ENUMERATION OF MATCHINGS IN FAMILIES OF SELF-SIMILAR GRAPHS

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Abstract. The number of matchings of a graph G is an important graph parameter in various contexts, notably in statistical physics (dimer-monomer model). Following recent research on graph parameters of this type in connection with self-similar, fractal-like graphs, we study the asymptotic behavior of the number of matchings in families of self-similar graphs that are constructed by a very general replacement procedure. Under certain conditions on the geometry of the graphs, we are able to prove that the number of matchings generally follows a doubly exponential growth. The proof depends on an independence theorem for the number of matchings that has been used earlier to treat the special case of Sierpiński graphs. We provide a variety of examples and also discuss the situation when our conditions are not satisfied.

1. Introduction

The number of matchings (also known as independent edge subsets) of a (finite, simple) graph G, henceforth denoted by $m(G)$, is a parameter that is of relevance, among others, in statistical physics (so-called dimer-monomer model, cf. [7, 10, 11] and other references provided in [4]) and combinatorial chemistry (there, $m(G)$ is known as Hosoya-index of a graph, cf. [9, 13]). Therefore, the enumeration of matchings has already been investigated for various classes of graphs, in particular trees, hexagonal chains, grid graphs, and random graphs [2, 4, 14, 15, 16, 23].

Recently, the enumeration of matchings has been considered in the physical literature for Sierpiński graphs [4]. Since other models from statistical physics have

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already been treated extensively for fractals and self-similar graphs, it is quite natural to study the dimer-monomer model for fractal-like graphs with scaling invariance (as opposed to the translational invariance of a grid) as well. Other graph parameters that are of interest in a physical context have also been investigated recently, we refer to [5, 4, 3, 6]. A parameter that is of particular interest is the asymptotic growth constant

$$\lim_{n\to\infty} \frac{\log m(X_n)}{|VX_n|}$$

for a growing sequence X_n of graphs. The approach used in [4] depends heavily on the analysis of large systems of recurrences, which can be quite tedious. In this paper, we aim to treat the problem in more generality and also exhibit how an independence theorem for the number of matchings that was proved in [22] can be applied to shorten the calculations. The specific case of two-dimensional Sierpiński graphs has already been treated there as an example, but we will show that the same approach is actually applicable to a fairly general family of self-similar graphs.

In the following section, we will describe the construction of the self-similar graphs that are discussed in this paper. This construction leads to a system of recurrences, whose asymptotics are studied in Section 4. In our final section, we provide a variety of examples, some of which also exhibit the difficulties that arise if our technical conditions are not satisfied any longer.

2. Construction

There are many different approaches to construct self-similar graphs. A construction that is specifically geared to be used in the context of enumeration was described in [21], and we will also make use of it here (not in the most general possible setting though, to keep the amount of notation reasonable). A sequence of self-similar graphs is described by the following ingredients:

- An initial graph X_0.
- A set of distinguished vertices on X_0, given by a one-to-one map $\varphi : \{1, 2, \dots, \theta\} \to VX_0$, where $\theta \geq 1$ is the number of distinguished vertices.

- A model graph G.
- A one-to-one map $\psi : \{1, 2, \ldots, \theta\} \to VG$, which defines θ distinguished vertices on G.
- The number $s \geq 1$ of substitutions associated to the model graph G as well as one-to-one maps $\sigma_i : \{1, \ldots, \theta\} \to VG$ for $i \in \{1, \ldots, s\}$, which describe each substitution.

Let us also introduce some more notation: for convenience, we write $\Theta = \{1, 2, \ldots, \theta\}$ and $S = \{1, \ldots, s\}$.

With this data we inductively construct a sequence $(X_n)_{n\geq 0}$ of graphs and maps $\varphi_n : \Theta \to VX_n$, which define distinguished vertices of the graph X_n: let $n > 0$. For $i \in S$ let $Z_{n,i}$ be an isomorphic copy of the graph X_{n-1}, where the isomorphism is given by $\gamma_{n,i} : X_{n-1} \to Z_{n,i}$. Additionally, we require that the vertex sets VG and $VZ_{n,1}, \ldots, VZ_{n,s}$ are mutually disjoint. Now let Y_n be the disjoint union of the graphs G and $Z_{n,1}, \ldots, Z_{n,s}$ and define the relation $\sim$ on the vertex set VY_n to be the reflexive, symmetric and transitive hull of

$$\bigcup_{i=1}^{s} \Big\{ \{\sigma_i(j), \gamma_{n,i}(\varphi_{n-1}(j))\} \,:\, j \in \Theta \Big\} \subseteq VY_n \times VY_n.$$

Then $X_n = Y_n/\!\sim$ and the map φ_n is defined by $\varphi_n(j) = \overline{\psi(j)} \in VX_n$. Furthermore, we call the subgraph $P_{n,i} = \overline{Z_{n,i}}$ of X_n (which is isomorphic to X_{n-1}) the *i-th part* of X_n, and $F_n = \overline{G}$ the *frame* of X_n.

It is easy to deduce from the construction that

$$|VX_n| = s|VX_{n-1}| + |VG| - s\theta,$$

so that

$$|VX_n| = |VX_0|s^n + \frac{|VG| - s\theta}{s-1} \cdot (s^n - 1). \tag{1}$$

2.1. **Examples.**

Example 2.1. In [18] spectral properties of the modified Koch curve, which is a minor but interesting variation of the fractal Koch curve, were studied. The first few graphs in the associated graph sequence are depicted in Figure 1. The model graph G is edgeless and has five vertices $\{1, \ldots, 5\}$. In each step, we amalgamate five copies of X_n—as indicated in the figure—to obtain X_{n+1}, where we take $X_0 = K_2$ as the initial graph. Formally, if $VX_0 = \{v_1, v_2\}$, we set $\varphi(i) = v_i$, and the maps ψ and $\sigma_1, \ldots, \sigma_5$ are given by the following table:

j	$\psi(j)$	$\sigma_1(j)$	$\sigma_2(j)$	$\sigma_3(j)$	$\sigma_4(j)$	$\sigma_5(j)$
1	1	1	2	3	2	3
2	4	2	3	4	5	5

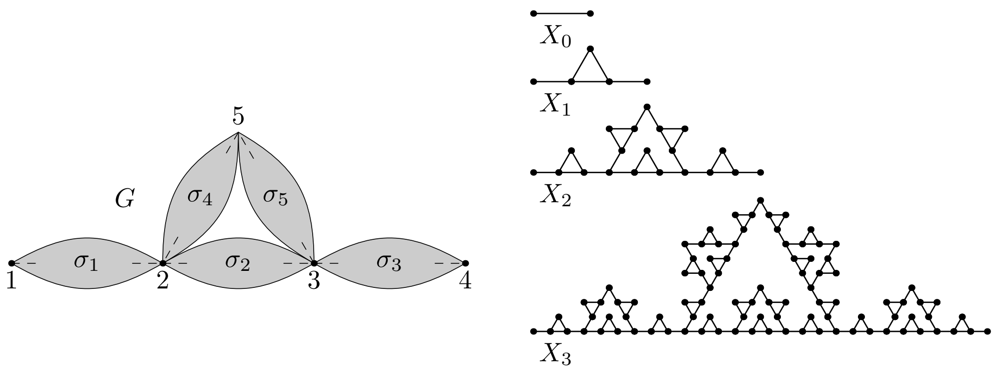

FIGURE 1. Model graph and finite modified Koch graphs X_0, X_1, X_2, and X_3.

Example 2.2 (The loop-erased Schreier graph of the Fabrykowski-Gupta group). Let $X_0 = K_3$, where $VX_0 = \{1, 2, 3\}$; furthermore, let $\theta = 3$ and $\varphi(i) = i$ for $i \in \{1, 2, 3\}$, define G by

$$VG = \{x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}\},$$

$$EG = \big\{\{x_{11}, x_{21}\}, \{x_{21}, x_{31}\}, \{x_{31}, x_{11}\}\big\},$$

and set $\psi(i) = x_{i2}$ for $i \in \{1, 2, 3\}$. Finally we set $s = 3$ and $\sigma_i(j) = x_{ij}$. See Figure 2 for a visualization of the model graph G and X_1, X_2. This graph sequence already served as an example in [21]; see [1] and [8] for a description of how these graphs arise in the context of infinite groups.

