knot diagram:



Theorem (Reidemeister, 1927)

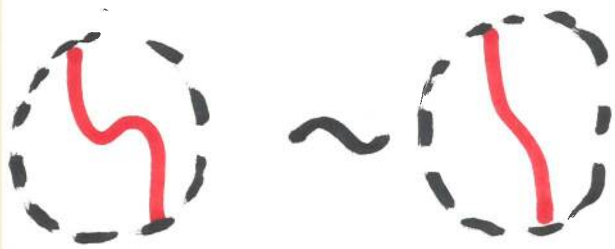
Two knots $K, K' \subseteq \mathbb{R}^3$ are equivalent

$$\Longleftrightarrow$$

their corresponding diagrams $D, D' \subseteq \mathbb{R}^2$ can be transformed into each other by the followi

local moves:

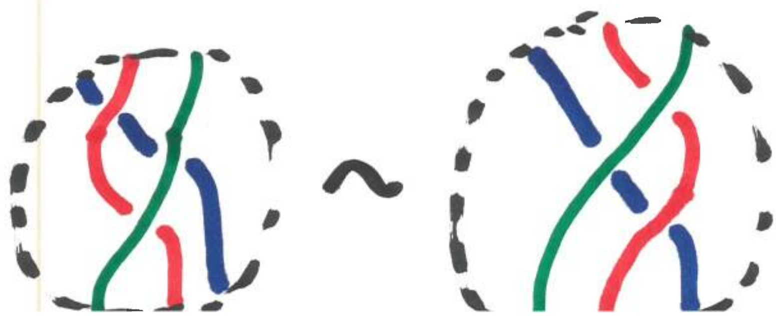
R0



R1

R2

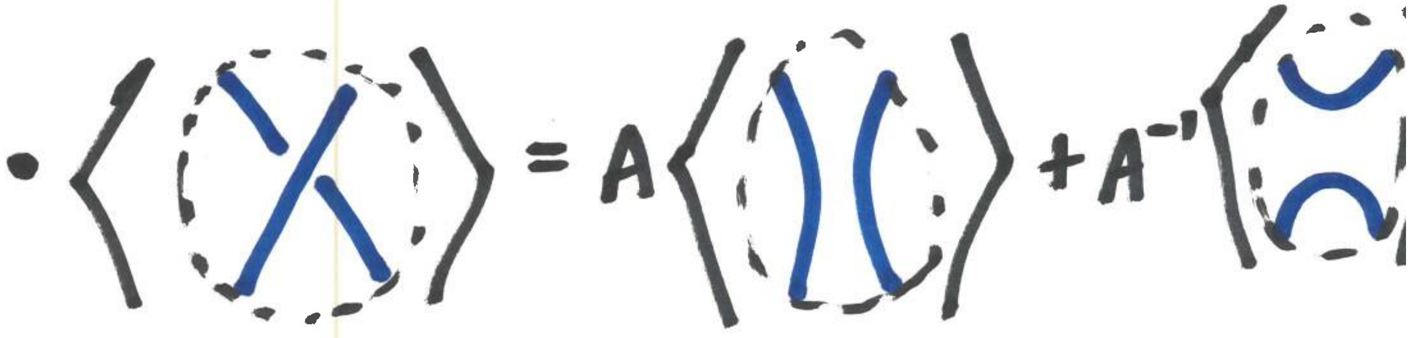
R3



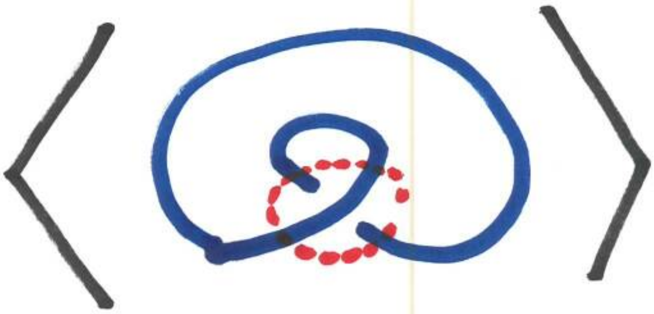
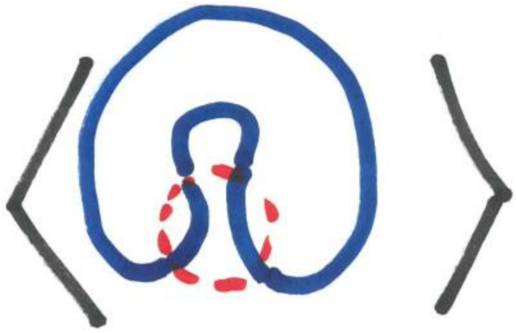
2. The Kauffman bracket

