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Elmar Teufl · **Stephan Wagner**[*]

Resistance scaling and the number of spanning trees in self-similar lattices

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Abstract The problem of enumerating spanning trees in self-similar lattices was recently introduced to the literature by Chang, Chen and Yang, who determined explicit formulae in the case of Sierpiński graphs and some of their generalizations. The aim of this note is to show that their results hold in more generality and that there is a strong relation between this enumeration problem and resistance scaling on self-similar lattices.

Keywords self-similar lattices · electrical networks · resistance scaling · spanning trees

Mathematics Subject Classification (2000) 05C30 · 05C05 · 82B20

1 Introduction

Enumeration of spanning trees and the analysis of electrical networks are closely related as it was already shown in the fundamental work of Kirchhoff [19]. This interplay was further explored in various directions, see for instance [26, 37]; in

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Elmar Teufl
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
Germany

Stephan Wagner
Department of Mathematical Sciences
Stellenbosch University
Private Bag X1
Matieland 7602
South Africa

particular, the famous Matrix-Tree-Theorem (see for instance [2]) yields a practical method to compute the number of spanning trees in graphs, which was used in numerous works. The number of spanning trees is also of interest in statistical physics, since it corresponds to a special $q \to 0$ limit of the partition function of the q-state Potts model [11,39]. There are also other interesting ties to dimer coverings [32] and sandpile models [8].

A large number of results in the physical literature are concerned with the number of spanning trees in two- and higher-dimensional lattices [7,30,31,38]; recently, Chang, Chen, and Yang [6] considered this problem for the Sierpiński gasket and its variants. The Sierpiński gasket is probably the most classical example of a self-similar fractal lattice: in contrast to the translational invariance of lattices such as the square lattice or the honeycomb, one of the main properties of self-similar lattices is scaling-invariance. Many other models of statistical physics have been investigated on self-similar lattices (in particular the Sierpinski gasket) as well, see [4,5,10,12,13,15,16].

The results of Chang, Chen, and Yang are mainly based on the analysis of systems of recurrences. The aim of this paper is a continuation of their research: especially we aim to

- generalize their results to an entire class of self-similar lattices,
- establish a relation between the asymptotic growth of the number of spanning trees and so-called (resistance) renormalization on these lattices, and
- prove a conjecture of [6].

The aforementioned conjecture was also proven by the authors in [35] using different methods that made use of the high degree of symmetry. Here we aim to treat the problem in more generality. The main tool that we are going to use is a technique that was recently developed by the authors in [36]. Shortly, the main theorem of [36] states the following: If a part of a graph is substituted by an electrically equivalent part, then the weighted number of spanning trees (where the weight of a spanning tree is the product of the conductances of its edges) changes by a factor depending on the substituted graphs only.

Our paper is organized as follows:

- In Section 2 basic notions concerning the theory of electrical networks are recalled and the authors' method from [36] is explained briefly.
- Section 3 provides an inductive construction scheme for self-similar lattices. Furthermore, renormalization of resistances/conductances on self-similar lattices is discussed.
- Section 4 contains the main results: The asymptotic growth of the number of spanning trees on self-similar lattices is determined and a relation to renormalization is revealed.

Several examples are provided for illustration.

2 Electrical networks

The vertex (site) set of a graph G is denoted by VG and the edge (bond) set is denoted by EG. In the following graphs are allowed to have parallel edges and loops. An (electrical) network is an edge-weighted graph, i. e., a weight (conductance) $c(e)$ is assigned to each edge e of G. Graphs without explicit conductances are considered as electrical networks with unit conductances, i. e., $c(e) = 1$ for each edge e. The (weighted) Laplace matrix $L = L_G$ of a network G is defined as follows:

$$L_{x,y} = -\sum_{\substack{e\in EG\\ e \text{ connects } x,y}} c(e) \qquad \text{and} \qquad L_{x,x} = -\sum_{\substack{z\in VG\\ z\neq x}} L_{x,z}$$

for distinct vertices x,y of G. We say that two networks F and G are *electrically equivalent* with respect to $B \subseteq VF \cap VG$, if they cannot be distinguished by applying voltages to B and measuring the resulting currents on B. As a consequence of Kirchhoff's current law two networks F and G are electrically equivalent if the rows corresponding to the vertex set B of the matrices $L_F H_B^F$ and $L_G H_B^G$ are equal, where H_B^F is the matrix associated to harmonic extension. A special situation of electrical equivalence is the trace operation on networks: If F and G are networks with $VF \subseteq VG$ and F and G are equivalent with respect to VF then the network F is called the *trace* of G with respect to the vertices of F. In terms of Laplace matrices traces are Schur complements: Write $B = VF$ and $C = VG \setminus B$, then

$$L_F = (L_G)_{BB} - (L_G)_{BC} \cdot (L_G)_{CC}^{-1} \cdot (L_G)_{CB}, \tag{1}$$

where $(L_G)_{BC}$ denotes the submatrix of L_G with rows corresponding to B and columns corresponding to C. If the inverse of $(L_G)_{CC}$ does not exist, it must be replaced by the Moore-Penrose generalized inverse, see [24].

A graph T is a *tree*, if T is connected and does not contain cycles. A subgraph H of a graph G is called *spanning* if $VH = VG$. See for example [2] for these and other graph-theoretical notions. Given a network G we write $N_{ST}(G)$ to denote the weighted number of spanning trees in G:

$$N_{ST}(G) = \sum_T \prod_{e\in ET} c(e),$$

where the sum is taken over all spanning trees T of G. If G is equipped with unit conductances then $N_{ST}(G)$ is the usual number of spanning trees. The following theorem was proven in [36] and is the main tool in the following.