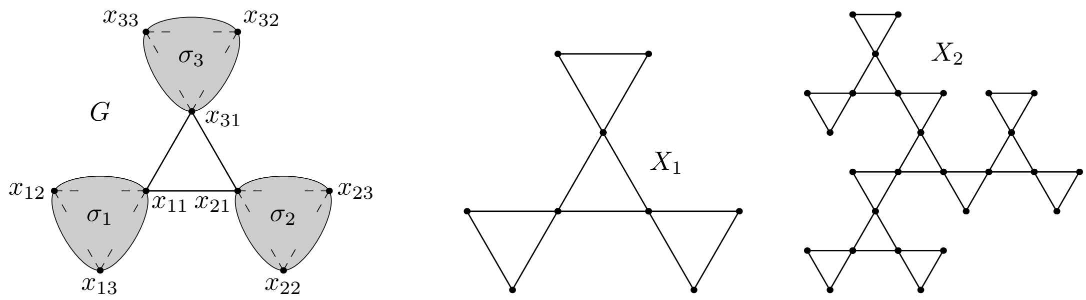

FIGURE 2. Model graph and X_1, X_2.

Example 2.3 (Sierpiński graphs, see [19]). The Sierpiński gasket and its higher-dimensional generalizations certainly belong to the most popular examples of fractals, and the graph-theoretical properties of its finite approximations have been thoroughly studied as well. The finite Sierpiński graphs can also be obtained by means of our construction as follows: Fix some $d > 1$ and let $s = \theta = d + 1$. Define the edgeless graph G by

$$VG = \left\{ \mathbf{x} \in \mathbb{N}_0^{d+1} \ : \ \sum_i x_i = 2 \right\}$$

and the map $\psi : \Theta \to VG$ by $\psi(i) = 2\mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th canonical basis vector of $\mathbb{R}^{d+1}$. In addition, set $\sigma_i(j) = \mathbf{e}_i + \mathbf{e}_j \in VG$ for $i \in S$ and $j \in \Theta$ (note that $\Theta = S = \{1, \ldots, d+1\}$). Finally, we use $X_0 = K_{d+1}$ as initial graph and define φ in the obvious way (each of the vertices becomes a distinguished vertex $\varphi(j)$ for some j). See Figure 3 for the case $d = 2$.

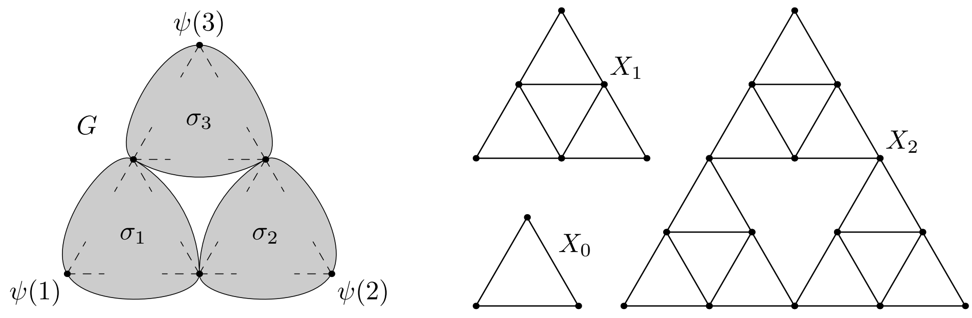

FIGURE 3. Model graph and finite Sierpiński graphs.

Example 2.4 (Pentagasket). A pentagonal analogue of the two-dimensional Sierpiński gasket is known as the Pentagasket, see Figure 4; it is essentially constructed in the same way, the figure shows the graphs X_0, X_1, X_2 in the sequence. A slight difference to the Sierpiński graphs lies in the fact that it is less symmetric: while the symmetry group with respect to the distinguished vertices of the Sierpiński graphs is the entire symmetric group S_3, the symmetry group is only the dihedral group D_5 in this example.

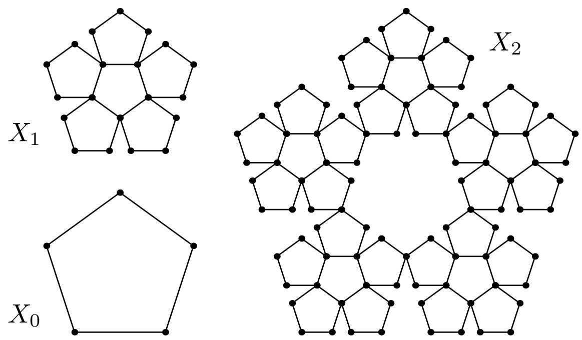

FIGURE 4. The Pentagasket—a pentagonal analogue of the Sierpiński gasket.

Example 2.5 (Complete d-ary trees)*.* Rooted d-ary trees play a role in many branches of graph theory and combinatorics and mathematics in general. A complete d-ary tree is a rooted tree with the property that each internal vertex has the same outdegree d and all leaves have the same distance to the root. Complete rooted trees are easily constructed by means of our procedure, as will be explained in the following: the initial graph $X_0 = K_1$ is a graph with only a single vertex that is also the only distinguished vertex $\varphi(1)$. The model graph G is a star with $d > 1$ leaves, i.e.

$$VG = \{0, 1, \ldots, d\}, \qquad EG = \big\{\{0, i\} \,:\, i = 1, 2, \ldots, d\big\},$$

and we set $\psi(1) = 0$ as well as $\sigma_i(1) = i$ for $i = 1, 2, \ldots, d$. See Figure 5 for the model graph and the first three graphs in the sequence in the case $d = 3$.

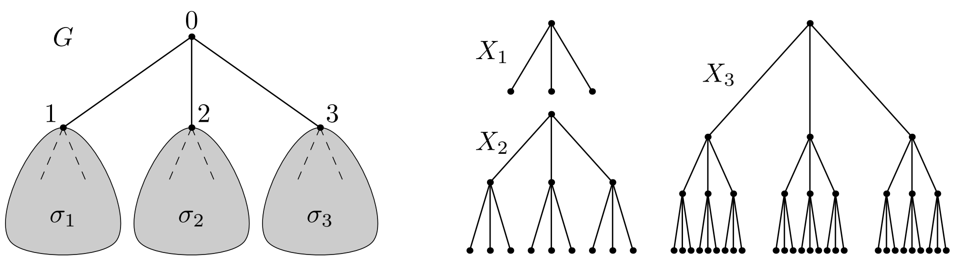

FIGURE 5. Complete ternary trees: model graph and X_1, X_2, X_3.

3. Matchings

Let $m(G)$ denote the number of matchings of G. In order to determine the asymptotic behavior of $m(X_n)$, where the sequence X_n is constructed as outlined in the previous section, we will first establish some recursive relations, as outlined in [21]. For a set $A \subseteq \Theta$, let $\mathcal{M}_n(A)$ be the set and $a_n(A)$ the number of matchings of X_n with the property that all vertices in $\varphi_n(A)$ are covered, while all other vertices in $\varphi_n(\Theta)$ are not. By means of the inclusion-exclusion principle, this could also be written as

$$a_n(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} m(X_n \setminus \varphi_n(\Theta \setminus B)). \tag{2}$$

Next, we construct a new graph H as follows: the vertex set of H is given by

$$VH = VG \cup \{(i,j) \,:\, i \in S, j \in \Theta\},$$

the edge set is

$$EH = EG \cup \{(\sigma_i(j),(i,j)) \,:\, i \in S, j \in \Theta\}.$$

Note that a matching $M \subseteq EX_n$ on X_n induces matchings on all s parts of X_n as well as on the frame F_n and consequently also on H—the corresponding matching M' on H can be constructed as follows:

- Any edge connecting two vertices in $F_n \simeq G$ is kept (i.e. the corresponding edge in EH is included in M').
- If $v \in F_n$ is covered by an edge in the i-th path $EP_{n,i}$ for some $i \in S$, then $v = \sigma_i(j)$ for some $j \in \Theta$, and we include the edge $(\sigma_i(j),(i,j))$ in M'.

