Feynman's Fabulous Formula in the Ising Model

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Mathematics Postgraduate Seminar

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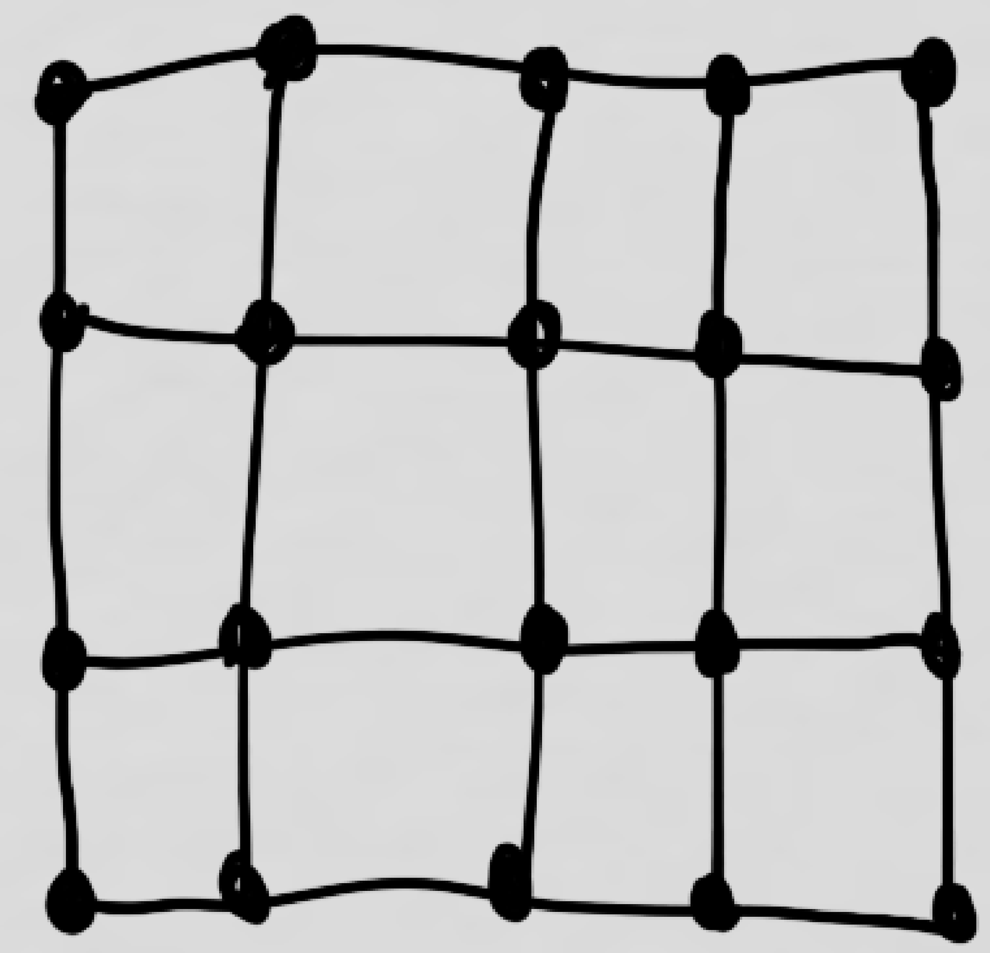
- Ising was his PhD student. He proved: the model doesn't work in 1d!

- But, in 1936, Peierls showed the 2d model <u>does</u> exhibit residual magnetization. Exact solution in 1944 by Onsager, for zero magnetic field.

Planar graph $G \subseteq \mathbb{R}^2$, eg.