Theorem 1 *Suppose that a network X can be decomposed into G and H, so that EG and EH are disjoint, $EX = EG \cup EH$, and $VX = VG \cup VH$. We set $B = VG \cap VH$. Let H' be a network with $EG \cap EH' = \varnothing$ and $VG \cap VH' = B$, such that H and H' are electrically equivalent with respect to B, and assume that $N_{ST}(H) \neq 0$ and $N_{ST}(H') \neq 0$. Then*

$$\frac{N_{ST}(X)}{N_{ST}(H)} = \frac{N_{ST}(X')}{N_{ST}(H')}.$$

3 Self-similar lattices and renormalization

We consider finite approximations $X_0, X_1, \ldots$ to self-similar lattices of the following type: let Z be a template graph with a tuple $\mathbf{z}$ of θ distinguished vertices and s "holes" described by a tuple of θ vertices for each hole. Let X_0 be a graph and $\mathbf{x}_0$ be a tuple of θ distinguished vertices. The graph X_1 is obtained by filling the holes of Z with s copies of X_0, i. e., the vertices of a hole are identified with the distinguished vertices of the associated copy of X_0. Furthermore, the vertices corresponding to those in $\mathbf{z}$ are used as tuple $\mathbf{x}_1$ of distinguished vertices for X_1. We write $Z(X_0)$ to denote the result X_1 of this construction (keeping distinguished vertices in mind). Now this procedure is repeated in order to get the graphs $X_2 = Z(X_1), X_3 = Z(X_2), \ldots$ with distinguished vertices $\mathbf{x}_2, \mathbf{x}_3, \ldots$ A rigorous description of this copy-construction can be found in [34]. In order to illustrate the construction above let us give the following examples.

Example 1 The modified Koch curve is a simple but interesting variation of the classical Koch curve, see Figure 1 for an illustration of the template graph Z and the construction of the associated graph sequence (distinguished vertices are drawn bold). The spectrum of the Laplace operator on these graphs was studied in [21].

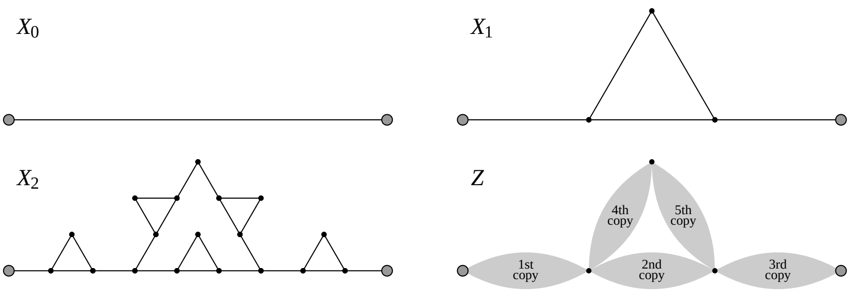

Fig. 1 Modified Koch graphs X_0, X_1, X_2 and their template graph Z.

Example 2 The construction of the Sierpiński graphs $X_0, X_1, X_2, \ldots$ and the corresponding template graph Z is outlined in Figure 2. Notice that the template graph Z is edgeless. The number of spanning trees $N_{ST}(X_k)$ in X_k and higher dimensional analogues are studied in [6]. Variants with a larger number of subdivisions on each side of the template graph are considered in [6] as well. This yields a family of lattices with two parameters: the dimension d and the number of subdivisions m. Notice that the number of distinguished vertices is given by $\theta = d + 1$ and the number of copies is given by $s = \binom{m+d-1}{d}$.

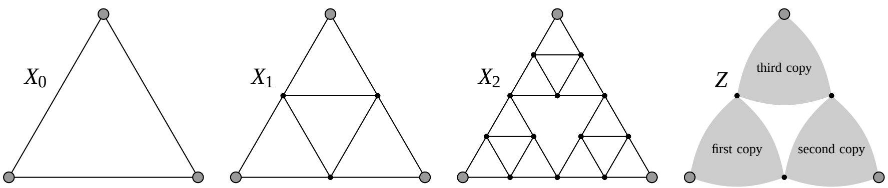

Fig. 2 Sierpiński graphs X_0, X_1, X_2 and their template graph Z.

Example 3 A slightly modified version of the Sierpiński graphs is given by the Towers of Hanoi graphs. The vertices of the graph X_k in this sequence correspond to all possible configurations of the game "Towers of Hanoi" with $k+1$ disks and three rods, whereas the edges describe transitions between configurations, see for example [17]. We remark that these graphs are finite Schreier graphs of the Hanoi tower group, see [14]. Their construction is outlined in Figure 3.

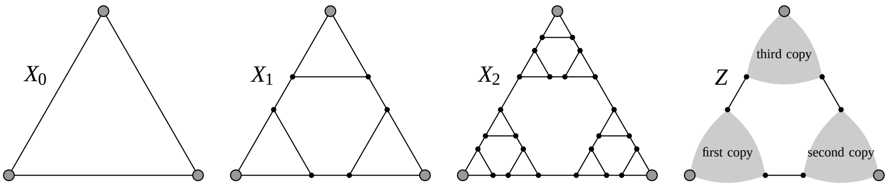

Fig. 3 The Towers of Hanoi graphs X_0, X_1, X_2 and their template graph Z.

Example 4 Another variation of the Sierpiński graphs (similar to the Towers of Hanoi graphs) is shown in Figure 4. The main point here is the existence of cycles in the template graph Z.

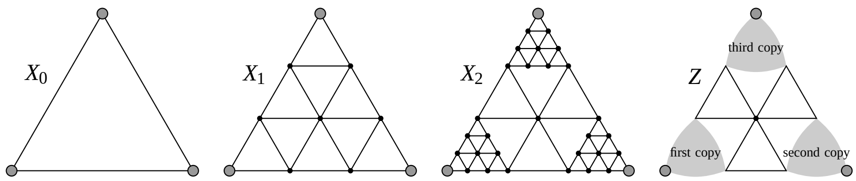

Fig. 4 The first three graphs X_0, X_1, X_2 constructed using the template Z.