This construction is essentially bijective: given a matching M' in H and appropriate matchings in all parts $P_{n,i}$ ("appropriate" meaning that if $(\sigma_i(j),(i,j)) \in M'$, there is an edge that covers $\sigma_i(j)$ in $P_{n,i}$), one can always combine them to form a matching of X_n. Hence, if $\mathcal{H}(A)$ denotes the set of matchings of H with the property that all vertices in $\psi(A)$, but not those in $\psi(\Theta \setminus A)$, are covered, and if $\eta_i(M')$ is the set of all $j \in \Theta$ such that $(\sigma_i(j),(i,j)) \in M'$, we have a bijection

$$\mathcal{M}_n(A) \longleftrightarrow \bigcup_{M' \in \mathcal{H}(A)} \prod_{i=1}^{s} \mathcal{M}_{n-1}(\eta_i(M')),$$

which leads to the recursion

$$a_n(A) = \sum_{M' \in \mathcal{H}(A)} \prod_{i=1}^{s} a_{n-1}(\eta_i(M')).$$

Our goal in the following section is to obtain the asymptotic behavior of $a_n(A)$ (and thus also $m(X_n) = \sum_{A \subseteq \Theta} a_n(A)$) from these recursions. Under some technical assumptions, this can be achieved by means of the following lemma (see [22]):

Theorem 1. *Let G be a graph and $B_1, B_2, \dots, B_\theta$ be disjoint sets of vertices. Furthermore, let*

$$r_G(B) = \frac{m(G \setminus B)}{m(G)}$$

denote the ratio of all matchings of G which cover no element of B. Then there are positive constants C and D, $D < 1$, which depend only on the maximum degree $\Delta = \Delta(G)$ of G and the sizes of the B_i such that

$$(1 + C\,D^{d-1})^{1-\theta} \leq \frac{r_G(\bigcup_{i=1}^{\theta} B_i)}{\prod_{i=1}^{\theta} r_G(B_i)} \leq (1 + C\,D^{d-1})^{\theta-1}$$

holds, where $d = \min_{i,j} d(B_i, B_j)$ is the minimal distance between two sets from our collection. The constants C and D can be taken as

$$C = (1 + \Delta)^{\max_i |B_i|} \qquad \text{and} \qquad D = 1 - C^{-1}.$$

Intuitively, this can be interpreted as follows: if the mutual distance d is large, both the upper and lower bound are close to 1, which means that the influences of the sets B_i on the number of matchings are approximately independent of each other.

4. Asymptotics

We will have to make some technical assumptions in order to obtain general results: in the following, we always assume the following:

- Each distinguished vertex belongs to a unique part and is also not incident to any edge of the model graph; formally, for every $j \in \{1, 2, \dots, \theta\}$, $\psi(j)$ is an isolated vertex of the model graph, and there are unique $\ell = \ell(j)$ and $h = h(j)$ such that $\sigma_\ell(h) = \psi(j)$.

- No part contains more than one distinguished vertex, i.e. $\ell(1), \ell(2), \ldots, \ell(\theta)$ are pairwise different.

These conditions are also quite natural from a geometric point of view, specifically if existence of a limiting structure is desired. In Section 5, we will also discuss specific examples where the conditions are not satisfied. We note two immediate consequences of our two conditions:

- The degrees of X_n stay bounded; this is due to the fact that only the degrees of the distinguished vertices can actually change in further steps. However, our first condition guarantees that the maximum degree of a distinguished vertex does not increase at any step.
- The distances between distinguished vertices tend to ∞ at an exponential rate: indeed, if we add an edge between $\sigma_i(a)$ and $\sigma_i(b)$ for all i and all pairs $1 \leq a, b \leq \theta$ to the model graph G, and let d be the minimum distance between vertices in $\psi(\Theta)$ in the resulting graph (note that this is at least 2!), it is easy to see that the minimum distance between vertices in $\varphi_n(\Theta)$ is at least d^n.

Let $\Delta = \max_n \Delta(X_n)$ be the upper bound for the maximum degree of X_n, and let

$$D_n = \min_{j,k} d(\varphi_n(j), \varphi_n(k))$$

be the minimum distance between distinguished vertices in X_n. Furthermore, set

$$q_n = \Delta(X_n)\left(1 - \frac{1}{1+\Delta(X_n)}\right)^{D_n - 1} \leq \Delta\left(1 - \frac{1}{1+\Delta}\right)^{d^n - 1} = (1+\Delta)\left(\frac{\Delta}{1+\Delta}\right)^{d^n},$$

which tends to 0 at a doubly exponential rate. This will enable us to make use of Theorem 1 in the same way as in [22]. To this end, we define further auxiliary parameters, namely

$$\rho_n(j) = r_{X_n}(\{\varphi_n(j)\}) = \frac{m(X_n \setminus \{\varphi_n(j)\})}{m(X_n)}.$$

Then, Theorem 1 implies that

$$m(X_n \setminus \varphi_n(\Theta \setminus B)) = m(X_n) \prod_{j \in \Theta \setminus B} \rho_n(j)(1 + O(q_n)),$$

and so (2) becomes

$$\begin{aligned}a_n(A) &= \sum_{B\subseteq A}(-1)^{|A|-|B|}m(X_n)\Bigg(\prod_{j\in\Theta\setminus B}\rho_n(j)\Bigg)(1+O(q_n))\\ &= m(X_n)\Bigg(\prod_{j\in\Theta\setminus A}\rho_n(j)\Bigg)\Bigg(\prod_{j\in A}(1-\rho_n(j))\Bigg)(1+O(q_n)).\end{aligned}$$

Noticing that

$$\rho_n(j) = \frac{m(X_n\setminus\{\varphi_n(j)\})}{m(X_n)} = \frac{\sum_{A\subseteq\Theta,\, j\notin A} a_n(A)}{\sum_{A\subseteq\Theta} a_n(A)},$$

we find that the parameters $\rho_n(j)$ satisfy the following property:

$$\begin{aligned}\rho_n(j) &= \frac{\sum_{A\subseteq\Theta,\, j\notin A} a_n(A)}{\sum_{A\subseteq\Theta} a_n(A)} = \frac{\sum_{A\subseteq\Theta,\, j\notin A}\sum_{M'\in\mathcal{H}(A)}\prod_{i=1}^{s} a_{n-1}(\eta_i(M'))}{\sum_{A\subseteq\Theta}\sum_{M'\in\mathcal{H}(A)}\prod_{i=1}^{s} a_{n-1}(\eta_i(M'))}\\ &= \frac{\sum_{A\subseteq\Theta,\, j\notin A}\sum_{M'\in\mathcal{H}(A)}\prod_{i=1}^{s}\big(\prod_{r\in\Theta\setminus\eta_i(M')}\rho_{n-1}(r)\big)\big(\prod_{r\in\eta_i(M')}(1-\rho_{n-1}(r))\big)(1+O(q_{n-1}))}{\sum_{A\subseteq\Theta}\sum_{M'\in\mathcal{H}(A)}\prod_{i=1}^{s}\big(\prod_{r\in\Theta\setminus\eta_i(M')}\rho_{n-1}(r)\big)\big(\prod_{r\in\eta_i(M')}(1-\rho_{n-1}(r))\big)(1+O(q_{n-1}))}.\end{aligned}$$

Our first condition implies that there is exactly one neighbor of $\psi(j)$ in H, namely $(\ell(j), h(j))$. There is an obvious bijection between those matchings of H that contain the edge between the two and those which do not contain it. The former belong to $\mathcal{H}(A)$ for some A with $j\in A$ (and thus contribute a factor $1-\rho_{n-1}(h(j))$), the latter belong to $\mathcal{H}(A)$ for some A with $j\notin A$ (and thus contribute a factor $\rho_{n-1}(h(j))$). Hence, the above quotient simply reduces to

$$\rho_n(j) = \rho_{n-1}(h(j))(1+O(q_{n-1})).$$

Let $\mathbf{r}_n$ be the column vector with entries $\rho_n(j)$, $j=1,\dots,\theta$. Then, the above equation can be written as

$$\mathbf{r}_n = T\cdot\mathbf{r}_{n-1}(1+O(q_{n-1})),$$

where T is a matrix with entries

$$t_{j,k} = \begin{cases}1 & \text{if } k = h(j),\\ 0 & \text{otherwise.}\end{cases}$$

