On the number of spanning trees on various lattices

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Abstract. We consider the number of spanning trees in lattices; for a lattice $\mathcal{L}$, one defines the bulk limit $z_{\mathcal{L}} = \lim_{|VG|\to\infty}(\log N_{ST}(G))/|VG|$, where $N_{ST}(G)$ is the number of spanning trees in a finite section G of $\mathcal{L}$. Explicit values for $z_{\mathcal{L}}$ are known in various special cases. In this note we describe a simple yet effective method to deduce relations between the values of $z_{\mathcal{L}}$ for different lattices $\mathcal{L}$ by means of electrical network theory.

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1. Introduction

Let G be a graph. We write VG to denote the vertex (site) set of G and EG to denote the edge (bond) set. All graphs are allowed to contain parallel edges and loops. A subgraph H of G is *spanning* if $VH = VG$. A graph T is a *tree*, if T is connected and does not contain cycles. A *spanning tree* of G is a spanning subgraph which is also a tree. Further information concerning graphs can be found for instance in [1].

We assume that G is edge-weighted: if e is an edge, then denote by $c(e)$ the edge weight (conductance). We write $N_{ST}(G)$ to denote the weighted number of spanning trees:

$$N_{ST}(G) = \sum_T \prod_{e \in ET} c(e),$$

where the sum is taken over all spanning trees T of G. We note that $N_{ST}(G)$ is the usual number of spanning trees, if unit conductances on G are used (i.e., $c(e) = 1$ for all edges e). If no conductances are explicitly given, we will always endow graphs with unit conductances. The enumeration of spanning trees has received much interest since the fundamental work of Kirchhoff, see [2]. Two well-known methods for computing $N_{ST}(G)$ are as follows: $N_{ST}(G)$ is equal to any cofactor of the Laplace matrix, see [1, Theorem 20.15], and $N_{ST}(G)$ is equal to a special value of the Tutte polynomial, see [1, Section 21.7].

Due to their structure, lattices are of special interest. Especially the number of spanning trees in a finite subgraph of a lattice was studied extensively, see for instance [3, 4, 5, 6, 7]. Let $\mathcal{L}$ be a lattice and let G be a finite section. It turns out that $N_{ST}(G)$ has asymptotically exponential growth; one defines the quantity $z_{\mathcal{L}}$ by

$$z_{\mathcal{L}} = \lim_{|VG| \to \infty} \frac{\log N_{ST}(G)}{|VG|}.$$

This limit is known as bulk, or thermodynamical limit. Closed form expressions for $z_{\mathcal{L}}$ have been obtained for many lattices, see [3, 5, 6, 7].

In this paper we show how to use the theory of electrical networks for counting spanning trees, and use this technique to provide relations between the values of $z_{\mathcal{L}}$ for several lattices.

2. Electrical networks

Let G be an edge-weighted graph. The Laplace matrix $L = L(G)$ is defined by its entries

$$L_{x,y} = -\sum_{\substack{e \in EG \\ e \text{ connects } x,y}} c(e) \qquad \text{and} \qquad L_{x,x} = -\sum_{\substack{z \in VG \\ z \neq x}} L_{x,z}$$

where x, y are vertices in VG, $x \neq y$. The well-known Matrix-Tree Theorem states that any cofactor of L is equal to $N_{ST}(G)$, see for instance [1, Theorem 20.15]. We say that two edge-weighted graphs (electrical networks) G and H are *electrically equivalent* with respect to $B \subseteq VG \cap VH$, if they cannot be distinguished by applying voltages to B and measuring the resulting currents on B. By Kirchhoff's current law this means that the rows corresponding to B of $L_G H_B^{VG}$ and $L_H H_B^{VH}$ are equal, where H_B^{VG} is the matrix associated to harmonic extension, see for instance [8, 9].

The following theorem was proved in [10]. It states that if a subgraph of G is replaced by an electrically equivalent graph, the number $N_{ST}(G)$ only changes by a factor which involves the subgraph to be replaced and the graph it is replaced by, but not G itself.

Theorem 1. *Suppose that an edge-weighted graph X can be decomposed into graphs G and H, so that EG and EH are disjoint, $EX = EG \cup EH$, and $VX = VG \cup VH$. We set $B = VG \cap VH$. Let H' be an edge-weighted graph with $EG \cap EH' = \varnothing$ and $VG \cap VH' = B$, such that H and H' are electrically equivalent with respect to B, and assume that $N_{ST}(H) \neq 0$ and $N_{ST}(H') \neq 0$. Then*

$$\frac{N_{ST}(X)}{N_{ST}(H)} = \frac{N_{ST}(X')}{N_{ST}(H')}.$$

As a consequence of this theorem we may use simplification techniques for electrical networks in order to compute the number of spanning trees. We write K_n for the complete graph with n vertices and $K_{1,n}$ for the star, see [1]. In the following we list some useful operations on electrical networks:

(i) Parallel edges: If two parallel edges with conductances a and b are merged into a single edge with conductance $a + b$, the (weighted) number of spanning trees remains the same.