Example 5 The sequence of graphs depicted in Figure 5 exhibits two phenomena, which have influence on the number of spanning trees. Firstly, the graphs in the sequence are less symmetric; secondly the template graph Z contains a cycle.

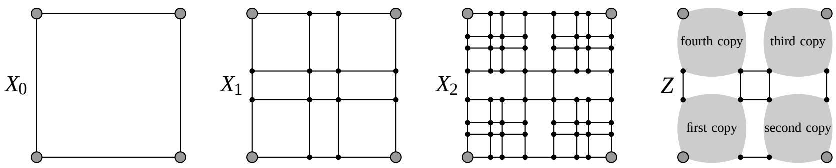

Fig. 5 The first three graphs X_0, X_1, X_2 constructed using the template Z.

Example 6 The Lindstrøm snowflake is a well-known self-similar fractal, see [20]. The approximating graphs and their template graph are shown in Figure 6.

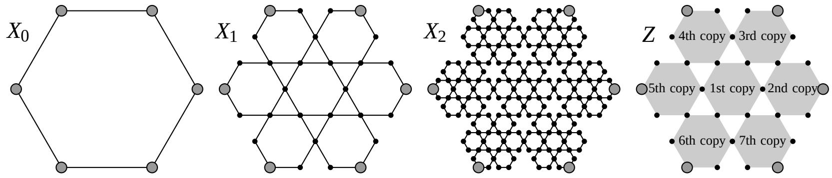

Fig. 6 The snowflake graphs X_0, X_1, X_2 and their template graph Z.

In the following we describe the notion of (conductance/resistance) renormalization on self-similar lattices, see for instance [1, 18, 27]. Let $X_0 = K_\theta$ be the complete graph with θ vertices, and fix a template graph Z and endow its edges with fixed conductances c_Z. Let $X_1, X_2, \dots$ be constructed as above. There are two natural operations for conductances on X_0 and X_1, respectively:

– Replication: If we are given conductances c_0 on X_0, then X_1 naturally inherits conductances from X_0 and Z. Let us denote these conductances on X_1 by $S(c_0)$.
– Traces: If we are given conductances c_1 on X_1, consider the trace of the network X_1 with respect to its distinguished vertices $\mathbf{x}_1$. The underlying graph of this trace is a complete graph with θ vertices, which can naturally be identified with the vertices of X_0. Hence the trace operation defines conductances on X_0, which we denote by $T(c_1)$.

The so-called renormalization map R is the composition of T and S, i. e.,

$$R = T \circ S \colon \mathbb{R}^{\binom{\theta}{2}} \to \mathbb{R}^{\binom{\theta}{2}}.$$

Note here that $X_0 = K_\theta$ has $\binom{\theta}{2}$ edges. Both the replication map S and the trace map T are rational in all coordinates, due to the representation (1) for the Laplace matrix of a trace. Thus, R is also rational in all coordinates. Moreover, if the template graph Z is edgeless, the renormalization map R is homogeneous, i. e., $R(\alpha c) = \alpha R(c)$. Generally, the renormalization map R is a rational function in the conductances c on $X_0 = K_\theta$ and c_Z on Z. Writing $R(c, c_Z)$ to emphasise the dependence on c and c_Z, we have $R(\alpha c, \alpha c_Z) = \alpha R(c, c_Z)$.

The basic question in renormalization is the dynamical behaviour of the iterated map R^n. Fix some conductances c_0 on X_0 and set $c_n = R^n(c_0)$ for $n > 0$. In well-behaved instances of the graph construction above (in particular in all our examples) it turns out that there exists a constant $\rho > 1$, so that the sequence $(\rho^n c_n)_{n \geq 0}$ is bounded from above and below by positive numbers. In this case we call ρ the *resistance scaling factor* of the self-similar lattice. Even more holds true for all examples above: There are conductances $c_\infty > 0$, so that $\rho^n c_n = c_\infty + o(1)$. Assume that the limit

$$R_\infty(c) = \lim_{\alpha \to \infty} R(c, \alpha c_Z)$$

exists and is continuous in c. Notice that this limit corresponds to the shortening (contraction) of all edges in Z. In this case $\rho^n c_n = c_\infty + o(1)$ implies

$$c_\infty = \rho R_\infty(c_\infty), \tag{2}$$

as $\rho^{n+1} c_{n+1} = \rho R(\rho^n c_n, \rho^n c_Z)$. Hence ρ and c_∞ form a solution of the non-linear eigenvalue problem above. Notice that if the template graph Z is edgeless, then $R(c) = R_\infty(c)$. Existence and uniqueness of solutions of this non-linear eigenvalue problem, as well as contractivity of R have been studied for a variety of self-similar lattices, see for instance [20,22,23,25,28]. We remark that the use of symmetries of the sequence $X_0, X_1, \dots$ often reduces the complexity of effective computations significantly.

Let us discuss renormalization and resistance scaling for the examples above:

Example 1 (continued from page 4) Endow $X_0 = K_2$ with conductance x, then the renormalization map is given by

$$R\colon x \longrightarrow \tfrac{3}{8}x.$$

Thus the resistance scaling factor is $\rho = \frac{8}{3}$.

Example 2 (continued from page 4) First, let us consider the usual (i. e., $d = 2$,

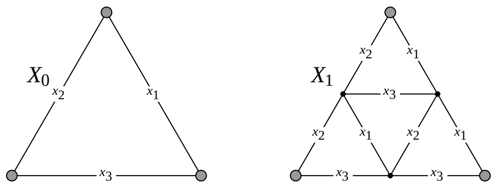

Fig. 7 Conductances on X_0 and one way to specify conductances on X_1 using replication.