The matrix T encodes the map $j\mapsto h(j)$, and it has the obvious property that every row contains exactly one entry 1, while the remaining entries are 0. All powers of T

have the same property, and so there have to be positive integers a and b such that

$$T^{a+b} = T^a,$$

which implies that

$$\mathbf{r}_{n+a+b} = T^{a+b}\mathbf{r}_n(1+O(q_n)) = T^a\mathbf{r}_n(1+O(q_n)) = \mathbf{r}_{n+a}(1+O(q_n)).$$

Here, we made use of the fact that q_n decreases at a doubly exponential rate, so that

$$(1+O(q_n))(1+O(q_{n+1}))\cdots(1+O(q_{n+a+b-1})) = 1+O(q_n).$$

Hence, the subsequence $\mathbf{r}_{nb+c}$ $(n \geq 0)$ converges for every $0 \leq c < b$. We denote the limit by $\mathbf{R}_c$. Then, there is a constant $0 < \kappa < 1$ such that

$$\mathbf{r}_n = \mathbf{R}_c + O\big(\kappa^{d^n}\big)$$

whenever $n \equiv c \bmod b$, with d as in the definition of q_n. Hence, if $n \equiv c \bmod b$, we have

$$\begin{aligned}
m(X_{n+1}) &= \sum_{A \subseteq \Theta} a_{n+1}(A) \\
&= \sum_{A \subseteq \Theta} \sum_{M' \in \mathcal{H}(A)} \prod_{i=1}^{s} a_n(\eta_i(M')) \\
&= m(X_n)^s \sum_{A \subseteq \Theta} \sum_{M' \in \mathcal{H}(A)} \prod_{i=1}^{s} \Bigg(\prod_{j \in \Theta \setminus \eta_i(M')} \rho_n(j) \Bigg) \Bigg(\prod_{j \in \eta_i(M')} (1-\rho_n(j)) \Bigg) (1+O(q_n)) \\
&= m(X_n)^s \sum_{A \subseteq \Theta} \sum_{M' \in \mathcal{H}(A)} \prod_{i=1}^{s} \Bigg(\prod_{j \in \Theta \setminus \eta_i(M')} R_c(j) \Bigg) \Bigg(\prod_{j \in \eta_i(M')} (1-R_c(j)) \Bigg) \big(1+O\big(\kappa^{d^n}\big)\big) \\
&= m(X_n)^s \big(B_c + O\big(\kappa^{d^n}\big)\big),
\end{aligned}$$

where $R_c(j)$ denotes the j-th component of $\mathbf{R}_c$ and B_c a constant that depends only on c. Now, it is easy to determine the asymptotics of $m(X_n)$: we take the logarithm to find

$$y_{n+1} = \log m(X_{n+1}) = s y_n + \log B_c + O\big(\kappa^{d^n}\big).$$

Let ε_n denote the error term. Then, iterating the recursion gives us

$$\begin{aligned}
y_n &= s^n y_0 + \sum_{c=0}^{b-1} \sum_{\substack{k=0 \\ k\equiv c \bmod b}}^{n-1} s^{n-k-1}(\log B_c + \varepsilon_k) \\
&= s^n\Bigg(y_0 + \sum_{c=0}^{b-1} \sum_{\substack{k=0 \\ k\equiv c \bmod b}}^{\infty} s^{-k-1}(\log B_c + \varepsilon_k)\Bigg) - \sum_{c=0}^{b-1} \sum_{\substack{k=n \\ k\equiv c \bmod b}}^{\infty} s^{n-k-1}(\log B_c + \varepsilon_k) \\
&= s^n\Bigg(y_0 + \sum_{c=0}^{b-1} \sum_{\substack{k=0 \\ k\equiv c \bmod b}}^{\infty} s^{-k-1}(\log B_c + \varepsilon_k)\Bigg) - \sum_{c=0}^{b-1} \sum_{\substack{k=0 \\ k+n\equiv c \bmod b}}^{\infty} s^{-k-1}\log B_c \\
&\qquad + O\Bigg(\sum_{k=n}^{\infty} s^{n-k-1}\kappa^{d^k}\Bigg) \\
&= C_1 s^n + C_2(n) + O\big(\kappa^{d^n}\big),
\end{aligned}$$

where C_1 is a constant C_2 only depends on the residue class of n modulo b. Hence,

$$m(X_n) = \alpha(n)\cdot \beta^{s^n}\big(1 + O\big(\kappa^{d^n}\big)\big),$$

where $\alpha(n)$ is periodic with period b. Let us state this as a theorem:

Theorem 2. *Suppose that X_n is a sequence of graphs that is constructed as described in Section 2, and that*

- *each distinguished vertex belongs to a unique part and is also not incident to any edge of the model graph, and*
- *no part contains more than one distinguished vertex.*

Then there are positive constants $\beta > 1$, $\kappa < 1$, and a periodic function $\alpha : \mathbb{N} \to \mathbb{R}^+$ such that

$$m(X_n) = \alpha(n)\cdot \beta^{s^n}\big(1 + O\big(\kappa^{d^n}\big)\big).$$

It also follows immediately that the growth constant

$$\lim_{n\to\infty} \frac{\log m(X_n)}{|VX_n|} = \frac{\log\beta}{|VX_0| + \frac{|VG| - s\theta}{s-1}}$$

always exists, since the number of vertices of X_n grows at an exponential rate of s^n, as can be seen from equation (1).

5. Examples

5.1. **Graph families that satisfy our conditions.**

Example 5.1. The Koch graphs have the nice property that we actually get an explicit formula for the number of matchings. Note that due to symmetry, we have $a_n(\{1\}) = a_n(\{2\})$ and $\rho_n(1) = \rho_n(2)$ for all n, which reduces our system of recurrences to three equations, namely

$$(a_{n+1}(\emptyset), a_{n+1}(\{1\}), a_{n+1}(\{1,2\})) = P(a_n(\emptyset), a_n(\{1\}), a_n(\{1,2\})),$$

where

$$P : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x^5 + 8x^4y + 18x^3y^2 + 10x^2y^3 + 3x^4z + 8x^3yz \\ x^4y + 7x^3y^2 + 13x^2y^3 + 5xy^4 + x^4z + 8x^3yz + 12x^2y^2z + x^3z^2 \\ x^3y^2 + 6x^2y^3 + 9xy^4 + 2y^5 + 2x^3yz + 11x^2y^2z + 12xy^3z + x^3z^2 + 4x^2yz^2 \end{pmatrix}$$

Then, an easy induction shows that

$$\rho_n(1) = \rho_n(2) = \tfrac{2}{3}$$

and even more precisely

$$a_n(\emptyset) = 2a_n(\{1\}) = 2a_n(\{2\}) = 4a_n(\{1,2\})$$

for all $n \geq 1$. This reduces the system to the simple equation

$$m(X_{n+1}) = a_{n+1}(\emptyset) + 2a_{n+1}(\{1\}) + a_{n+1}(\{1,2\}) = \frac{2^7\,5^2\,m(X_n)^5}{3^8},$$

with the explicit solution

$$m(X_n) = 9 \cdot 2^{\frac{7}{4}(5^{n-1}-1)} \cdot 5^{\frac{1}{2}(5^{n-1}-1)} = \alpha \cdot \beta^{5^n},$$

where $\alpha = 9 \cdot 2^{-7/4} \cdot 5^{-1/2}$ and $\beta = 2^{7/20} \cdot 5^{1/10}$.

Example 5.2. The graphs described in Example 2.2 were also used as an example in the authors' paper [21], where it was shown, among other results, that the number of matchings in X_n is given by

$$m(X_n) = \frac{2}{\sqrt{7}} \cdot (2\sqrt{7})^{3^n}.$$

Again, we have an explicit formula that exhibits the usual doubly exponential growth.