(ii) Serial edges: If two serial edges with conductances a and b are merged into a single edge with conductance $\frac{ab}{a+b}$, then:

$$N_{ST}(X') = \frac{1}{a+b} \cdot N_{ST}(X).$$

(iii) Wye-Delta transform: if a star with conductances a, b, c (see Figure 1) is changed into an electrically equivalent triangle with conductances $x = \frac{bc}{a+b+c}$, $y = \frac{ac}{a+b+c}$, and $z = \frac{ab}{a+b+c}$, the weighted number of spanning trees changes as follows:

$$N_{ST}(X') = \frac{1}{a+b+c} \cdot N_{ST}(X).$$

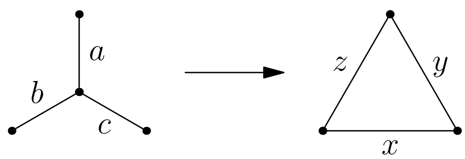


Figure 1: Wye-Delta transform.

(iv) If a star $K_{1,n}$ with conductances a is changed into an electrically equivalent complete graph K_n with conductances $\frac{a}{n}$, then:

$$N_{ST}(X') = \frac{1}{an} \cdot N_{ST}(X).$$

Notice that the last operation is a generalization of Wye-Delta (in the case of equal conductances). Of course there are converses to all four operations. It is known that parallel and serial reductions hold in the more general setting of the q-Potts model (multivariate Tutte polynomial), see [11].

For later use we need the following simple result that estimates the change in $N_{ST}(G)$ if conductances are modified. In order to emphasize the edge weights c, we write $N_{ST}(G,c)$ instead of $N_{ST}(G)$.

Theorem 2. *Let c and $\tilde{c}$ be two sets of conductances on a graph G. Assume that there is a subset $F \subseteq EG$, such that $c(e)=\tilde{c}(e)$ for $e \in EG \setminus F$ and $0<\tilde{c}(e)/c(e)<\infty$ for $e \in F$. Set*

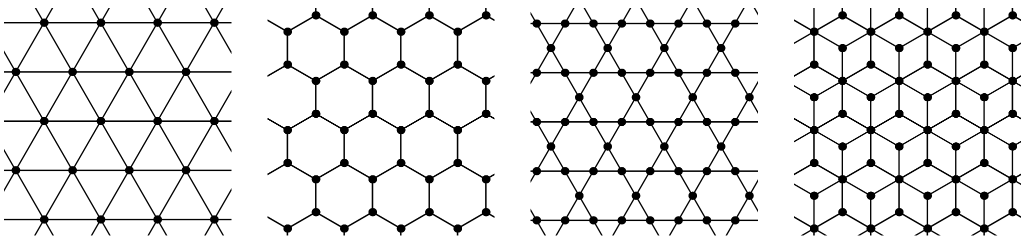
$$m = \min\Big\{\frac{\tilde{c}(e)}{c(e)} \;:\; e \in EG\Big\} \qquad \text{and} \qquad M = \max\Big\{\frac{\tilde{c}(e)}{c(e)} \;:\; e \in EG\Big\}.$$

Then

$$\min\{m,1\}^k N_{ST}(G,c) \le N_{ST}(G,\tilde{c}) \le \max\{M,1\}^k N_{ST}(G,c),$$

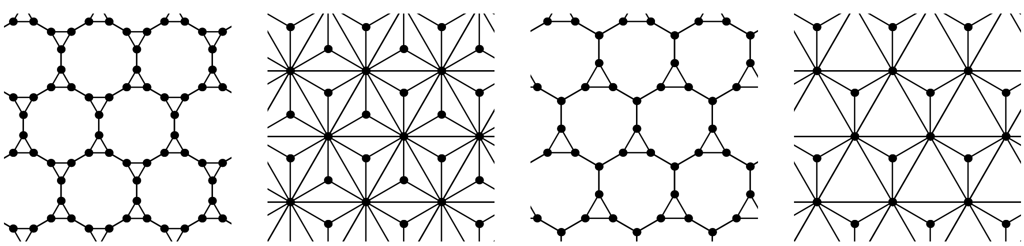
where $k=\min\{|F|,|VG|-1\}$.