$m = 2$) Sierpiński graphs $K_3 = X_0, X_1, \dots$ equipped with conductances as indicated in Figure 7, i. e., $c_0 = (x_1, x_2, x_3)$. The renormalization map is then given by

$$R\colon \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} \frac{(x_1x_2+x_2x_3+x_3x_1)(3x_1(x_1+x_2+x_3)+x_2x_3)}{6(x_1^2(x_2+x_3)+x_2^2(x_1+x_3)+x_3^2(x_1+x_2))+14x_1x_2x_3} \\ \frac{(x_1x_2+x_2x_3+x_3x_1)(3x_2(x_1+x_2+x_3)+x_1x_3)}{6(x_1^2(x_2+x_3)+x_2^2(x_1+x_3)+x_3^2(x_1+x_2))+14x_1x_2x_3} \\ \frac{(x_1x_2+x_2x_3+x_3x_1)(3x_3(x_1+x_2+x_3)+x_1x_2)}{6(x_1^2(x_2+x_3)+x_2^2(x_1+x_3)+x_3^2(x_1+x_2))+14x_1x_2x_3} \end{pmatrix}$$

Now, let us take symmetry into account: assume that $x = x_1 = x_2 = x_3$. Then the renormalization map reduces to

$$R\colon x \mapsto \tfrac{3}{5}x,$$

whence $c = (1,1,1)$ and $\rho = \frac{5}{3}$ is a solution of (2). For arbitrary dimension d and number of subdivisions m unit conductances $c = (1,\dots,1) \in \mathbb{R}^{d+1}$ always yield an eigenvalue for some $\rho > 1$, but no explicit formula for ρ is known. However, in the special case $m = 2$, it is known that $\rho = \frac{d+3}{d+1}$.

Example 3 (continued from page 4) Let us fix constant conductances $c_Z = (z,z,z)$ on the template graph Z and equip $X_0 = K_3$ with constant conductances $c_0 = (x,x,x)$. Then the renormalization map R is given by

$$R\colon x \mapsto \frac{3xz}{3x+5z}.$$

Note that the Sierpiński graphs are obtained by the limit $z \to \infty$. Although the renormalization map is slightly more complicated, one still obtains an explicit formula for the iterates $x_n = R^n(x)$:

$$x_n = \left(\frac{3}{5}\right)^n \cdot \frac{xz}{\frac{3}{2}\left(1-\left(\frac{3}{5}\right)^n\right)x+z},$$

as the reciprocal of x_n satisfies the linear recursion

$$\frac{1}{x_{n+1}} = \frac{1}{z} + \frac{5}{3x_n}.$$

In particular, the resistance scaling factor is given by $\rho = \frac{5}{3}$.

Example 4 (continued from page 5) As before we fix $c_Z = (z,z,z)$ and equip $X_0 = K_3$ with conductances $c_0 = (x,x,x)$. Then the renormalization map R is given by

$$R\colon x \mapsto \frac{xz(5x+2z)}{3(x^2+3xz+z^2)}.$$

In this and in the following example the iterates of the renormalization map cannot be given explicitly any longer. However, it is possible to derive information about the asymptotic behaviour. Since

$$R(x) = \frac{2x}{3}\left(1 - \frac{x}{2}\cdot\frac{2x+z}{x^2+3xz+z^2}\right)$$

it follows that $R(x) \leq \frac{2}{3}x$ for $x,z > 0$. Thus $x_n = R^n(x)$ satisfies

$$x_n = \left(\frac{2}{3}\right)^n \cdot \prod_{j=0}^{n-1}\left(1 - \frac{x_j}{2}\cdot\frac{2x_j+z}{x_j^2+3x_jz+z^2}\right).$$

The infinite product

$$C_{x,z} = \prod_{j=0}^{\infty}\left(1 - \frac{x_j}{2}\cdot\frac{2x_j+z}{x_j^2+3x_jz+z^2}\right)$$

converges since its factors tend to 1 at an exponential rate. Therefore the resistance scaling factor is $\rho = \frac{3}{2}$ and $\rho^n x_n \to C_{x,z}$ as $n \to \infty$.

Example 5 (continued from page 5) Assign conductance z to all edges in the template graph, conductance x to the "side" edges and y to the diagonal edges of the initial graph $X_0 = K_4$, see Figure 8. The renormalization map is given by

$$R : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{z(2x^2+2yx+2zx+yz)}{x^2+yx+6zx+4z^2+yz} \\ \frac{z^2(9x^4+19yx^3+12zx^3+11y^2x^2+4z^2x^2+24yzx^2+y^3x+8yz^2x+12y^2zx+2y^2z^2)}{(x^2+yx+6zx+4z^2+yz)(2x^3+4yx^2+7zx^2+2y^2x+4z^2x+8yzx+2yz^2+y^2z)} \end{pmatrix}.$$

For the sake of simplicity, we assume $z = 1$, which does not actually mean a loss of generality. We write R_1 and R_2 to denote the first and second coordinate

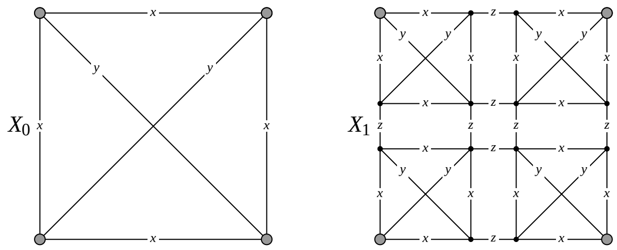

Fig. 8 Conductances on $X_0 = K_4$ and X_1.