Example 5.3. Sierpiński graphs were discussed in the aforementioned paper [4], and the two-dimensioqnal case was also treated as an example in [22]. However, our general theorem is applicable to higher dimensions as well. In the two-dimensional case, the growth constants were found to be

$$m(X_n) \sim 1.4277123849 \cdot 2.6763163570^{3^n}$$

in [22]. In the three- and four-dimensional case, we obtain

$$m(X_n) \sim 1.8506206904 \cdot 4.7697931292^{4^n}$$

and

$$m(X_n) \sim 2.4910066647 \cdot 8.9526404164^{5^n},$$

respectively. The asymptotic growth constants, without the precise asymptotics, were also given in [4].

Example 5.4. Note that the model graph G is allowed to contain edges, as long as they are not incident with distinguished vertices. Example 5.2 is of this type, and we would like to describe another one that is closely related to the Sierpiński graphs. The only modification is that we connect the parts by edges instead of amalgamating them at their (distinguished) corner vertices. This gives us the sequence of graphs shown in Figure 6. The analysis is essentially the same as in the previous example, and it is also possible to make use of symmetries. We end up with the following asymptotic behavior:

$$m(X_n) \sim 0.6971213284 \cdot 5.6372305346^{3^n}.$$

Example 5.5. For the analysis of the Pentagasket (see Example 2.4), it is also possible to make use of symmetry properties. For instance, it is easy to see that

$$a_n(\{1,2\}) = a_n(\{2,3\}) = a_n(\{3,4\}) = a_n(\{4,5\}) = a_n(\{5,1\}).$$

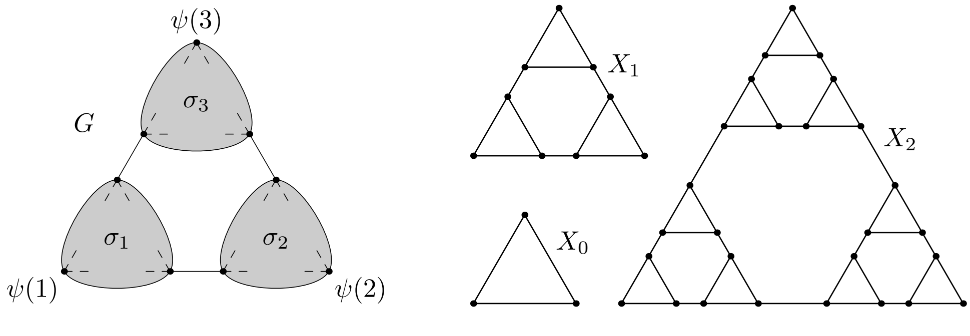

FIGURE 6. Sierpiński graphs with additional edges

Note, however, that $a_n(\{1,2\}) \neq a_n(\{1,3\})$. The system of recurrences can thus be reduced to a system that involves only 8 variables (rather than 32!). Since all conditions are satisfied, the asymptotics are of the form given in Theorem 2; we have

$$m(X_n) \sim 1.6806194435 \cdot 7.2081354456^{5^n}.$$

Example 5.6. Our final example in this subsection deals with a sequence of trees that is constructed from the model graph depicted in Figure 7; to be precise, we start with $X_0 = K_2$ ($\varphi(i) = v_i$ if $VX_0 = \{v_1, v_2\}$) and define ψ and σ_i for $1 \leq i \leq 4$ as follows:

j	$\psi(j)$	$\sigma_1(j)$	$\sigma_2(j)$	$\sigma_3(j)$	$\sigma_4(j)$
1	1	2	3	4	5
2	4	1	2	3	2

Note that there is no symmetry, so the orientation of each part plays a role. In this particular case, note that we have $h(1) = 2$ and $h(2) = 1$ in the notation of Section 4. This yields a periodic function α (with nontrivial period 2) in Theorem 2, i.e. we have

$$m(X_n) \sim \begin{cases} 1.1705265656 \cdot 1.5613328336^{4^n} & \text{if } n \text{ is even,} \\ 1.1505965967 \cdot 1.5613328336^{4^n} & \text{if } n \text{ is odd.} \end{cases}$$

5.2. **Model graphs with edges incident to distinguished vertices.**

Example 5.7. Complete d-ary trees (see Example 2.5) are a very important example—the asymptotic behavior of the number of matchings in these trees was also discussed in the recent paper [12]. Since there is only one distinguished vertex, independence

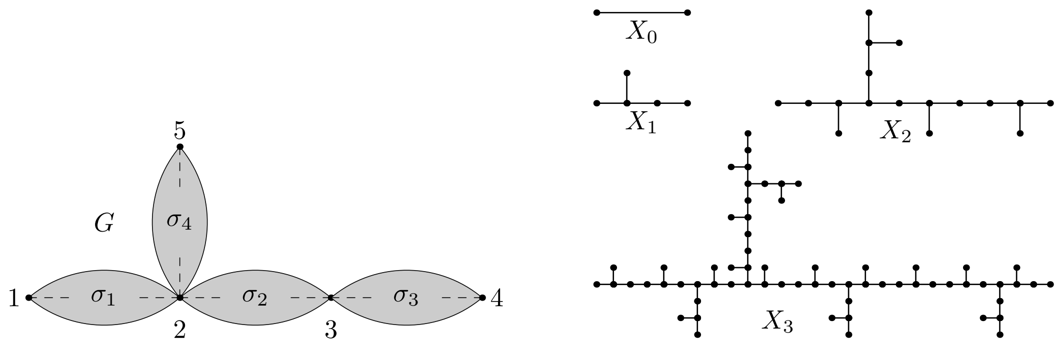

FIGURE 7. Model graph and first three steps in Example 5.6.

is not a question in this example, but the parameter $\rho_n(1)$ shows some interesting dynamical behavior again.

First of all, we obtain the simple recurrences

$$a_{n+1}(\emptyset) = m(X_n)^d \qquad \text{and} \qquad a_{n+1}(\{1\}) = dm(X_n)^{d-1}a_n(\emptyset),$$

from which it follows that

$$\rho_{n+1}(1) = \frac{a_{n+1}(\emptyset)}{m(X_{n+1})} = \frac{m(X_n)^d}{m(X_n)^d + dm(X_n)^{d-1}a_n(\emptyset)} = \frac{1}{1+d\rho_n(1)},$$

which means that the values $\rho_n(1)$ are now the iterates of a rational function, so it is not immediate that the sequence $\rho_n(1)$ converges. However, straightforward induction (note also that $\rho_0(1) = 1$) shows that

$$\begin{aligned}\rho_n(1) &= \frac{\left(\frac{1}{2}\left(1+\sqrt{1+4d}\right)\right)^{n+1} - \left(\frac{1}{2}\left(1-\sqrt{1+4d}\right)\right)^{n+1}}{\left(\frac{1}{2}\left(1+\sqrt{1+4d}\right)\right)^{n+2} - \left(\frac{1}{2}\left(1-\sqrt{1+4d}\right)\right)^{n+2}} \\ &= \frac{2}{1+\sqrt{1+4d}}\left(1 + O\left(\frac{\sqrt{1+4d}-1}{\sqrt{1+4d}+1}\right)^n\right).\end{aligned}$$

From this, we obtain

$$m(X_{n+1}) = \frac{a_{n+1}(\emptyset)}{\rho_{n+1}(1)} = m(X_n)^d \cdot \frac{1+\sqrt{1+4d}}{2}\left(1 + O\left(\frac{\sqrt{1+4d}-1}{\sqrt{1+4d}+1}\right)^n\right),$$

and the usual method of taking logarithms yields

$$\log m(X_{n+1}) = d\log m(X_n) + \log\frac{1+\sqrt{1+4d}}{2} + O\left(\frac{\sqrt{1+4d}-1}{\sqrt{1+4d}+1}\right)^n.$$

Iterating this recurrence gives us

$$\log m(X_n) = d^n \log m(X_0) + \sum_{i=0}^{n-1} d^{n-i-1} \log \frac{1+\sqrt{1+4d}}{2} + \sum_{i=0}^{n-1} d^{n-i-1}\varepsilon_i,$$

where ε_i is the error term in the i-th step. This can be rewritten as

$$\begin{aligned}\log m(X_n) &= \frac{d^n-1}{d-1} \log \frac{1+\sqrt{1+4d}}{2} + d^n \sum_{i=0}^{\infty} d^{-i-1}\varepsilon_i - \sum_{i=n}^{\infty} d^{n-i-1}\varepsilon_i \\ &= \frac{d^n-1}{d-1} \log \frac{1+\sqrt{1+4d}}{2} + C(d)d^n + O\bigg(\frac{\sqrt{1+4d}-1}{\sqrt{1+4d}+1}\bigg)^n.\end{aligned}$$

Finally, this yields the asymptotics for the number of matchings:

$$m(X_n) \sim \left(\frac{1+\sqrt{1+4d}}{2}\right)^{-1/(d-1)} \cdot \beta(d)^{d^n}$$

for some constant $\beta(d)$.

