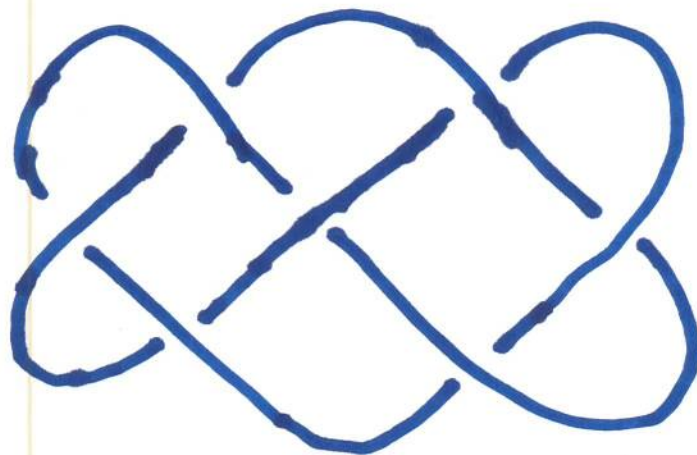


# The Jones polynomial of knots

Stellenbosch University  
Mathematics Society  
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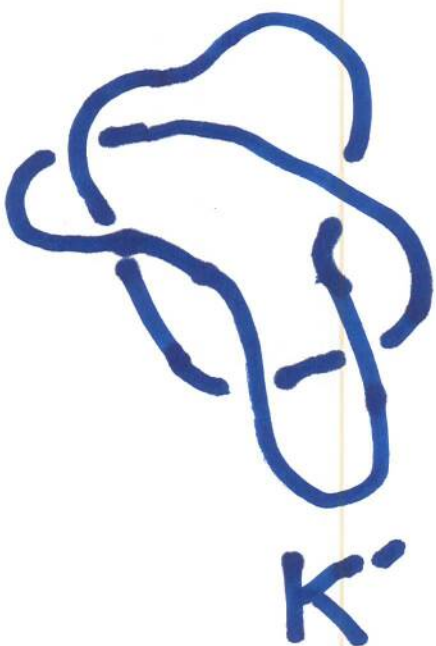




A knot  $K$  is a closed loop of string in space.



Two knots  $K, K'$  are equivalent if  $K$  can be deformed into  $K'$  (without cutting the string).



$\sim$

$K$

A knot  $K \subseteq \mathbb{R}^3$  is the image of a map

$$f: [0, 1] \longrightarrow \mathbb{R}^3$$

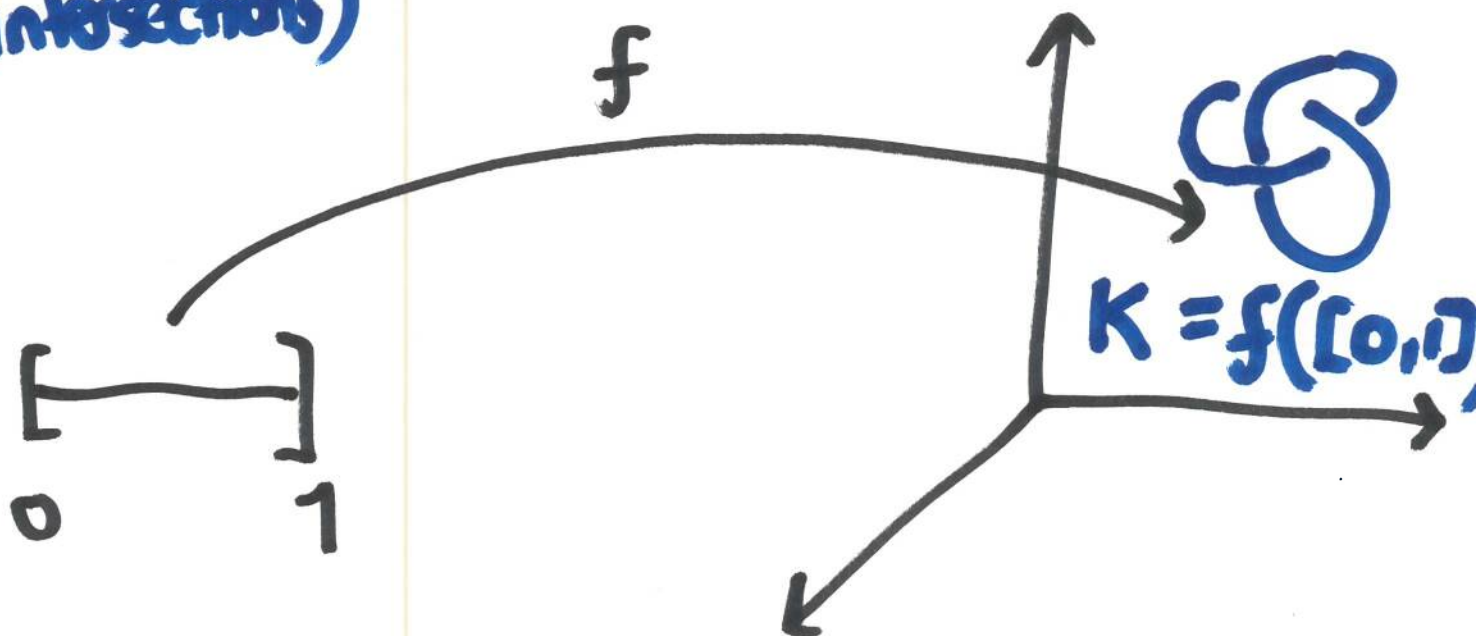
Such that:

(closed) •  $f(0) = f(1)$

(smooth) [ •  $\frac{d^n f}{dt^n}$  exists for all  $n$

[ •  $f'(t) \neq \vec{0}$  for all  $t \in [0, 1]$

(no self-intersections) •  $f$  is injective on  $(0, 1)$



Two knots  $K, K'$  are equivalent if there exists a smooth map ("isotopy")

$$D: [0,1] \times [0,1] \longrightarrow \mathbb{R}^3$$

Such that :

$$\bullet D(t, -) : [0,1] \longrightarrow \mathbb{R}^3$$

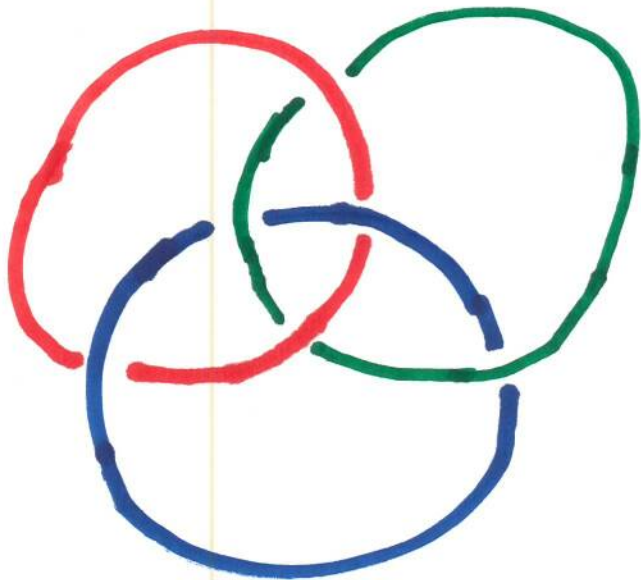
is a knot for each  $t \in [0,1]$

$$\bullet \text{Im}(D(0, -)) = K$$

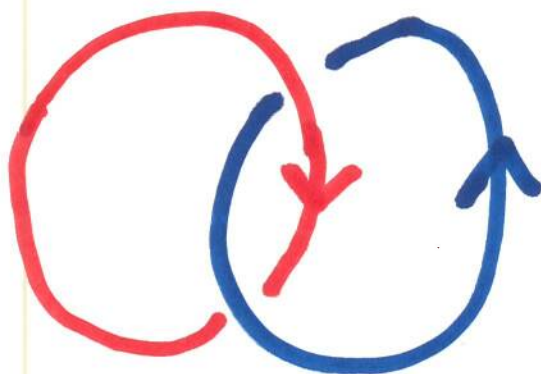
$$\text{Im}(D(1, -)) = K'$$

A link is a disjoint union of knots,

e.g.