of $R(x,y)$, respectively. Since the difference $R_1(x,y) - R_2(x,y)$ is a rational function in x,y without negative coefficients, $x,y \geq 0$ implies $R_1(x,y) \geq R_2(x,y) \geq 0$. Furthermore, if $x \geq y \geq 0$, then

$$\begin{aligned} R_1(x,y) &= \frac{2x^2+2yx+2x+y}{x^2+yx+6x+y+4} \leq \frac{x(4x+3)}{x^2+6x+4} \\ &= \frac{3x}{4}\left(1 - \frac{x(3x+2)}{3(x^2+6x+4)}\right) \leq \frac{3x}{4}. \end{aligned}$$

As a consequence the iterates $(x_n, y_n) = R^n(x,y)$ satisfy $0 \leq y_n \leq x_n \leq (\frac{3}{4})^{n-1} x_1$ for $n \geq 1$. Now consider the quotient $t_n = y_n/x_n$: since

$$\frac{R_2(x,y)}{R_1(x,y)} = \frac{9x^4+19x^3y+12x^3+11x^2y^2+24x^2y+4x^2+12xy^2+xy^3+8xy+2y^2}{(2x^2+2xy+2x+y)(2x^3+4x^2y+7x^2+2xy^2+8xy+4x+y^2+2y)}$$

it follows that

$$t_{n+1} = \frac{2x_n^2+4x_ny_n+y_n^2}{(2x_n+y_n)^2}(1+O((3/4)^n)) = \frac{t_n^2+4t_n+2}{(t_n+2)^2}(1+O((3/4)^n)).$$

The function

$$t \mapsto \frac{t^2+4t+2}{(t+2)^2}$$

is a contraction on $[0,1]$ with Lipschitz constant $\frac{1}{2}$ and unique fixed point $\sqrt{3}-1$, which shows that

$$t_n = \sqrt{3} - 1 + O((3/4)^n).$$

Thus we finally obtain

$$x_{n+1} = \tfrac{1}{4}(1+\sqrt{3})\, x_n \,(1 + O((3/4)^n)),$$

which implies that the resistance scaling factor in this example is

$$\rho = \frac{4}{\sqrt{3}+1} = 2(\sqrt{3}-1).$$

Furthermore, $\rho^n(x_n, y_n) \to C_{x,y}(1, \sqrt{3}-1)$ for $n \to \infty$ and some constant $C_{x,y}$.

Example 6 (continued from page 5) As in the previous example, one needs several variables: we assign conductance x to the sides, y to the shorter diagonals and z to the main diagonals of a hexagon. The resulting renormalization map is rather complicated, the entries are rational functions whose numerators and denominators are of degree 7 and 6 respectively. Some numerical details are given in [29]. It can be shown that the resistance scaling factor ρ is an algebraic number of degree 8 whose numerical value is 1.841467.

4 Counting spanning trees in self-similar lattices

In the following, we exhibit the relationship between spanning tree enumeration and the renormalization map. We fix a template graph Z and conductances on this template graph, and define a sequence of graphs $X_0, X_1, \ldots$ as shown in the preceding section. Then by the above considerations, X_n is electrically equivalent to a complete graph K_θ with suitable conductances c_n, which are given as iterates of the renormalization map:

$$c_n = R(c_{n-1}).$$

Assuming the existence of a resistance scaling factor, we obtain the following very general theorem:

Theorem 2 *Suppose that the factor $\rho > 1$ and the vector $c_\infty > 0$ are such that $\lim_{n\to\infty} \rho^n c_n = c_\infty$. Then the number of spanning trees of X_n satisfies the asymptotic formula*

$$N_{ST}(X_n) \sim A \cdot \rho^{-\kappa n/(s-1)} \cdot B^{s^n} \tag{3}$$

for certain constants A and B, where κ is defined as follows: fill the holes of the template graph with copies of K_θ to obtain the graph $Z(K_\theta)$, and let r be the smallest possible number of edges in a spanning tree of $Z(K_\theta)$ that are not edges of Z. Then

$$\kappa = s\theta - s - r.$$

Furthermore, the formula (3) *is exact (i. e., it holds with $=$ instead of $\sim$) if $c_n = \rho^{-n} c_\infty$ for all n and the template graph Z does not contain any edges, or if $\kappa = 0$.*

Remark 1 The parameter r can also be defined as follows: contract all edges in $Z(K_\theta)$ that already belong to Z. Then r is the number of edges in a spanning tree of the resulting graph. Clearly, the order of the contracted graph is at most $s\theta - (s-1)$ (for otherwise the contracted graph could not be connected), and thus $r \le s\theta - (s-1) - 1 = s\theta - s$, so that $\kappa \ge 0$.

Proof Let Y_n be a complete graph on θ vertices endowed with conductances c_n, so that Y_n is electrically equivalent to X_n. Note that X_{n+1} comprises of the template graph Z and s copies of X_n, each of which is now replaced by a copy of Y_n. The resulting graph is denoted by $R_{n+1} = Z(Y_n)$ (keeping conductances in mind). By Theorem 1, we have

$$N_{ST}(X_{n+1}) = N_{ST}(R_{n+1}) \cdot \left(\frac{N_{ST}(X_n)}{N_{ST}(Y_n)} \right)^s. \tag{4}$$