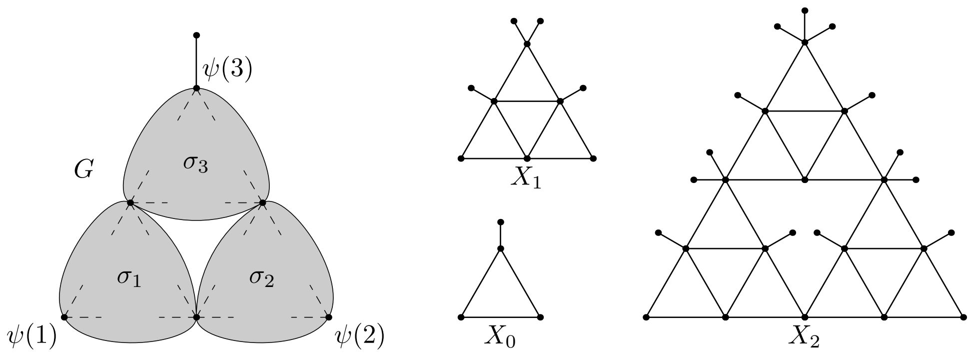

FIGURE 8. A "coronated" version of the Sierpiński graphs.

Example 5.8. Consider the following slight modification of the Sierpiński graphs: in the model graph, we add a single edge incident with the top vertex $\psi(3)$. This means that one edge is added to the three parts in each step, and the number of pendant edges incident with a vertex v in the resulting graph depends on the largest complete triangle of which v is the top. Note that we can still make use of our independence theorem (Theorem 1): the maximum degree Δ only grows linearly with n (i.e. $\Delta(X_n) = O(n)$), while the distance between distinguished vertices grows exponentially with n (it is equal to 2^n), which shows that the auxiliary parameter q_n that was introduced in Section 4 still tends to 0 at a doubly exponential rate. To be precise, we have

$$q_n = O\big(C_1^{(2-\varepsilon)^n}\big)$$

for arbitrary $\varepsilon > 0$ and a constant $C_1 < 1$ that only depends on ε. Hence, the only part of the argument that does not hold any longer is the result that $\rho_n(j)$ converges for all j. We have

$$\rho_{n+1}(1) = \rho_n(1) + O(q_n),$$

by the same argument as in the proof of Theorem 2, which shows that $\rho_n(1) = \rho_n(2)$ converges to a value $R_1 = 0.5599790429$ as in Section 4, i.e. with a doubly exponential error term:

$$\rho_n(1) = R_1 + O\big(C_2^{(2-\varepsilon)^n}\big). \tag{3}$$

On the other hand, $\rho_n(3)$ shows a different asymptotic behavior: we have

$$\rho_{n+1}(3) = \frac{\rho_n(3)}{1+\rho_n(3)}(1 + O(q_n)), \tag{4}$$

from which it can be easily deduced that $\rho_n(3)$ behaves like

$$\rho_n(3) = \frac{1}{n + R_3} + O\big(C_2^{(2-\varepsilon)^n}\big)$$

for a constant $R_3 = 2.6460653132$, again with a very small error term. The recurrence for $\rho_n(3)$ can be explained as follows: Consider matchings in X_{n+1} that do not contain the new edge added to the three parts. Of these, approximately $\rho_n(3)$ do not cover $\varphi_{n+1}(3)$ by the same argument that was also used to prove (3). Additionally, there are matchings that do not contain the new edge, and there is an obvious bijection between these matchings and those that do not cover $\varphi_{n+1}(3)$, which shows why (4) holds.

The main asymptotics remain the same in this example, but the error term becomes much weaker now: we get

$$m(X_{n+1}) = \frac{R_1(2-R_1)(n+1+R_3)(1+R_1(n+R_3-1))^2}{(n+R_3)^3}\, m(X_n)^3\big(1 + O\big(C_2^{(2-\varepsilon)^n}\big)\big)$$

from the system of recurrences that is satisfied by the quantities $a_n(A)$ and therefore

$$m(X_n) = \alpha \cdot \beta^{3^n}\Big(1 + \frac{\gamma}{n} + O(n^{-2})\Big),$$

where $\alpha = 1.9886480689$, $\beta = 4.1803026218$, $\gamma = -1.2857811157$ can be determined by the same technique as in Section 4 (i.e. taking logarithms and iterating). It is even possible to compute further terms in the asymptotic expansion in this way.

5.3. **At least one distinguished vertex belongs to more than one part.** In this case, the maximum degree usually tends to ∞. This has two consequences:

- Theorem 1 might not be applicable any longer (however, if the distance between distinguished vertices tends faster to ∞ than the maximum degree, it can still be used—see Examples 5.8 and 5.9), and
- as in the previous subsection, the quotients $\rho_n(j)$ show a more complicated dynamical behavior.

In the following example, we will show the effect of these facts. It is chosen in such a way that very explicit results can be given, so as to simplify the asymptotic discussion.

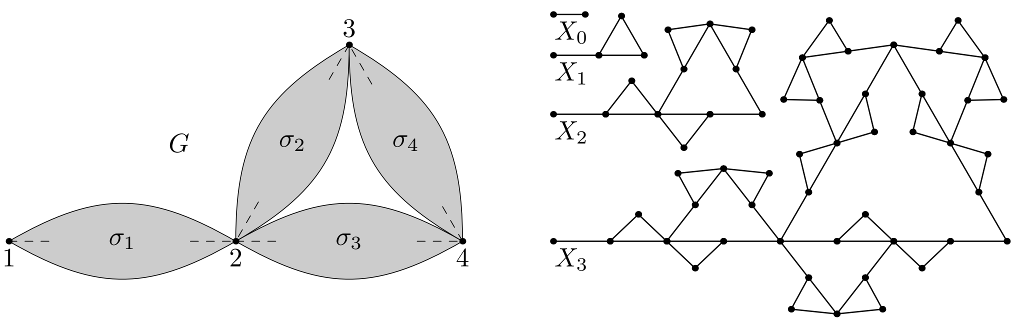

FIGURE 9. Model graph and resulting Austria graphs X_0, X_1, X_2, X_3.