N vertices

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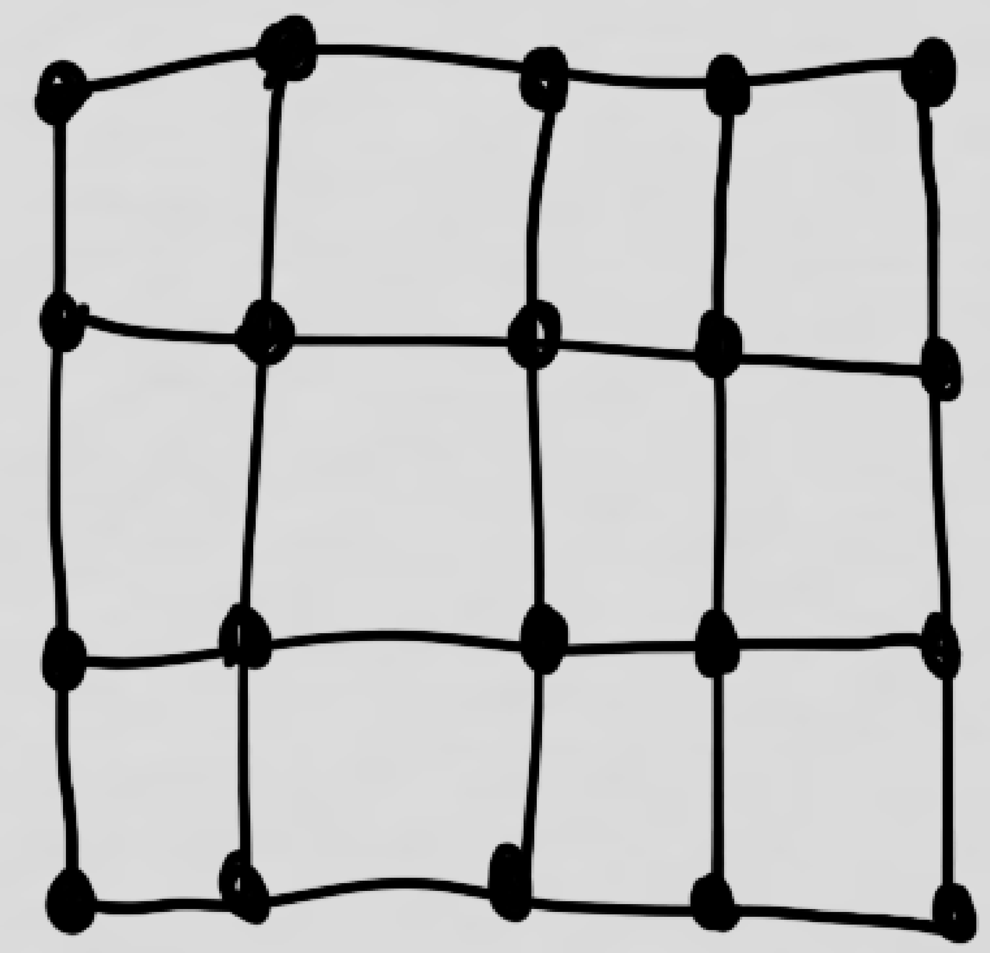

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Spin configuration $\sigma : V(G) \longrightarrow \{-1, +1\}$

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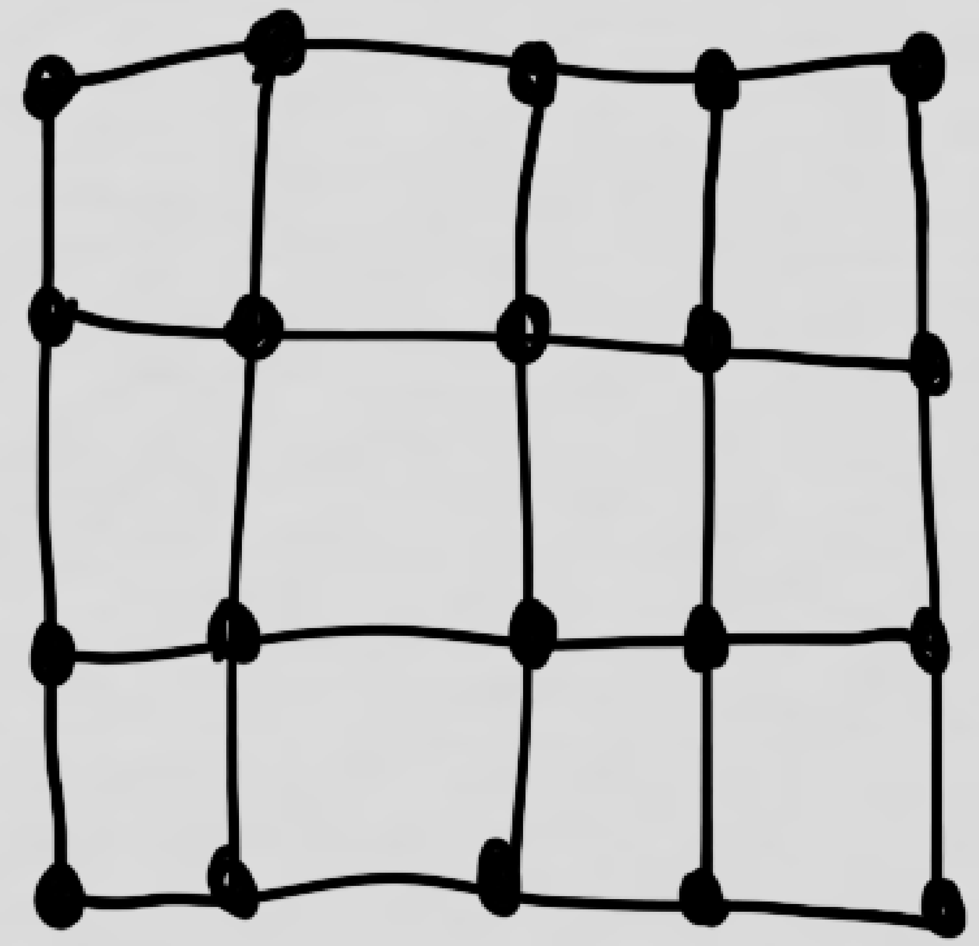