3. Lattices



(a) triangular lattice $\mathcal{L}_{tri}$ (b) honeycomb lattice $\mathcal{L}_{hc}$ (dual of $\mathcal{L}_{tri}$) (c) kagomé lattice $\mathcal{L}_{kag}$ (d) diced lattice $\mathcal{L}_{dic}$ (dual of $\mathcal{L}_{kag}$)

Figure 2: Archimedean lattices and their duals with symmetry of order 3.



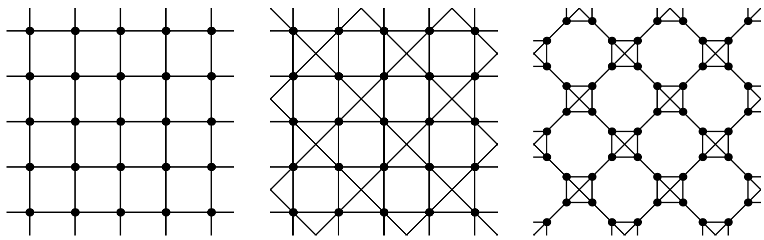
(a) $3\cdot 12\cdot 12$ lattice $\mathcal{L}_{3\cdot 12\cdot 12}$ (b) triakis lattice $\mathcal{L}_{triak}$ (dual of $\mathcal{L}_{3\cdot 12\cdot 12}$) (c) lattice with nonagons and triangles $\mathcal{L}_{nine}$ (d) lattice $\mathcal{L}^*_{nine}$ (dual of $\mathcal{L}_{nine}$)

Figure 3: More lattice with a symmetry of order 3.

In this section we compute relations between the bulk limits $z_{\mathcal{L}}$ of different lattices $\mathcal{L}$ using the connection between electrical network theory and the combinatorial problem of counting spanning trees outlined in the previous section. First some lattices with symmetry of order 3 are considered: the triangular lattice $\mathcal{L}_{tri}$, the honeycomb lattice $\mathcal{L}_{hc}$, the kagomé lattice $\mathcal{L}_{kag}$, the diced lattice $\mathcal{L}_{dic}$ (see Figure 2), the $3 \cdot 12 \cdot 12$ lattice $\mathcal{L}_{3\cdot 12\cdot 12}$, the triakis lattice $\mathcal{L}_{triak}$, a lattice consisting of nonagons and triangles and its dual ($\mathcal{L}_{nine}$ and $\mathcal{L}^*_{nine}$), see Figure 3. Furthermore, relations between the square lattice $\mathcal{L}_{sq}$ and modifications of the square lattice and of the bathroom lattice ($\mathcal{L}_{msq}$ and $\mathcal{L}_{mbr}$), see Figure 4, are established. As can be seen from the detailed calculation for the triakis lattice below, this method will always give relations of the form

$$z_{\mathcal{L}'} = a z_{\mathcal{L}} + b \log c$$

for some rational numbers a, b, c.



(a) square lattice $\mathcal{L}_{sq}$ (b) modified square lattices $\mathcal{L}_{msq}$ (c) modified bathroom lattice $\mathcal{L}_{mbr}$

Figure 4: Some lattices with a symmetry of order 4.

Let us remark that planar duality yields another method in order to obtain such relations, see [5]. Let $\mathcal{L}$ be a two-dimensional lattice and $\mathcal{L}^*$ its planar dual, then

$$z_{\mathcal{L}^*} = z_{\mathcal{L}}/\nu_{\mathcal{L}}.$$

The constant $\nu_{\mathcal{L}}$ is defined by the bulk limit

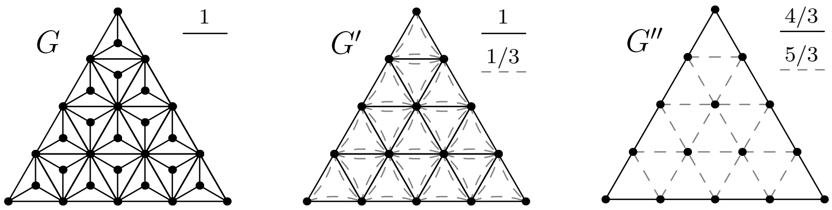
$$\nu_{\mathcal{L}} = \lim_{|VG|\to\infty} \frac{|VG^*|}{|VG|},$$

where G is a finite section of $\mathcal{L}$ and G^* is its planar dual (note that $\nu_{\mathcal{L}}\nu_{\mathcal{L}^*} = 1$). Short calculations yield

$$\nu_{hc} = \tfrac{1}{2}, \quad \nu_{dic} = 1, \quad \nu_{3\cdot 12\cdot 12} = \tfrac{1}{2}, \quad \nu_{nine} = \tfrac{1}{2}.$$