The **Kauffman bracket** $\langle D \rangle$ of an unoriented link diagram D is a polynomial in A, A^{-1} defined recursive[ly] via:

$\langle D \rangle$ is

- invariant under planar deformation of D
- 
- $\langle D \sqcup U \rangle = (-A^2 - A^{-2})\langle D \rangle$

  $\sqcup$ = "disjoint union"
  U = "unknot" $\bigcirc$
- $\langle U \rangle = 1.$

 $= A$ 

$+ A^{-1}$ 

$$= A \underbrace{\langle \bigcirc \rangle}_{=1} + A^{-1} \underbrace{\langle \bigcirc \rangle}_{=1} \cdot (-A^{2} - A^{-2})$$

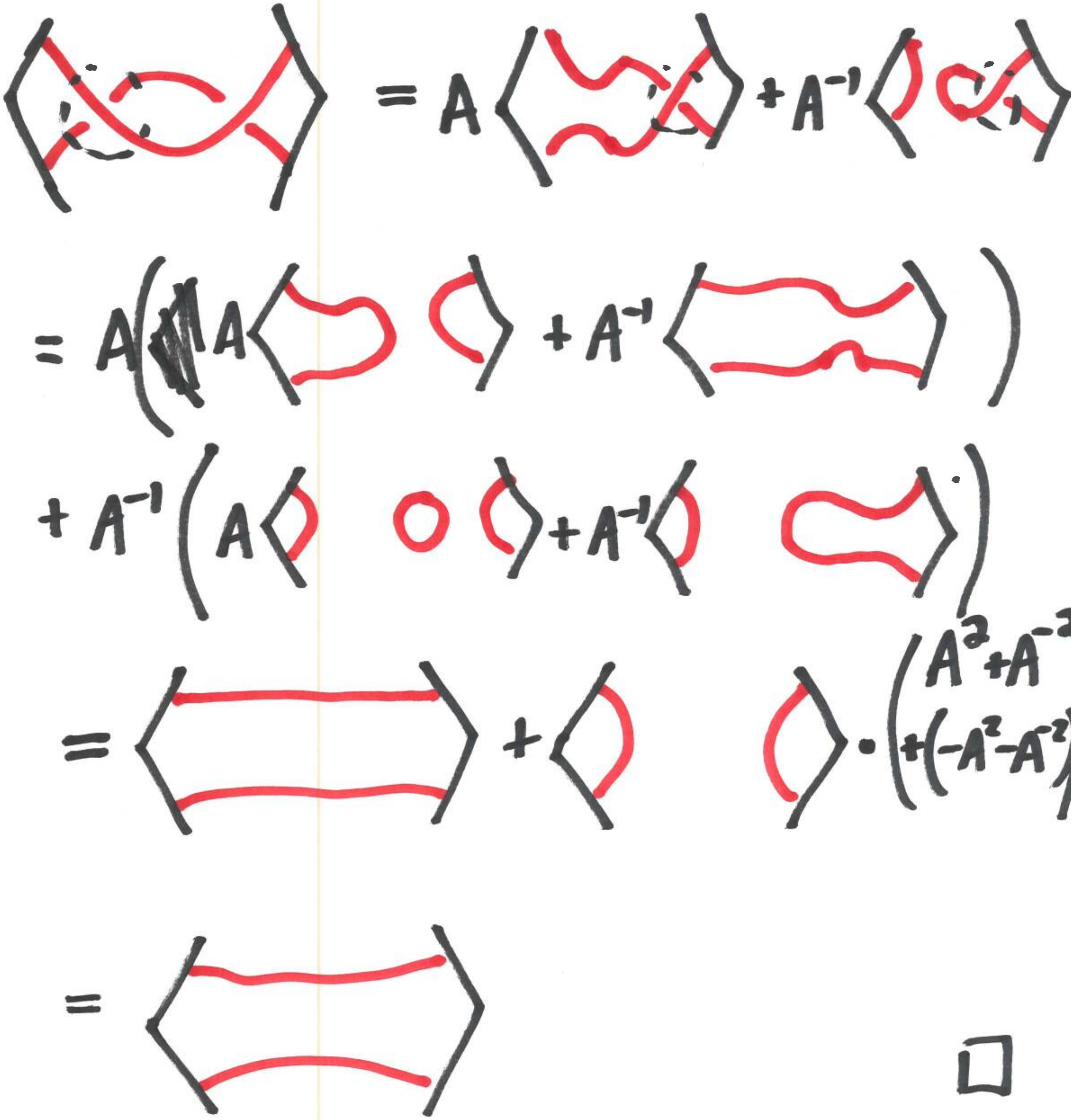
$\langle \bigcirc \rangle = 1$

$$= A \cancel{+} - A^{+1} - A^{-3}$$

$$= -A^{-3}.$$

<u>Lemma</u> $\langle\ \rangle$ is invariant under R2 and R

<u>Proof</u> (for R2)



Note: $\langle \ \rangle$ is <u>not</u> invariant under R1, since

$\langle$  $\rangle = -A^{-3} \langle$  $\rangle$

(1984)

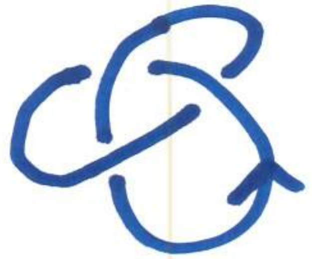
The <u>Jones polynomial</u> of an oriented link $\vec{L} \subseteq \mathbb{R}^3$ is defined as:

$$V(\vec{L}) := \left(-A^3\right)^{-w(\vec{D})} \langle D \rangle$$

where:

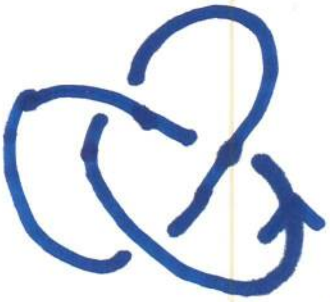
- $\vec{D} \subseteq \mathbb{R}^2$ is the oriented link diagram representing $\vec{L} \subseteq \mathbb{R}^3$
- $w(\vec{D}) := \#($  crossings $)$ $- \#($  crossings