Example 5.9. The so-called "Austria graphs" (their shape resembles a map of Austria) were introduced in [17]. Its model graph (shown in Figure 9) is edgeless, in each step four copies of X_n are amalgamated to form X_{n+1}. The initial graph is $X_0 = K_2$, with the usual definition for the function φ that describes the distinguished vertices on the initial graph. In view of the missing symmetry, it makes an important difference how the two distinguished vertices of each part are identified with the vertices in the model graph. We show three different possibilities, each of which yields to a quite different behavior for the number of matchings. First, let ψ and $\sigma_1, \dots, \sigma_4$ defined by

j	$\psi(j)$	$\sigma_1(j)$	$\sigma_2(j)$	$\sigma_3(j)$	$\sigma_4(j)$
1	1	1	2	4	4
2	4	2	3	2	3

The first three graphs in the resulting sequence are depicted in Figure 9. We obtain the following system of recurrences:

$$(a_{n+1}(\emptyset), a_{n+1}(\{1\}), a_{n+1}(\{2\}), a_{n+1}(\{1,2\})) = P(a_n(\emptyset), a_n(\{1\}), a_n(\{2\}), a_n(\{1,2\})),$$

where

$$P : \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} w^4 + w^3x + 4w^3y + w^2xy + 4w^2y^2 + w^3z \\ w^3x + w^2x^2 + 3w^2xy + wx^2y + 2wxy^2 + w^3z + w^2xz + 2w^2yz \\ 2w^3x + 2w^2x^2 + 6w^2xy + wx^2y + 4wxy^2 + 2w^3z + 3w^2xz + 4w^2yz \\ 2w^2x^2 + 2wx^3 + 4wx^2y + x^3y + x^2y^2 + 4w^2xz + 3wx^2z + 6wxyz + w^2z^2 \end{pmatrix}$$

Note that the maximum degree is still bounded while the distance between distinguished vertices tends to ∞; hence, we can expect the two distinguished vertices to be asymptotically independent. Indeed, it is easy to prove by means of induction that an even stronger result holds: we have

$$a_n(\emptyset) = 2a_n(\{1\}) = a_n(\{2\}) = 2a_n(\{1,2\})$$

for $n \geq 1$ (implying $\rho_n(\{1\}) = \frac{2}{3}$ and $\rho_n(\{2\}) = \frac{1}{2}$), and it follows immediately that

$$m(X_{n+1}) = \frac{7}{18} m(X_n)^4$$

for $n \geq 1$ and thus

$$m(X_n) = 3 \cdot 2^{\frac{4^n+2}{6}} \cdot 21^{\frac{4^{n-1}-1}{3}} = \alpha \cdot \beta^{4^n}$$

with $\alpha = \left(\frac{18}{7}\right)^{1/3}$ and $\beta = 84^{1/12}$.

Things change immediately if we slightly modify the construction by reversing one of the parts: we set $\sigma_1(1) = 2$ and $\sigma_1(2) = 1$ now to obtain a new sequence of graphs (see Figure 10).

This changes the asymptotic behavior of the number of matchings dramatically: making use of the same argument as in the proof of Theorem 2, we should have $\rho_{n+1}(1) \approx \rho_n(2)$, with only a small error term. As in Example 5.6, this leads to

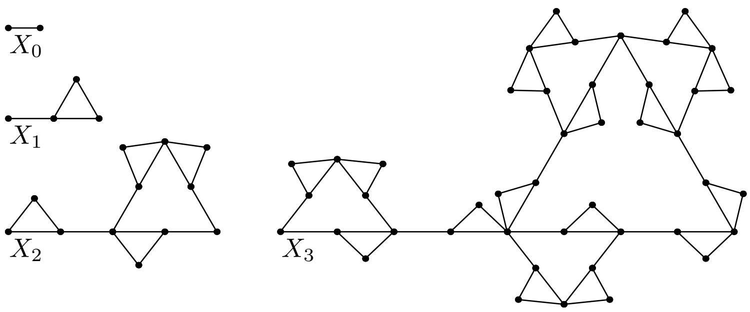

FIGURE 10. Modified Austria graphs X_0, X_1, X_2, X_3.

periodic behavior. The polynomial P is now given by

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} w^4+2w^3x+3w^3y+3w^2xy+2w^2y^2+w^3z \\ w^3y+w^2xy+3w^2y^2+wxy^2+2wy^3+w^3z+3w^2yz \\ 2w^3x+4w^2x^2+4w^2xy+4wx^2y+wxy^2+2w^3z+4w^2xz+3w^2yz \\ 2w^2xy+2wx^2y+4wxy^2+x^2y^2+xy^3+2w^2xz+2w^2yz+6wxyz \\ \quad +3wy^2z+w^2z^2 \end{pmatrix}$$

and an inductive argument shows that for $n \geq 1$,

$$a_n(\emptyset) = \begin{cases} 2^{1-n/2}a_n(\{1\}) = 2^{1-n/2}a_n(\{2\}) = 2^{2-n}a_n(\{1,2\}) & \text{if } n \text{ is even,} \\ 2^{(3-n)/2}a_n(\{1\}) = 2^{(1-n)/2}a_n(\{2\}) = 2^{2-n}a_n(\{1,2\}) & \text{if } n \text{ is odd.} \end{cases}$$

It follows that

$$m(X_{x+1}) = \begin{cases} \frac{(1+2^{n/2})^2(1+3\cdot 2^{n/2-1})}{(1+2^{n/2-1})^7}\, m(X_n)^4 & \text{if } n \text{ is even,} \\ \frac{(1+2^{(n+1)/2})^2}{(1+2^{(n-1)/2})^2(1+2^{(n-3)/2})^4}\, m(X_n)^4 & \text{if } n \text{ is odd.} \end{cases}$$

These recurrences lead to the following explicit formulæ: for even n, we have

$$m(X_n) = (1+2^{n/2-1})^2(1+2^{n/2})^2 \cdot 2^{4^{n-1}} \cdot \prod_{k=1}^{n/2-1} (1+2^k)^{9\cdot 4^{n-1-2k}}(1+3\cdot 2^{k-1})^{4^{n-1-2k}}.$$

Rewriting this as

$$m(X_n) = (1+2^{n/2-1})^2(1+2^{n/2})^2 \cdot 2^{4^{n-1}} \cdot 2^{\sum_{k=1}^{n/2-1}(10k-1)4^{n-1-2k}} \cdot 3^{\sum_{k=1}^{n/2-1} 4^{n-1-2k}}$$
$$\times \prod_{k=1}^{n/2-1} \Big((1+2^{-k})^9\big(1+\tfrac{2}{3}\cdot 2^{-k}\big)\Big)^{4^{n-1-2k}},$$

we find the asymptotics

$$m(X_n) \sim 2^{-86/45}\, 3^{-4/15} \cdot 2^{2n/3} \cdot \beta^{4^n}(1 + O(2^{-n/2})),$$

where

$$\beta = 2^{37/90}\, 3^{1/60} \cdot \prod_{k=1}^{\infty} \Big((1 + 2^{-k})^9 \big(1 + \tfrac{2}{3}\, 2^{-k}\big) \Big)^{4^{-1-2k}} = 1.4433516328.$$

Likewise, the following formula holds for odd n:

$$m(X_n) = (1 + 2^{(n-1)/2})(1 + 2^{(n-3)/2}) \cdot 2^{4^{n-1}} \cdot \prod_{k=1}^{(n-1)/2} (1 + 2^k)^{9 \cdot 4^{n-1-2k}} (1 + 3 \cdot 2^{k-1})^{4^{n-1-2k}},$$

yielding the asymptotics

$$m(X_n) \sim 2^{-104/45} 3^{-1/15} \cdot 2^{2n/3} \cdot \beta^{4^n}(1 + O(2^{-n/2}))$$

with the same β as before. Hence, we see two important effects of the slight modification on the asymptotic behavior—we observe an additional exponential factor that does not occur in Theorem 2 and also not in the first version of the Austria graphs as well as periodicity (with period 2), which was to be expected (since the two distinguished vertices are essentially interchanged at each step).

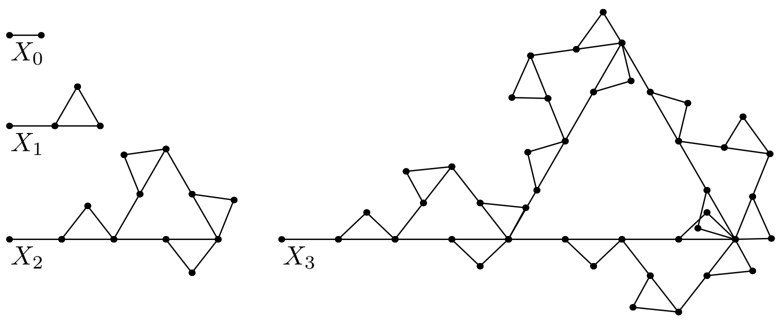

Figure 11. Another modification of the Austria graphs.