By our assumptions, $c_n \to 0$ holds componentwise as $n \to \infty$. Note that both $N_{ST}(R_{n+1}) = P(c_n)$ and $N_{ST}(Y_n) = Q(c_n)$ are polynomials in c_n. Thus the quotient $N_{ST}(R_{n+1})/N_{ST}(Y_n)^s$ is a rational function. Furthermore, $N_{ST}(Y_n)$ is even homogeneous of degree $\theta - 1$, so that

$$N_{ST}(Y_n) = Q(c_n) = \rho^{-(\theta-1)n} Q(\rho^n c_n).$$

On the other hand, the smallest total degree of a monomial in P is r (by definition of r), and so we have

$$N_{ST}(R_{n+1}) = P(c_n) = \rho^{-rn} P_{(r)}(\rho^n c_n)(1 + O(\rho^{-n})),$$

where $P_{(r)}$ is the polynomial that consists of all monomials of total degree r in P, which correspond to all spanning trees in the graph R_{n+1} (or, if conductances are neglected, $Z(K_\theta)$) that have r edges not belonging to Z. Hence we obtain

$$\begin{aligned} N_{ST}(X_{n+1}) &= \rho^{(s\theta-s-r)n} \cdot N_{ST}(X_n)^s \cdot \frac{P_{(r)}(\rho^n c_n)}{Q(\rho^n c_n)^s} \cdot (1 + O(\rho^{-n})) \\ &= \rho^{\kappa n} \cdot N_{ST}(X_n)^s \cdot \frac{P_{(r)}(c_\infty)}{Q(c_\infty)^s} \cdot (1 + \delta_n), \end{aligned}$$

where δ_n tends to 0. Set $u_n = \log N_{ST}(X_n)$, $a = \log P_{(r)}(c_\infty) - s \log Q(c_\infty)$, and $\varepsilon_n = \log(1 + \delta_n)$ to obtain

$$u_{n+1} = \kappa n \log \rho + s u_n + a + \varepsilon_n.$$

Iteration yields

$$\begin{aligned} u_n &= s^n u_0 + \sum_{j=0}^{n-1} s^{n-1-j}(\kappa j \log \rho + a + \varepsilon_j) \\ &= s^n u_0 + \frac{a(s^n - 1)}{s - 1} + \frac{\kappa(s^n - ns + n - 1)\log \rho}{(s-1)^2} + s^n \sum_{j=0}^{\infty} s^{-j-1}\varepsilon_j - \sum_{j=n}^{\infty} s^{n-1-j}\varepsilon_j. \end{aligned}$$

The sum $\sum_{j=0}^{\infty} s^{-j-1}\varepsilon_j$ converges since $\varepsilon_j \to 0$, and the sum $\sum_{j=n}^{\infty} s^{n-1-j}\varepsilon_j$ tends to 0 for the same reason. Thus we end up with

$$u_n = \log A - \frac{\kappa n \log \rho}{s - 1} + s^n \log B + o(1)$$

with

$$\begin{aligned} A &= \rho^{-\kappa/(s-1)^2} \cdot \left(\frac{Q(c_\infty)^s}{P_{(r)}(c_\infty)} \right)^{1/(s-1)}, \\ B &= \frac{N_{ST}(X_0)}{A} \cdot \exp\left(\sum_{j=0}^{\infty} s^{-j-1}\varepsilon_j \right), \end{aligned}$$

which proves the asymptotic result. It remains to show that the formula is exact in the two cases mentioned in the statement of the theorem. If the template graph does not contain any edges, then P is homogeneous as well, and the condition

$\rho^n c_n = c_\infty$ implies that $\delta_n = \varepsilon_n = 0$ in the above argument. It follows that the formula is exact.

On the other hand, if $\kappa = 0$, then every spanning tree of R_{n+1} contains at least $r = s\theta - s = s(\theta - 1)$ edges in the s copies of Y_n. This is also an upper bound, since each of these copies has θ vertices, so that more edges would necessarily result in a cycle. Hence every spanning tree of R_{n+1} is composed of some edges in Z that connect the s parts and spanning trees in the s copies of Y_n. This implies that

$$N_{ST}(R_{n+1}) = C \cdot N_{ST}(Y_n)^s$$

for some constant C that only depends on Z, and thus

$$N_{ST}(X_{n+1}) = C \cdot N_{ST}(X_n)^s,$$

from which an exact formula follows immediately. □

Remark 2 It is possible that the formula (3) is exact even if none of the two stated conditions holds. An example is given below by the Towers of Hanoi graphs.

In the case $\kappa = 0$, the structure of the resulting sequence of graphs is "tree-like", and a spanning tree in X_{n+1} induces spanning trees on each of the copies of X_n it comprises of.

If Z is edgeless and the automorphism group of X_n acts with full symmetry (or at least 2-homogeneously) on the set of distinguished vertices, then the condition $c_n = \rho^{-n} c_\infty$ is always satisfied, since X_n is electrically equivalent to a complete graph with constant conductances in this case, and the renormalization map reduces to a one-dimensional linear map.

Remark 3 If (3) only holds asymptotically, then the constants A and B can generally only be determined numerically.

Let us now determine the number of spanning trees in our examples.

Example 1 (continued from page 7) In this case Theorem 2 yields an exact result, since $R^n(x) = \rho^{-n} x$ with $\rho = \frac{8}{3}$, and since the template graph is edgeless. Obviously $s = 5$, $\theta = 2$, $\kappa = 1$. Furthermore, $Q(c) = x$ and $P(c) = 3x^4$, so that

$$N_{ST}(X_n) = \left(\frac{8}{3}\right)^{-n/4} \cdot 6^{3(5^n-1)/16}.$$

Example 2 (continued from page 7) For the sequence of Sierpiński graphs, the formula is exact for the same reason as before. Indeed, we have

$$N_{ST}(X_n) = \sqrt[4]{\frac{3}{20}} \cdot \left(\frac{5}{3}\right)^{-n/2} \cdot \left(\sqrt[4]{540}\right)^{3^n}, \tag{5}$$

which was obtained in different ways in [6,33,35]. It is clear that Theorem 2 also applies more generally to Sierpiński graphs in higher dimension with an arbitrary number of subdivisions. For arbitrary dimension $d \geq 2$ and the simplest case of only two subdivisions, one obtains

$$N_{ST}(X_n) = \left(2^{d((d+1)^n-1)} (d+1)^{(d+1)^{n+1}+dn+d-1} (d+3)^{(d+1)^n-dn-1}\right)^{\frac{d-1}{2d}}, \tag{6}$$

which was conjectured in [6] and proven by means of a different method that depends on the high degree of symmetry in [35]. In order to derive it from Theorem 2 and its proof, one needs to determine the resistance scaling factor and the polynomials $P(c)$ and $Q(c)$ ($s = \theta = d+1$, $\kappa = \frac{1}{2}d(d-1)$, and $N_{ST}(X_0) = N_{ST}(K_{d+1}) = (d+1)^{d-1}$ are easy to obtain).