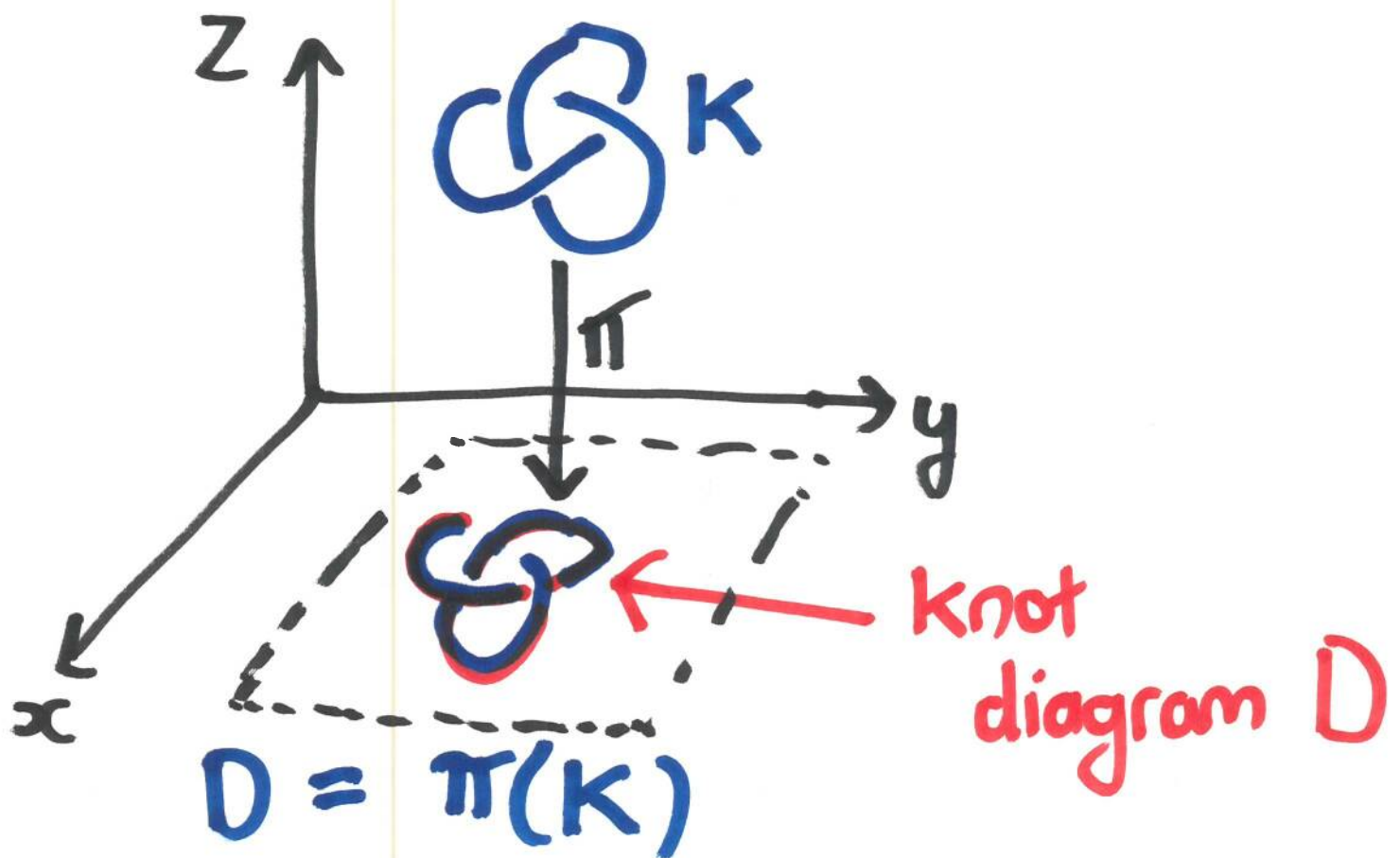
A link is oriented if each component is oriented, e.g.





A knot diagram is a closed loop in the plane with over/under crossing info indicated.