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coupling   external field

Energy $$H(\sigma) := -\sum_{\text{edges } e=\{i,j\}} J_e \, \sigma_i \sigma_j \; - L \sum_{\text{vertices } i} \sigma_i \quad ; \; \{J_e\}, L \geq 0$$

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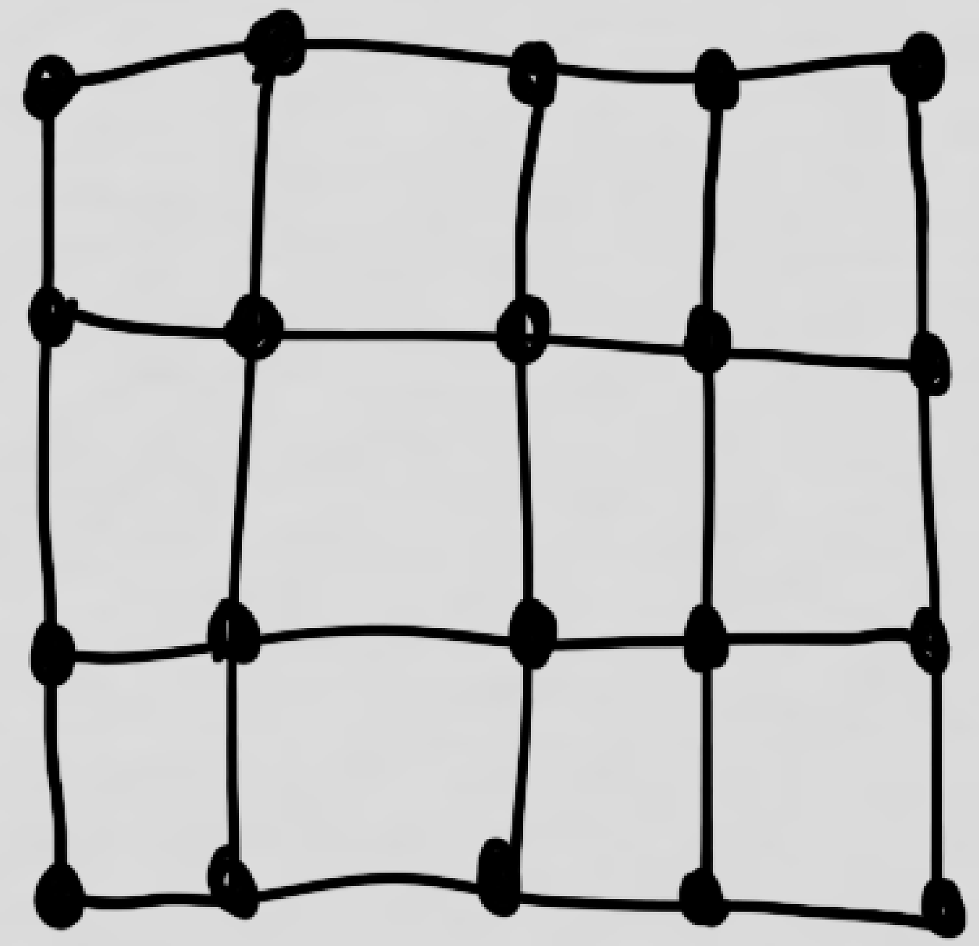