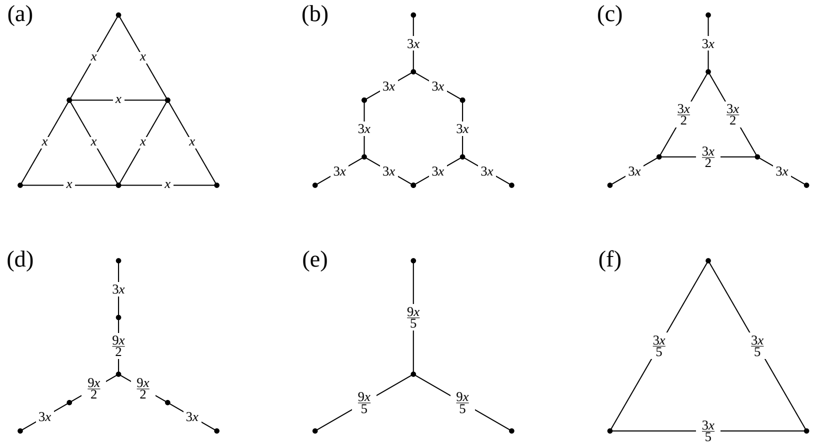

Fig. 9 Starting graph (a) and all steps (b)–(f) in the simplification for two-dimensional Sierpiński graphs ($\theta = 3$).

As mentioned in the preceding section, the resistance scaling factor is $\frac{d+3}{d+1}$, which can be seen as follows (see Figure 9): let $c = (x, x, \ldots, x)$ be constant conductances on $X_0 = K_\theta$; we substitute $s = d+1$ copies of X_0 into the template graph (which is edgeless). Each of these copies is now replaced by an electrically equivalent star with conductances θx. The centers of these stars form a complete graph with subdivided edges of conductances θx. These can be reduced to single edges of conductance $\theta x/2$. The resulting complete graph can now be transformed to a star with conductances $\theta^2 x/2$. The new graph is a star whose edges are all subdivided into two parts whose conductances are θx and $\theta^2 x/2$. These are reduced to single edges of conductance $\theta^2 x/(\theta+2)$. Finally, the star is transformed back to a complete graph with conductances $\theta x/(\theta+2)$. This shows that the renormalization map is given by

$$x \mapsto \frac{\theta}{\theta+2} \cdot x = \frac{d+1}{d+3} \cdot x,$$

so that $\rho = \frac{d+3}{d+1}$ must be the resistance scaling factor.

It remains to determine the polynomials P and Q. Let again $c = (x, x, \ldots, x)$ be the conductances. Then Q is easily found to be

$$Q(c) = x^{\theta-1} N_{ST}(K_\theta) = x^{\theta-1}\theta^{\theta-2} = x^d (d+1)^{d-1},$$

and P can be determined by means of the same transformations that were used to determine the resistance scaling factor together with Theorem 1: the subdivided star with conductances θx and $\theta^2 x/2$ has only one spanning tree whose weight is $(\theta^3 x^2/2)^\theta$. The first replacement step yields a factor $(1/(x\theta^2))^\theta$ from Theorem 1, the serial replacements $(2\theta x)^{\theta(\theta-1)/2}$, and the final transformation from a complete graph to a star a factor of $2/(\theta^3 x)$. Therefore, we have

$$\begin{aligned} P(c) &= \left(\frac{1}{x\theta^2}\right)^\theta \cdot (2\theta x)^{\theta(\theta-1)/2} \cdot \frac{2}{\theta^3 x} \cdot (\theta^3 x^2/2)^\theta \\ &= 2^{(\theta-1)(\theta-2)/2} \theta^{(\theta+3)(\theta-2)/2} x^{(\theta-1)(\theta+2)/2} \\ &= 2^{d(d-1)/2} (d+1)^{(d+4)(d-1)/2} x^{d(d+3)/2}. \end{aligned}$$

Putting everything together, one obtains formula (6).

Example 3 (continued from page 7) In this example, the two polynomials P and Q are given by

$$P(c) = 27x^5 z^2 (2z+3x) \qquad \text{and} \qquad Q(c) = 3x^2$$

if $c = (x,x,x)$. Write $c_n = (x_n, x_n, x_n) = R^n(c)$ for the iterates. Then

$$\frac{P(c_n)}{Q(c_n)^3} = z^2 \left(\frac{2z}{x_n} + 3\right) = \frac{z^2(3x+2z)}{x} \cdot \left(\frac{5}{3}\right)^n.$$

The recursion for $N_{ST}(X_n)$ thus reduces to

$$N_{ST}(X_{n+1}) = z^2 \left(\frac{2z}{x_n} + 3\right) = \frac{z^2(3x+2z)}{x} \cdot \left(\frac{5}{3}\right)^n N_{ST}(X_n)^3.$$

In view of the cancellations, the formula we obtain in this example is exact even though the conditions for an exact formula given in Theorem 2 are not satisfied, which shows that these conditions are sufficient, but not necessary:

$$N_{ST}(X_n) = \sqrt[4]{\frac{3}{5}} \sqrt{\frac{x}{z^2(3x+2z)}} \cdot \left(\frac{5}{3}\right)^{-n/2} \cdot \left(\sqrt[4]{135}\sqrt{x^3 z^2 (3x+2z)}\right)^{3^n}.$$

In particular, $x = z = 1$ yields a formula for "ordinary" Towers of Hanoi graphs:

$$N_{ST}(X_n) = \sqrt[4]{\frac{3}{125}} \cdot \left(\frac{5}{3}\right)^{-n/2} \cdot \left(\sqrt[4]{3375}\right)^{3^n}.$$

The limit

$$\lim_{z\to\infty} z^{-3/2(3^n-1)} N_{ST}(X_n)$$

gives the number of spanning trees in X_n which contain all edges of weight z. This corresponds exactly to spanning trees in the associated Sierpiński graphs that result from contracting these edges, so that we obtain the formula of the previous example as a special case.

