SOUTH AFRICAN TERTIARY MATHEMATICS OLYMPIAD

27 August 2016

Rules:

- Time allowed: 120 minutes.
- No calculators allowed.
- 1 mark for each correct answer.

1. Determine the following value: $\Big||2-\pi| - |2+\pi|\Big|$.
2. Determine the positive real number x that satisfies the equation
$$\ln(x^x) = \ln(20^x) + \ln(16^x).$$
3. Determine the positive real number a for which
$$\sqrt{\int_0^a x\,dx} = \int_0^a \sqrt{x}\,dx.$$
4. Which real numbers α in the interval $[0, 2\pi]$ satisfy the equation
$$\cos\alpha = \sin 2\alpha = \cos 3\alpha = \sin 4\alpha = \cos 5\alpha = \sin 6\alpha = \cos 7\alpha = \sin 8\alpha = \cos 9\alpha = \sin 10\alpha?$$
5. Find the smallest positive integer with the following property: it is divisible by 7, the sum of its digits is divisible by 7, but none of its digits is 7.
6. Caster and Wayde run ten practice laps in an athletics stadium. They start at the same point and at the same time, and they run in the same direction. They also complete their ten laps at exactly the same time. Caster runs the first five laps twice as fast as the last five laps, and Wayde runs the last five laps twice as fast as the first five laps. How often does it happen during this practice session that one of the two overtakes the other?
7. A collection of 2016 balls is arranged in a triangle, with one ball in the first row, two balls in the second, three balls in the third, etc. Now all rows with an even number of balls are removed. How many balls are still left after this?
8. It is known that the three straight lines in the Cartesian plane that are given by the equations $2x + 3y = a$, $3x + ay = 2$ and $ax + 2y = 3$ pass through a common point. What is the value of a?

9. Determine the limit
$$\lim_{x\to 3} \frac{x^x - x^3}{x^x - 3^x}.$$

10. Positive integers (not necessarily distinct) are written on the faces of a cube in such a way that the difference of the numbers on adjacent faces is never a multiple of 3. The sum of all six numbers is S. Determine the number of possible values of S that are less than 100.

11. Determine all complex numbers z that satisfy the equation $4|z| - 2z = 2 - i$.

12. How many triplets (A, B, C) of three sets are there for which the two properties $A \cup B \cup C = \{1, 2, \ldots, 100\}$ and $A \cap B \cap C = \emptyset$ hold simultaneously?

13. Suppose that A is a 3×3-matrix such that $\det(A) = 1$, $\det(A + I) = 3$ and $\det(A + 2I) = 5$, where I is the 3×3 identity matrix. Find $\det(A + 3I)$.

14. A sequence $f_0, f_1, f_2, \ldots$ of real functions is defined as follows: $f_0(x) = x$, and
$$f_{n+1}(x) = \begin{cases} x f_n'(x) & n \text{ even}, \\ \int_0^x f_n(t)\, dt & n \text{ odd}. \end{cases}$$
Determine $f_{2016}(1)$.

15. The sides of a die are numbered from 1 to 6. It is rolled three times, and the outcomes are multiplied. What is the probability that the product is not divisible by 6?

16. Determine the smallest prime that does not occur as a factor of any of the numbers $1! + 2016$, $2! + 2016$, $3! + 2016$, ...

17. Find the maximum value of the expression
$$xy + x\sqrt{1 - y^2} + y\sqrt{1 - x^2} - \sqrt{(1 - x^2)(1 - y^2)},$$
where x and y are two real numbers in the interval $[-1, 1]$.

18. Determine all matrices X that satisfy the equation
$$X^2 + X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

19. A circle of radius 1 rolls along the x-axis until it has made one complete revolution after one minute. A point on the circle is marked and its position in the plane is recorded as the circle moves. What is the maximum speed (in units per minute) at which this point moves along its journey?

20. A horizontal line cuts the parabola that is given by the equation $y = x^2$. The area of the finite region enclosed by the line and the parabola is denoted by A. A circle is inscribed in this region in such a way that it touches the parabola in two points and the horizontal line in one point. If B is the area of this circle, determine the maximum of the ratio B/A.