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$$\mathrm{Prob}(\sigma) := \frac{e^{-\beta H(\sigma)}}{Z}, \quad \beta := \frac{1}{kT}$$

$$\boxed{Z(\beta, J, L) := \sum_{\sigma} e^{-\beta H(\sigma)} \qquad \text{partition function}}$$

average value of f

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square lattice

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$$x_e = \tanh(\beta J_e)$$

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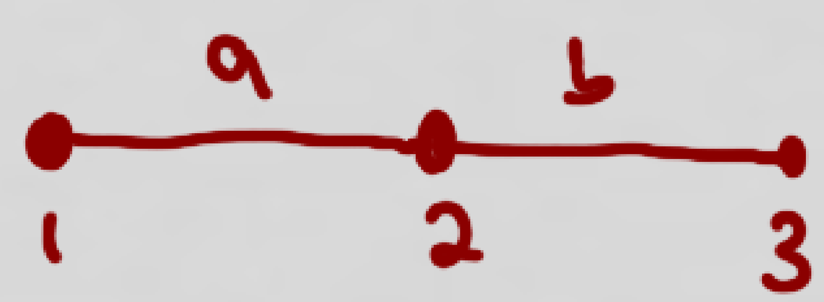
Consider eg. $G =$ 

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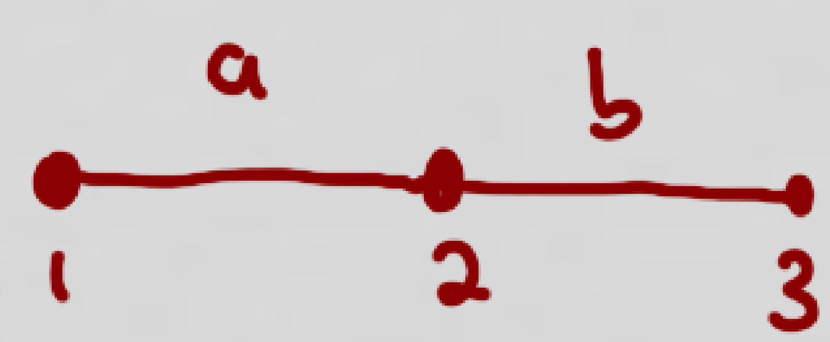
Consider eg. $G =$ 

$$(1 + x_a \sigma_1 \sigma_2)(1 + x_b \sigma_2 \sigma_3)$$

$$= 1 + x_a \sigma_1 \sigma_2 + x_b \sigma_2 \sigma_3 + x_a x_b (\sigma_1 \sigma_2)(\sigma_2 \sigma_3)$$

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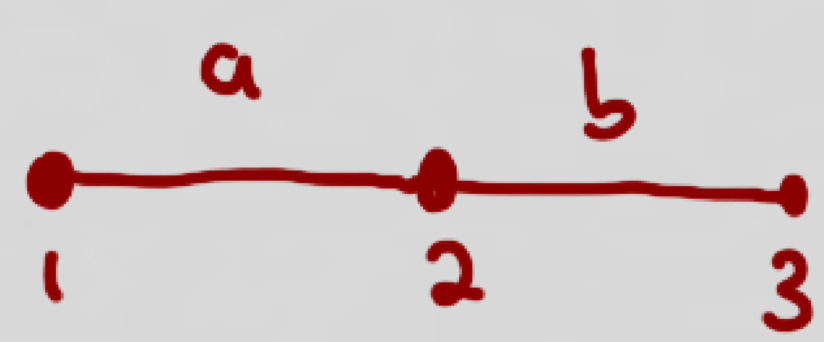
Consider eg. $G =$ 

$$(1 + x_a \sigma_1 \sigma_2)(1 + x_b \sigma_2 \sigma_3)$$

$$= 1 + x_a \sigma_1 \sigma_2 + x_b \sigma_2 \sigma_3$$

$$+ x_a x_b (\cancel{\sigma_1} \cancel{\sigma_2})(\cancel{\sigma_2} \cancel{\sigma_3})$$