By projecting a knot onto the plane, we get a knot diagram:



# Theorem (Reidemeister, 1927)

Two knots  $K, K' \subseteq \mathbb{R}^3$  are equivalent



their corresponding diagrams  $D, D' \subseteq \mathbb{R}^2$  can be transformed into each other by the following

local moves :

R0



R1



R2



R3





## 2. The Kauffman bracket

The Kauffman bracket  $\langle D \rangle$  of an unoriented link diagram  $D$  is a polynomial in  $A, A^{-1}$  defined recursively via:

- $\langle D \rangle$  is invariant under planar deformation of  $D$

- $\langle \text{crossing} \rangle = A \langle \text{positive crossing} \rangle + A^{-1} \langle \text{negative crossing} \rangle$

- $\langle D \sqcup U \rangle = (-A^2 - A^{-2}) \langle D \rangle$

$\sqcup$  = "disjoint union"

$U$  = "unknot" 

- $\langle U \rangle = 1$

$$\langle \text{Diagram 1} \rangle = A \langle \text{Diagram 2} \rangle + A^{-1} \langle \text{Diagram 3} \rangle$$

Diagram 1: A blue circle with a smaller blue circle inside, and a red dashed circle around the inner one.

Diagram 2: A blue circle with a blue figure-eight shape inside, and a red dashed circle around the figure-eight.

Diagram 3: A blue circle with a blue circle inside, and a red dashed circle around the inner one.



$$= A \underbrace{\langle \text{Diagram 4} \rangle}_{=1} + A^{-1} \underbrace{\langle \text{Diagram 5} \rangle}_{=1} \cdot (-A^2 - A^{-3})$$

Diagram 4: A simple blue circle.

Diagram 5: A simple blue circle.

$$\langle \text{Diagram 6} \rangle = 1$$

Diagram 6: A simple blue circle.

$$= A - A^{-1} - A^{-3}$$

$$= -A^{-3}$$



Lemma  $\langle \rangle$  is invariant under  $R_2$  and  $R$

Proof (for  $R_2$ )

$$\langle \text{Diagram 1} \rangle = A \langle \text{Diagram 2} \rangle + A^{-1} \langle \text{Diagram 3} \rangle$$

$$= A \left( A \langle \text{Diagram 4} \rangle + A^{-1} \langle \text{Diagram 5} \rangle \right)$$

$$+ A^{-1} \left( A \langle \text{Diagram 6} \rangle + A^{-1} \langle \text{Diagram 7} \rangle \right)$$

$$= \langle \text{Diagram 8} \rangle + \langle \text{Diagram 9} \rangle \cdot \left( \begin{matrix} A^2 + A^{-2} \\ +(-A^2 - A^{-2}) \end{matrix} \right)$$

$$= \langle \text{Diagram 10} \rangle$$

□

Note:  $\langle \rangle$  is not invariant under R1, since

$$\langle \text{b} \rangle = -A^{-3} \langle \text{I} \rangle$$

(1984)

The Jones polynomial of an oriented link  $\vec{L} \subseteq \mathbb{R}^3$  is defined as:

$$V(\vec{L}) := (-A^3)^{-w(\vec{D})} \langle D \rangle$$

where:

- $\vec{D} \subseteq \mathbb{R}^2$  is the oriented link diagram representing  $\vec{L} \subseteq \mathbb{R}^3$

- $w(\vec{D}) := \# \left( \begin{array}{c} \nearrow \\ \searrow \end{array} \text{ crossings} \right) - \# \left( \begin{array}{c} \nwarrow \\ \nearrow \end{array} \text{ crossings} \right)$