In order to demonstrate the use of electrical network theory we discuss the triakis lattice $\mathcal{L}_{triak}$ in detail, see Figure 3b. Consider a triangular section G of $\mathcal{L}_{triak}$ with k edges on each side of the boundary (see Figure 5a) and apply the Wye-Delta transform, see Figure 5b. Finally, parallel edges are replaced by single edges, see Figure 5c. Simple calculations yield

$$|VG| = \tfrac{3}{2}k^2 + \tfrac{3}{2}k + 1 \qquad \text{and} \qquad |VG''| = \tfrac{1}{2}(k+1)(k+2).$$



(a) A section of $\mathcal{L}_{triak}$ (b) Wye-Delta simplification (c) Parallel edge simplification

Figure 5: Electrical transformations proving a relation between z_{triak} and z_{tri}. Conductances on edges are indicated to the right.

Notice that we have applied k^2 Wye-Delta transforms, thus

$$N_{ST}(G'', c'') = (\tfrac{1}{3})^{k^2} N_{ST}(G).$$

Using Theorem 2 we get

$$(\tfrac{4}{5})^{3k} N_{ST}(G'', \tfrac{5}{3}) \leq N_{ST}(G'', c'') \leq N_{ST}(G'', \tfrac{5}{3}),$$

where $N_{ST}(G'', \frac{5}{3})$ denotes the weighted number of spanning trees with respect to constant conductance equal to $\frac{5}{3}$. Obviously,

$$N_{ST}(G'', \tfrac{5}{3}) = (\tfrac{5}{3})^{|VG''|-1} N_{ST}(G'').$$

Collecting the pieces yields

$$\begin{aligned}
z_{triak} &= \lim_{|VG|\to\infty} \frac{\log N_{ST}(G)}{|VG|} \\
&= \lim_{|VG|\to\infty} \frac{\log N_{ST}(G'') + k^2 \log 3 + (|VG''|-1)\log \frac{5}{3} + O(k)}{|VG|} \\
&= \lim_{|VG|\to\infty} \frac{|VG''|}{|VG|} \Big(\frac{\log N_{ST}(G'')}{|VG''|} + \frac{k^2 \log 3 + (|VG''|-1)\log \frac{5}{3}}{|VG''|} \Big) \\
&= \tfrac{1}{3}(z_{tri} + \log 15).
\end{aligned}$$

Using similar calculations we obtain expressions for z_{hc}, z_{kag}, z_{dic}, z_{triak}, $z_{3\cdot 12\cdot 12}$, z_{nine}, z^*_{nine} in terms of z_{tri}. Let us summarize these relations:

$$\begin{aligned}
z_{hc} &= \tfrac{1}{2} z_{tri}, \\
z_{kag} &= z_{dic} = \tfrac{1}{3}(z_{tri} + \log 6), \\
z_{triak} &= 2 z_{3\cdot 12\cdot 12} = \tfrac{1}{3}(z_{tri} + \log 15), \\
z^*_{nine} &= 2 z_{nine} = \tfrac{1}{2} z_{tri} + \log 2.
\end{aligned}$$

The first equation in each line can also be obtained by means of planar duality. The actual value of z_{tri} is determined in [5, 7]:

$$z_{tri} = \frac{3\sqrt{3}}{\pi} \Big(1 - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{13^2} \mp \cdots \Big) = 1.615329\ldots$$

Likewise, using the substitution $K_{1,4} \leftrightarrow K_4$, we get

$$z_{msq} = \tfrac{1}{2} z_{sq} + \tfrac{3}{2} \log 2, \qquad z_{mbr} = \tfrac{1}{4}(z_{sq} + \log 24).$$

Note that in this case the thermodynamic limits involved in left and right hand side of the above equations are based on different shapes of the sections. When using square sections in the limit on the one side, the limit of the other side is defined by diamond sections. The value of z_{sq} is determined in [5, 6, 7]:

$$z_{sq} = \frac{4C}{\pi} = \frac{4}{\pi}\Big(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} \pm \cdots\Big) = 1.166243\ldots,$$

where C is the Catalan constant.