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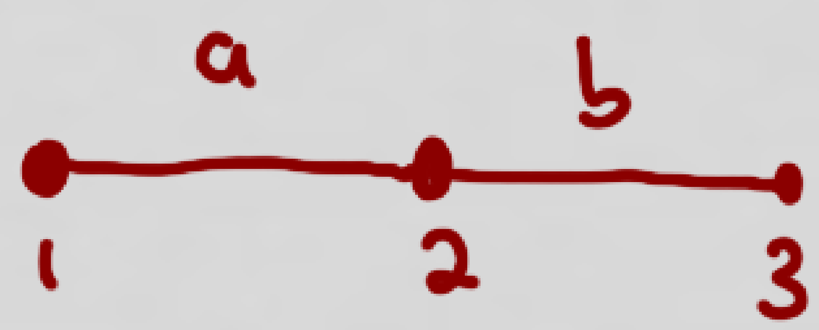
Consider eg. $G =$ 

$$(1 + x_a \sigma_1 \sigma_2)(1 + x_b \sigma_2 \sigma_3)$$

$= 0$ when we sum over all σ

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$$\therefore \boxed{Z = c\, 2^N \sum_{\substack{\text{even subgraphs} \\ H \subseteq G}} \prod_{e \in E(H)} x_e}$$

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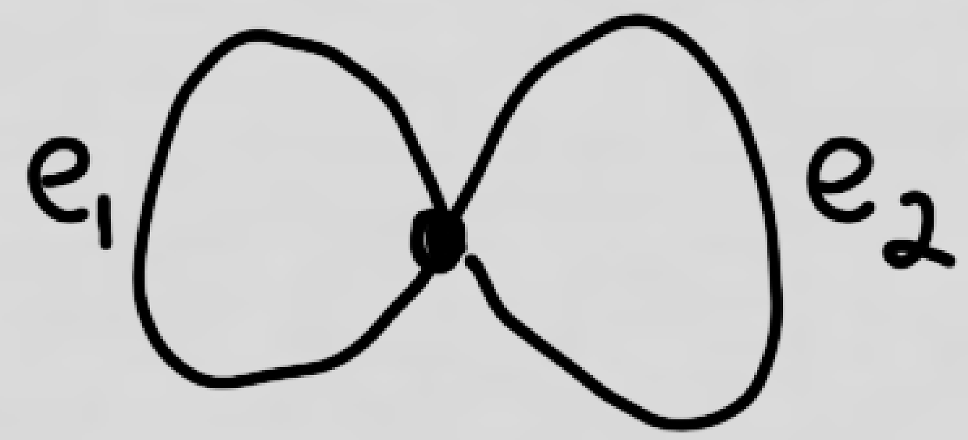
$G =$ 

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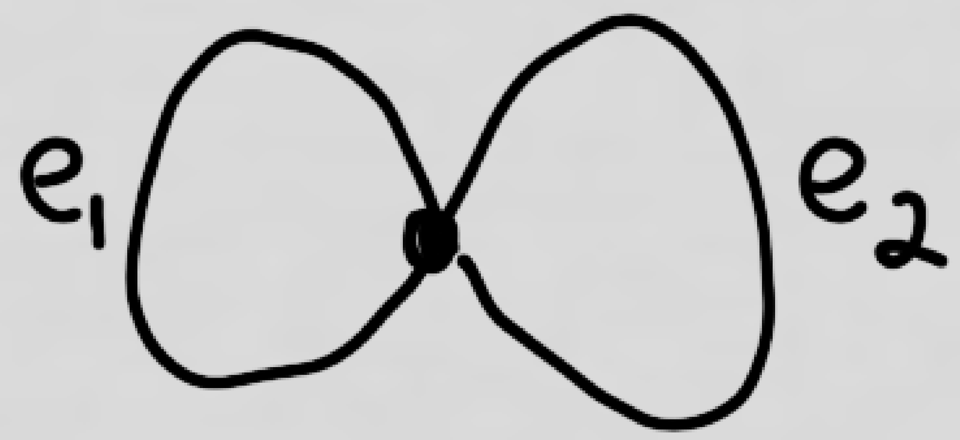
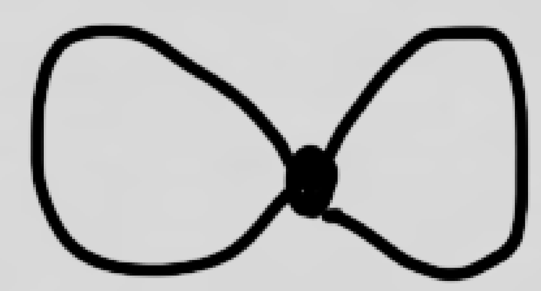
$G =$ 