writhe

$V\Big($  $\Big) = (-A^{3})^{-\boxed{-3}}(A^{7}-A^{3}-A^{-5}$

left-handed trefoil

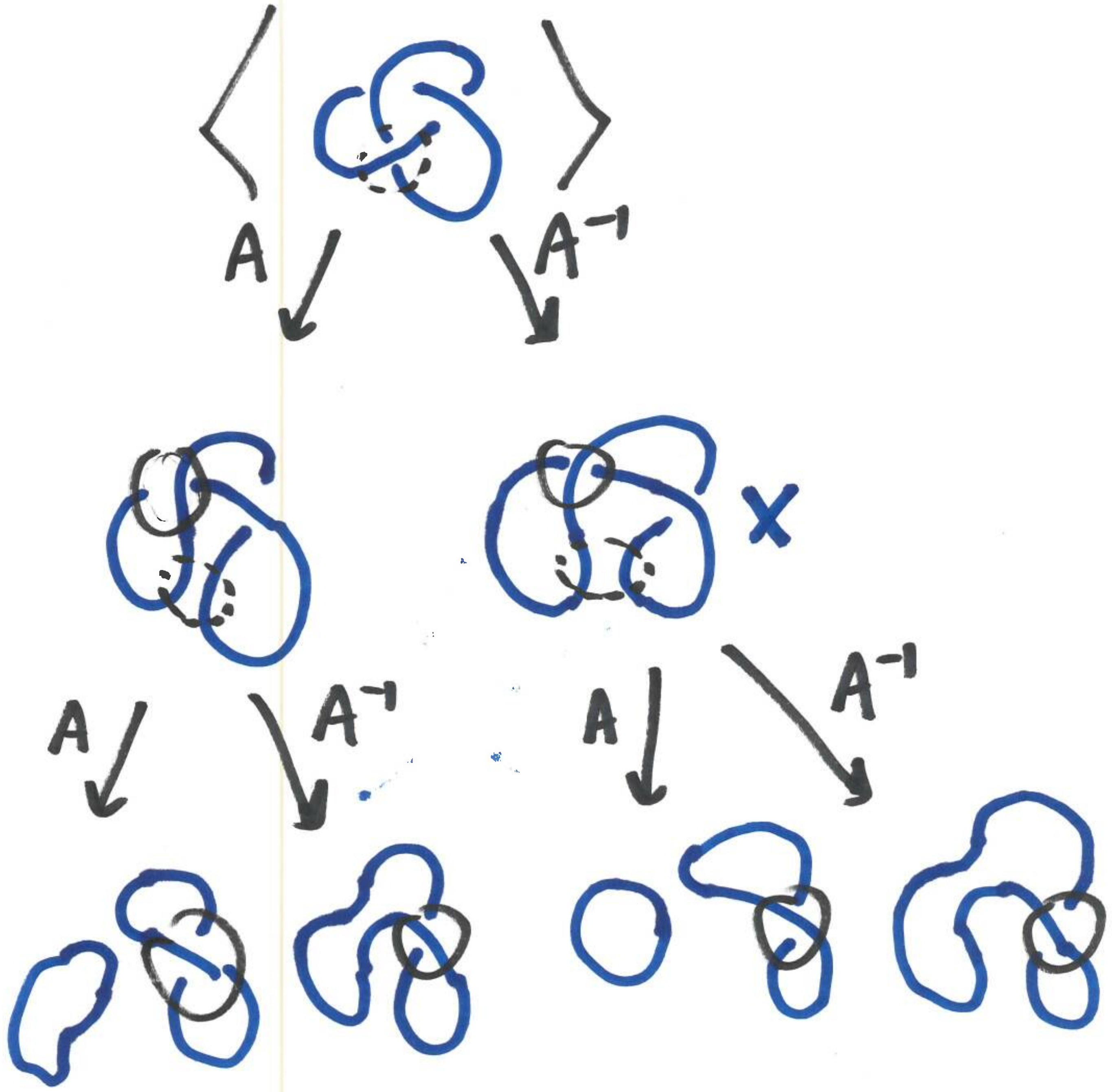
$= -A^{16} + A^{12} + A^{4}$

$V\Big($  $\Big) = (-A^{3})^{-\boxed{+3}}(A^{-7}-A^{-3}-A^{5}$

right-handed trefoil

$= -A^{-16} + A^{-12} + A^{-4}$

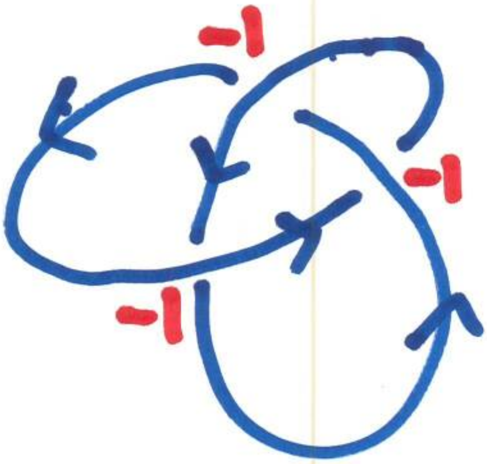
$\therefore$ left-handed trefoil $\not\simeq$ right-handed trefoil



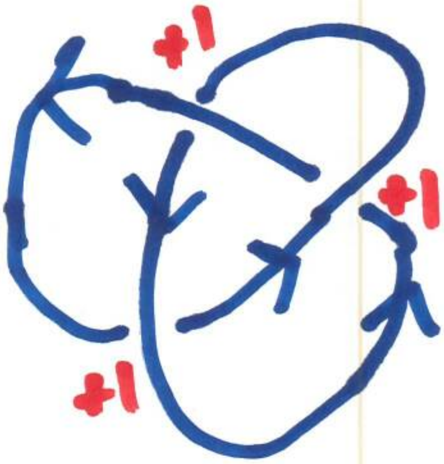
$$= A^{2}\cdot(-A^{3})\cdot(-A^{2}-A^{-2}) + 1\cdot(-A^{3}) + 1(-A^{3}) + A^{-2}(-A^{3})\cdot(-A^{2}-A^{-2})$$

$$\vdots$$

$$= A^{7} - A^{3} - A^{-5}$$



Writhe = -3



Writhe = 3