Example 4 (continued from page 8) Here one easily finds $\kappa = 3$, and the constants A and B can be determined numerically:

$$N_{ST}(X_n) \sim A \cdot \left(\frac{3}{2}\right)^{-3n/2} \cdot B^{3^n}$$

with $A \approx 0.071944$ and $B \approx 54.521061$.

Example 5 (continued from page 8) Let c_Z be unit conductances and $c_0 = (1,0)$ be fixed. Then all requirements of Theorem 2 are satisfied, and we obtain

$$N_{ST}(X_n) \sim A \cdot \rho^{-4n/3} \cdot B^{4^n}$$

with $A \approx 0.105066$ and $B \approx 35.126433$.

Example 6 (continued from page 10) We take the initial graph to be the cycle C_6, as shown in Figure 6 (i. e., the initial conductances are $(1,0,0)$). As mentioned before, the resistance scaling factor ρ has numerical value 1.841467, and one obtains

$$N_{ST}(X_n) \sim A \cdot \rho^{-n} \cdot B^{7^n}$$

with $A \approx 0.257362$ and $B \approx 16.887511$.

5 A final remark

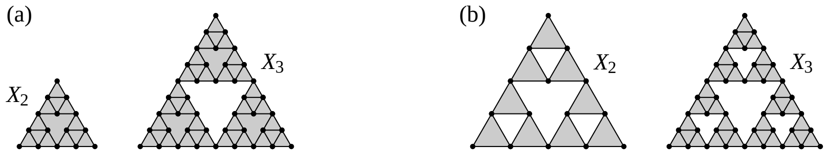

Fig. 10 Two ways to construct the Sierpiński graph X_3: (a) X_3 is obtained by glueing three copies of X_2, (b) X_3 is obtained by replacing each upright triangle (copy of X_0) by X_1.

The self-similarity of a sequence of graphs as defined in Section 3 may be used in two ways: first by using the copy-construction directly; second by inserting “microstructure” at the right places, see Figure 10 for an illustration in the case of Sierpiński graphs. Both variants were used to study several problems (spin models, random walks, spectral theory, etc.). In order to illustrate these two perspectives we quote two different descriptions of the partition function of the Ising modell on the sequence of Sierpiński graphs. Consider the Ising modell with nearest neighbour interactions only and constant interaction strength J. Let β be the “inverse” temperature and write $Z_n(\beta J)$ for the partition function. Then using the “high temperature expansion” and the first construction method, it was shown in [9] that

$$Z_n(\beta J) = 2^{3(3^n+1)/2} \cosh(\beta J)^{3^{n+1}} \Gamma_n(\tanh(\beta J)),$$

where $\Gamma_n(z)$ is defined by the recursion

$$\begin{pmatrix}\Gamma_0(z)\\ \Upsilon_0(z)\end{pmatrix} = \begin{pmatrix}1+z^3\\ z+z^2\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\Gamma_{n+1}(z)\\ \Upsilon_{n+1}(z)\end{pmatrix} = \begin{pmatrix}\Gamma_n(z)^3+\Upsilon_n(z)^3\\ \Upsilon_n(z)^2\Gamma_n(z)+\Upsilon_n(z)^3\end{pmatrix}.$$

On the other hand, using the second construction it was shown in [3] that, for $y=e^{\beta J}$, the quite different recurrence equation

$$Z_{n+1}(y) = (c(y))^{3^{n-1}} Z_n(f(y))$$

with

$$f(y) = \left(\frac{y^8-y^4+4}{y^4+3}\right)^{1/4} \quad \text{and} \quad c(y) = \frac{y^4+1}{y^3}\cdot\left((y^4+3)(y^8-y^4+4)\right)^{1/4}$$

holds. In this note we have studied the number of spanning trees using the copy-construction. Of course, one can also use the second one. In either cases we obtain a recurrence equation for $N_{ST}(X_n)$. Let us write down this equation for the case of two-dimensional Sierpiński graphs. Equation (4) implies

$$N_{ST}(X_{n+1}) = \frac{N_{ST}(X_1,(\frac{3}{5})^n)}{N_{ST}(X_0,(\frac{3}{5})^n)^3}\cdot N_{ST}(X_n)^3 = 2\,(\tfrac{5}{3})^n N_{ST}(X_n)^3,$$

since Y_n is nothing else but X_0 with constant conductances $(\frac{3}{5})^n$ and R_{n+1} is X_1 with constant conductances $(\frac{3}{5})^n$. On the other hand, we obtain

$$N_{ST}(X_{n+1}) = \left(\frac{N_{ST}(X_1)}{N_{ST}(X_0,\frac{3}{5})}\right)^{3^n}\cdot N_{ST}(X_n,\tfrac{3}{5}) = (\tfrac{3}{5})^{1/2}\,540^{3^n/2} N_{ST}(X_n)$$

by 3^n substitutions $X_1 \to (X_0,\frac{3}{5})$ in X_{n+1}. Again note the difference between these two recurrence equations. Furthermore, equating these equations directly yields Formula (5).

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