4 even subgraphs : empty ,  ,  , 

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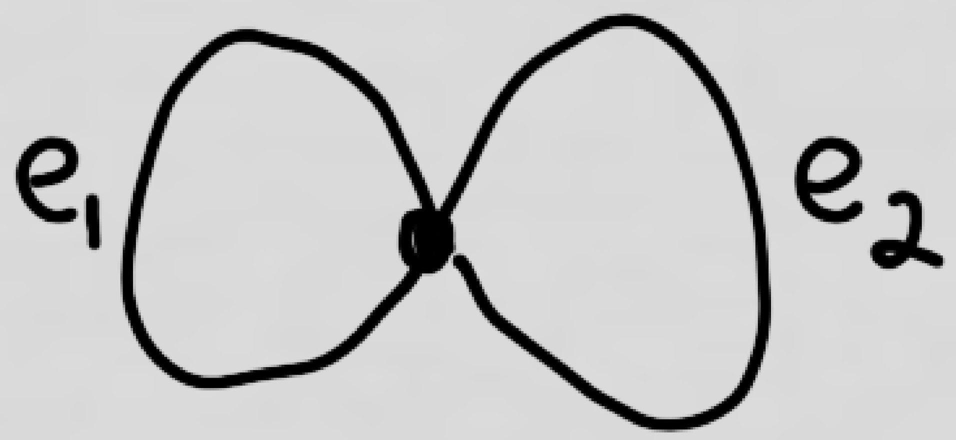
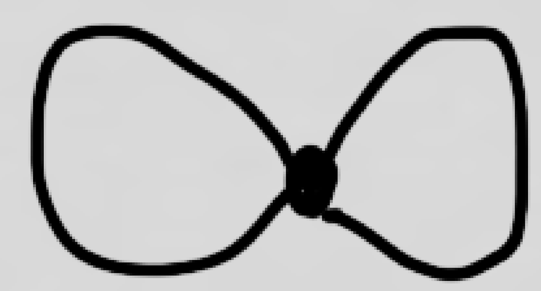
$G =$ 

4 even subgraphs : empty ,  ,  , 

$$P = 1 + x_1 + x_2 + x_1 x_2$$

Feynman's Fabulous Formula For any finite planar graph G,

$$\sum_{\substack{\text{even subgraphs} \\ H \subseteq G}} x(H) = \prod_{[\vec{\gamma}] \in P(G)} \left(1 - (-1)^{w[\vec{\gamma}]} x[\vec{\gamma}] \right)$$

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set of prime, reduced, unoriented closed paths in G.

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winding number of $\vec{\gamma}$

Feynman's Fabulous Formula For any finite planar graph G,

product of x_e along γ

$$\sum_{\substack{\text{even subgraphs} \\ H \subseteq G}} x(H) = \prod_{[\vec{\gamma}] \in P(G)} \left(1 - (-1)^{w[\vec{\gamma}]} x[\vec{\gamma}] \right)$$

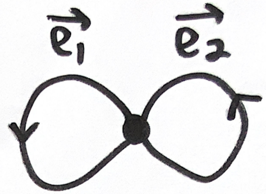
$P(G)$ = all prime, reduced, unoriented closed paths in G.

an <u>oriented edge</u> $\vec{e}$:= an unoriented edge e equipped with an orientation

An <u>oriented closed path</u> $\vec{\gamma}$ is a word

$$\vec{e_1}\,\vec{e_2}\,\ldots\,\vec{e_n}$$

of composable oriented edges.



G

$$p = 1 + x_1 + x_2 + x_1 x_2$$

$[\vec{\gamma}]$	$1-(-1)^{w(\gamma)}x(\gamma)$
$[\vec{e_1}]$	$1+x_1$
$[\vec{e_2}]$	$1+x_2$
$[\vec{e_1}\vec{e_2}]$	$1+x_1x_2$
$[\vec{e_1}\vec{e_2}^{-1}]$	$1-x_1x_2$
$[\vec{e_1}^{2}\vec{e_2}^{-1}]$	$1+x_1^2x_2$
$\vdots$	

Cimasoni's generalization

For any finite graph G,

$$\sum_{\substack{\text{even subgraphs} \\ H \subseteq G}} x(H) = \frac{1}{2^g} \sum_{\lambda \in \mathrm{Spin}(\Sigma)} (-1)^{\mathrm{Arf}(\lambda)} \cdots \prod_{[\vec{\gamma}] \in P(G)} \left(1 - (-1)^{w_\lambda[\vec{\gamma}]} x(\gamma)\right).$$

where Σ is any surface in which G embeds.

Kelvin-Stokes Theorem