writhe



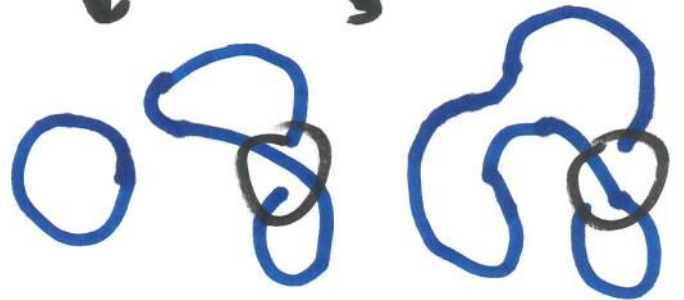
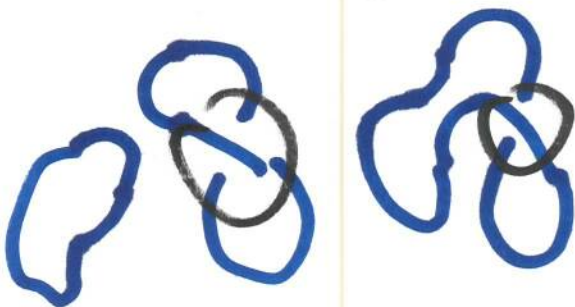
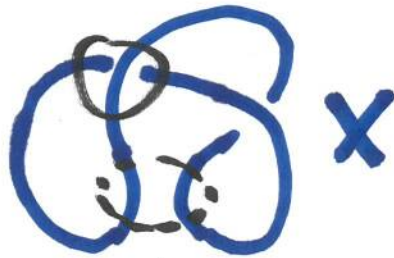
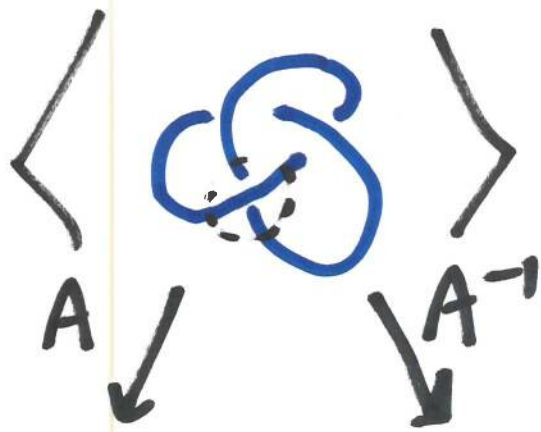
$$V \left( \begin{array}{c} \text{left-handed} \\ \text{trefoil} \end{array} \right) = (-A^3)^{\boxed{-3}} (A^7 - A^3 - A^{-5})$$

$$= \cancel{A^{17}} - A^{16} + A^{12} + A^4$$

$$V \left( \begin{array}{c} \text{right-handed} \\ \text{trefoil} \end{array} \right) = (-A^3)^{\boxed{+3}} (A^{-7} - A^{-3} - A^{-5})$$

$$= -A^{-16} + A^{-12} + A^{-4}$$

$\therefore$  left-handed trefoil  $\neq$  right-handed trefoil

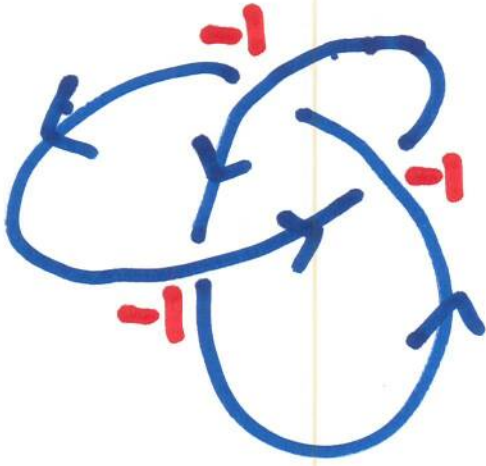


$$= A^2 \cdot (-A^3) + 1 \cdot (-A^3) + 1(-A^3) + A^{-2}(-A^3) \\ \cdot (-A^2 - A^{-2}) \quad \cdot (-A^2 - A^{-2})$$

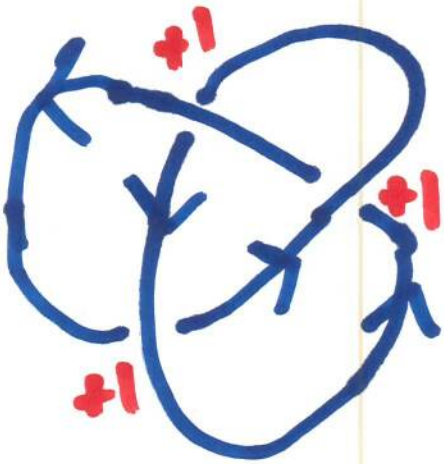
⋮

$$= A^7 - A^3 - A^{-5}$$





writhe = -3



writhe = 3