In these calculations, it is advantageous to have a few general considerations at hand. It is easy to see that scaling by a factor c is equivalent to adding $\log c$ to the bulk limit: if $\mathcal{L}(c)$ denotes the lattice resulting from $\mathcal{L}$ by multiplying all conductances by c, one has

$$z_{\mathcal{L}(c)} = z_{\mathcal{L}} + \log c.$$

Another useful observation is the following:

Theorem 3. *Suppose that the lattice $\mathcal{L}'$ is obtained from a lattice $\mathcal{L}$ by subdividing each edge into s edges. Write $\lambda_{\mathcal{L}}$ for the bulk limit*

$$\lambda_{\mathcal{L}} = \lim_{|VG| \to \infty} \frac{|EG|}{|VG|},$$

where G is a finite section of $\mathcal{L}$. Then one has

$$z_{\mathcal{L}'}(1 + (s-1)\lambda_{\mathcal{L}}) = z_{\mathcal{L}} + (\lambda_{\mathcal{L}} - 1)\log s.$$

A common construction that covers various examples is the following: start with an r-regular lattice. Replace each vertex by a complete graph K_r; each of the r new vertices corresponds to one of the incident edges. Furthermore, each edge of the original lattice is replaced by $s \geq 0$ serial edges connecting the two vertices in the new lattice corresponding to the edge. For example, the kagomé lattice $\mathcal{L}_{kag}$ and the $3 \cdot 12 \cdot 12$ lattice $\mathcal{L}_{3 \cdot 12 \cdot 12}$ are obtained from the honeycomb lattice in this way (with $s = 0$ and $s = 1$ respectively), and the modified square and modified bathroom lattices $\mathcal{L}_{msq}$ and $\mathcal{L}_{mbr}$ are obtained from the square lattice (again, with $s = 0$ and $s = 1$ respectively). The following general connection holds:

Theorem 4. *If the lattice $\mathcal{L}'$ results from an r-regular lattice $\mathcal{L}$ by the above construction, then the relation*

$$z_{\mathcal{L}'} = \frac{2}{r(s+1)} z_{\mathcal{L}} + \frac{1 - 2/r}{s+1} \log(r(2 + rs))$$

holds.

It should also be noted that $s = 0$ yields exactly the line graph (so that the theorem follows from general results on line graphs of regular graphs—see for instance [12, Lemma 8.2.5]—in this special case), and that $s = 1$ corresponds to truncation of lattices (in particular in the case $r = 3$; for instance, the $3 \cdot 12 \cdot 12$ lattice is a truncated honeycomb lattice).

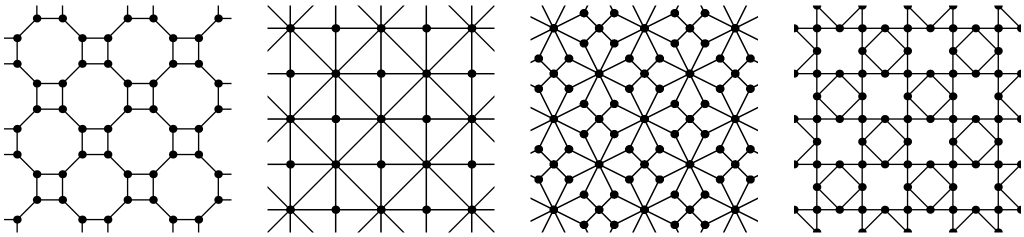
Last but not least, let us mention four more nice lattices allowing simple relations (see Figure 6). Note that the lattice $\mathcal{L}_{kite}$ results from $\mathcal{L}_{tetra}$ by splitting all edges into two parallel edges and applying the Delta-Wye transform to all triangles afterwards. Hence

$$z_{br} = 2z_{tetra} \qquad \text{and} \qquad z_{kite} = z_{abr} = \tfrac{1}{3}(z_{tetra} + \log 6)$$

using planar duality twice ($\nu_{br} = \frac{1}{2}$, $\nu_{abr} = 1$). We note that the value of z_{br} is determined in [3]:

$$z_{br} = \frac{C}{\pi} + \frac{1}{2}\log(\sqrt{2}-1) + \frac{1}{\pi}\int_0^{3+2\sqrt{2}} \frac{\arctan t}{t} dt = 0.786684\ldots,$$

where C is the Catalan constant.



(a) bathroom lattice L_{br}

(b) tetrakis lattice $\mathcal{L}_{tetra}$ (dual of $\mathcal{L}_{br}$)

(c) $\mathcal{L}_{kite}$: a lattice composed of kites

(d) another bathroom lattice $\mathcal{L}_{abr}$ (dual of $\mathcal{L}_{kite}$)

Figure 6: Another set of lattices with a symmetry of order 4 allowing simple relations.

These are just but a few representative examples and the list could be extended infinitely, not only to relations between two-dimensional lattices but also in higher dimensions.

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