SOUTH AFRICAN TERTIARY MATHEMATICS OLYMPIAD

26 August 2017

Rules:

- Time allowed: 120 minutes.
- No calculators allowed.
- 1 mark for each correct answer.

1. Determine the value of $2^{\ln e} + e^{\ln 2}$.

2. Find all real solutions to the equation $x^2 + 2|x| = 8$.

3. Determine all values of a for which the lines $y = (3a+4)x - 5$ and $y = (\frac{3}{a} - 4)x + 5$ are parallel.

4. Professor Linear accidentally spilled coffee over the system of equations

$$\begin{aligned} x + 2y + ??z &= ?? \\ 2x + 3y + ??z &= ?? \\ 3x + 4y + ??z &= ?? \end{aligned}$$

so that the six coefficients indicated by question marks are no longer visible. Fortunately, she remembers that the general solution was $x = t$, $y = -2t + 1$, $z = t + 2$. What is the sum of the six missing numbers?

5. Find the smallest possible value of the function

$$f(x) = \cos^2 x + \operatorname{cosec}^2 x + \sec^2 x + \sin^2 x.$$

6. If $f(x) = 2^{3^{4^x}}$ and $g(x) = 4^{3^{2^x}}$, what is $\dfrac{g'(0)}{f'(0)}$?

7. The product

$$17! \cdot \left(1 + \frac{1}{1}\right)^1 \cdot \left(1 + \frac{1}{2}\right)^2 \cdot \left(1 + \frac{1}{3}\right)^3 \cdots \left(1 + \frac{1}{17}\right)^{17}$$

can be written as a^b for certain integers $a, b > 1$. What are these two integers?

8. Cheslin writes the powers of 2 on a chessboard: 1 on the first square, 2 on the second square, 4 on the third square, and so forth, until he reaches the last (64th) square. What is the last digit that he writes on that last square?

9. Find the missing entries a and b in the following matrix equation:

$$\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}.$$

10. Find the equation of the parabola which passes through $(0, 1)$ and is tangent to the lines $y = x$ and $y = -x$.

11. There are sixteen football clubs in the South African Premier Division. Each club plays each of the others twice. Three points are awarded for a match win, one point for a draw, and no points for a loss. What is the lowest possible score of a club winning the league if no tiebreakers are needed to determine the champion (i.e., the winning club has more points that any other club)?

12. A polynomial $P(x)$ has nonnegative integer coefficients and satisfies $P(1) = 7$ as well as $P(10) = 124$. Determine all possibilities for P.

13. A contest involves a prize locked in a chest. There are 10 keys (indistinguishable to the naked eye) exactly one of which will open the chest. The winner draws three keys and tries them all. If he opens the chest he wins the prize, if not the keys are replaced and shuffled. The second place winner now draws two keys and tries them. If he opens the chest he wins the prize, if not the keys are reshuffled and the third place finisher gets to draw one key and try it. Find the probability that someone wins the prize.

14. A complex number z satisfies $z^4 + \frac{1}{z^4} = 6$. Determine the value of $\left(z + \frac{i}{z}\right)^8$.

15. Determine the following limit:

$$\lim_{n\to\infty} n^{3/2}\left(\sqrt{n+1} - 2\sqrt{n} + \sqrt{n-1}\right).$$

16. What is the smallest positive integer that can be expressed as a fraction $\frac{b!}{a!}$ with $1 \leq a < b$ in at least three different ways?

17. We have an unlimited supply of building blocks in two colours (red and blue) and sizes $1\times1\times2$, $1\times1\times3$, $1\times1\times4$, ... How many possible ways are there to arrange one or more of these blocks to form a $1 \times 1 \times 10$ shape?

18. The sequence of functions f_n is defined by $f_0(x) = \sqrt{x}$ and $f_{n+1}(x) = \sqrt{x + f_n(x)}$ for $n \geq 0$. Determine

$$\int_0^6 \lim_{n\to\infty} f_n(x)\, dx.$$

19. The function $f(x) = e^{1/x^2 - 6/x - 2x}$ has three local extrema (minima/maxima). Determine the product of the three values of f at these points.

20. A legal set is a set that can be constructed in finitely many steps from the following definition:

    - the empty set is legal,
    - a finite set whose elements are previously constructed legal sets is again a legal set.

    The power set $\mathcal{P}(A)$ is the set of all subsets of A. A legal set A is called magic if $A \subseteq \mathcal{P}(A)$. How many magic sets with four elements are there?