Now, let us modify the construction in another way. This time, we reverse parts 3 and 4, i.e. the substitutions are defined by the following table:

j	$\psi(j)$	$\sigma_1(j)$	$\sigma_2(j)$	$\sigma_3(j)$	$\sigma_4(j)$
1	1	1	2	2	3
2	4	2	3	4	4

This yields the sequence of graphs that is depicted in Figure 11. The corresponding system of recurrences is given by

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} w^4+3w^3x+2w^2x^2+2w^3y+2w^2xy+w^2y^2+w^3z \\ w^3x+3w^2x^2+2wx^3+w^2xy+wx^2y+w^3z+2w^2xz+w^2yz \\ 2w^3y+4w^2xy+wx^2y+4w^2y^2+2wxy^2+2wy^3+2w^3z+3w^2xz \\ \quad +4w^2yz \\ 2w^2xy+4wx^2y+x^3y+2wxy^2+x^2y^2+2w^2xz+3wx^2z+2w^2yz \\ \quad +4wxyz+2wy^2z+w^2z^2 \end{pmatrix}$$

Note that Theorem 1 is actually not applicable in this example, since the maximum degree and the distance between the two distinguished vertices are both equal to 2^n at step n, so that the sequence q_n that was defined in Section 4 does not tend to 0 any longer. However, we still obtain independence, and even explicit values for $\rho_n(1)$ and $\rho_n(2)$ again: we have

$$a_n(\emptyset) = 2a_n(\{1\}) = 2^{1-n}a_n(\{2\}) = 2^{2-n}a_n(\{1,2\}),$$

and in a similar manner as in the previous example, we obtain

$$m(X_n) = 3\cdot 2^{4^{n-1}-1}\cdot(2^{n-1}+1)\cdot\prod_{k=1}^{n-1}\big((2^{k-2}+1)(2^k+3)\big)^{4^{n-k-1}},$$

which is asymptotically

$$m(X_n) \sim 3\cdot 2^{-14/9}\cdot 2^{n/3}\cdot \beta^{4^n},$$

where

$$\beta = 2^{11/36}\prod_{k=1}^{\infty}\big((1+2^{2-k})(1+3\cdot 2^{-k})\big)^{4^{-k-1}} = 1.4341501552.$$

It is quite remarkable that slight changes can result in entirely different asymptotic behavior, especially if this is compared to the fact that was noted in [20]: the number of spanning trees is independent of the orientations of the four parts, i.e. the number of spanning trees is the same for all three sequences. Note also that, even though the conditions of Theorem 1 were not satisfied, we obtained

$$\frac{a_n(\emptyset)a_n(\{1,2\})}{a_n(\{1\})a_n(\{2\})} = 1$$

(even with identity, not only asymptotically). Indeed, it is quite possible that a stronger version of Theorem 1 (that does not involve the maximum degree) can be proved. However, we will see in our final example that we cannot expect such an independence of distinguished vertices if their distance remains bounded.

5.4. **A part contains more than one distinguished vertex.** In this case, the distance between distinguished vertices does not necessarily tend to ∞ any longer, and so we cannot expect (asymptotic) independence. The following sequence of graphs exhibits the consequences—the recurrence can be simplified by means of certain ad hoc arguments and solved explicitly, which greatly simplifies the analysis.

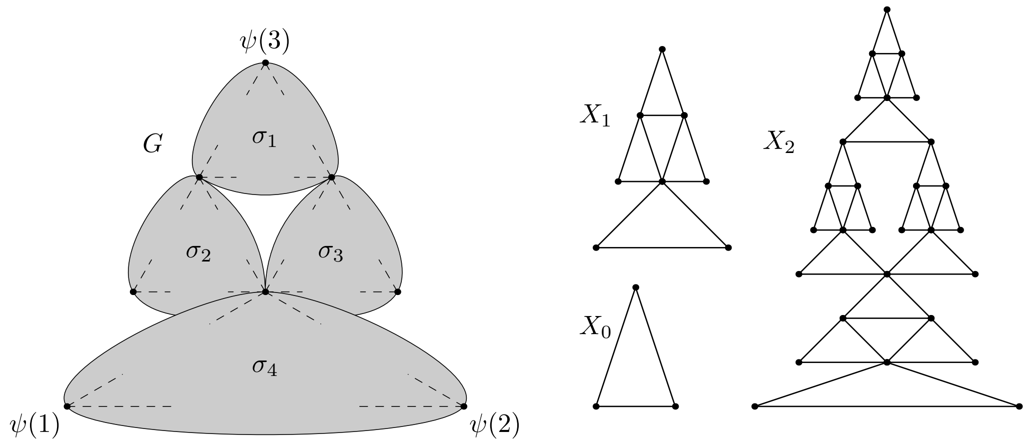

FIGURE 12. Model graph and Rocket graphs X_0, X_1, X_2.

Example 5.10. We slightly modify the model graph for the Sierpiński graphs by adding one more part and changing the choice of distinguished vertices (see Figure 12). Apart from not being completely symmetric any longer, the graph has the property that two of its distinguished vertices belong to the same part, and it is easy to see that their distance remains 1 throughout the whole sequence. In view of their shape, we will henceforth refer to this sequence as "Rocket graphs".

Note that the distinguished vertices $v_{1,n} = \varphi_n(1)$ and $v_{2,n} = \varphi_n(2)$ have a unique common neighbor w_n at any stage. There is an obvious bijection between all matchings that contain the edge between $v_{1,n}$ and $v_{2,n}$ and those which do not contain any edge incident with $v_{1,n}$ or $v_{2,n}$ at all. Furthermore, there is also a bijection between matchings that contain the edge between $v_{1,n}$ and w_n and those that contain the edge between $v_{2,n}$ and w_n. These are independent of the third distinguished vertex $\varphi_n(3)$, and so we have

$$a_n(\emptyset) = a_n(\{1,2\}), \qquad a_n(\{3\}) = a_n(\{1,2,3\}),$$
$$a_n(\{1\}) = a_n(\{2\}), \qquad a_n(\{1,3\}) = a_n(\{2,3\}).$$

This simplifies the recurrences quite a lot, and so we obtain a simplified system with only 4 auxiliary sequences, namely

$$(a_{n+1}(\emptyset), a_{n+1}(\{1\}), a_{n+1}(\{3\}), a_{n+1}(\{1,3\})) = P(a_n(\emptyset), a_n(\{1\}), a_n(\{3\}), a_n(\{1,3\})),$$

where

$$P : \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} (3w+y)(2w^3+6w^2x+6wx^2+2x^3+2w^2y+4wxy+2x^2y+wy^2 \\ \qquad +2w^2z+4wxz+2x^2z+2wyz+wz^2) \\ (3x+z)(2w^3+6w^2x+6wx^2+2x^3+2w^2y+4wxy+2x^2y+wy^2 \\ \qquad +2w^2z+4wxz+2x^2z+2wyz+wz^2) \\ (3w+y)(y+z)(2w^2+4wx+2x^2+2wy+2xy+y^2+2wz+2xz+yz) \\ (3x+z)(y+z)(2w^2+4wx+2x^2+2wy+2xy+y^2+2wz+2xz+yz) \end{pmatrix}$$

Now it can be proved by induction that

$$4a_n(\emptyset) = 12a_n(\{1\}) = 5a_n(\{3\}) = 15a_n(\{1,3\})$$

for all $n \geq 1$, which reduces the recurrence to a very simple one:

$$m(X_{n+1}) = \frac{323}{972} m(X_n)^4$$

with the explicit solution

$$m(X_n) = 72 \cdot 124032^{\frac{4^{n-1}-1}{3}} = \alpha \cdot \beta^{4^n},$$

where $\alpha = \left(\frac{972}{323}\right)^{1/3}$ and $\beta = 124032^{1/12}$. Note, however, that

$$a_n(\emptyset) \cdot a_n(\{1,2\}) \qquad \text{and} \qquad a_n(\{1\}) \cdot a_n(\{2\})$$

are not even asymptotically equal, which they would have to be if the conditions of Theorem 1 were satisfied. Thus, we cannot expect independence of distinguished vertices if the distance remains bounded.

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