SOUTH AFRICAN TERTIARY MATHEMATICS OLYMPIAD

22 August 2015

Rules:

- Time allowed: 120 minutes.
- No calculators allowed.
- 1 mark for each correct answer.

1. If $\ln x + \ln y = 0$ for positive real numbers x and y, what is the minimum value of $x + y$?

2. Find the value of the limit $\lim_{x \to 0} \dfrac{\frac{d}{dx}\left(\frac{\sin x}{x}\right)}{x}$.

3. If
$$x \cdot \binom{5}{5} \cdot \binom{10}{5} \cdot \binom{15}{5} \cdots \binom{2015}{5} = \binom{6}{5} \cdot \binom{11}{5} \cdot \binom{16}{5} \cdots \binom{2016}{5},$$
what is the value of x?

4. Determine the definite integral $\displaystyle\int_0^1 e^{\sqrt{x}}\, dx$.

5. The solution set of the equation
$$\tan(\arctan x) = \arctan(\tan x)$$
is an interval. How long is this interval?

6. In the popular number-placement puzzle Sudoku, one has to fill a 9×9-grid with numbers from 1 to 9 in such a way that each row, each column and each of nine 3×3 sub-grids contains all the numbers from 1 to 9. After having completed a Sudoku, Suzy erases two of the numbers. The average of the remaining numbers is 7/79 greater than the average of all numbers in the completed Sudoku grid. Which two numbers did Suzy erase?

7. Let R be the region consisting of the points (x, y) of the Cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Find the area of R.

8. How many complex numbers z satisfy the equation $\overline{z} = z^{2015}$?

9. Given a regular 100-gon, how many ways are there to draw a rectangle whose vertices are vertices of the 100-gon?

10. Point P lies in the Cartesian plane, but not on the x-axis, and three straight lines ℓ_1, ℓ_2 and ℓ_3 pass through it. The x-axis forms a triangle with the lines ℓ_1 and ℓ_2 that is divided into two smaller triangles of equal area by ℓ_3. If the gradients of ℓ_1 and ℓ_2 are 1 and -2 respectively, what is the gradient of ℓ_3?

11. Two random numbers are chosen (independently of each other) from the interval $[0, 1]$. What is the probability that they differ by more than their average?

12. In the solution to the system of equations

$$\begin{aligned}
20x_1 + x_2 + x_3 + \cdots + x_{14} + x_{15} &= -7, \\
x_1 + 20x_2 + x_3 + \cdots + x_{14} + x_{15} &= -6, \\
x_1 + x_2 + 20x_3 + \cdots + x_{14} + x_{15} &= -5, \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad & \quad \vdots \\
x_1 + x_2 + x_3 + \cdots + 20x_{14} + x_{15} &= 6, \\
x_1 + x_2 + x_3 + \cdots + x_{14} + 20x_{15} &= 7,
\end{aligned}$$

what is the value of x_{15}?

13. Which pairs (a, b) of positive integers satisfy the equation $(-2)^a + 228 = b^2$?

14. The sum of k consecutive squares is equal to the sum of the following $k-1$ consecutive squares. The last of these $2k-1$ squares is 2015^2. What is the first one?

15. How many 2×2-matrices with determinant 1 are there whose entries are (not necessarily distinct) elements of the set $\{1, 2, 3, 4\}$?

16. How many polynomials $P(x)$ of degree 4 with real coefficients satisfy the equation $P(x^2) = P(x)P(-x)$ for all x?

17. Using the fact that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3}$.

18. A mathdie has the following six matrices on its faces:

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad M_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad M_6 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Gus the gambler throws a mathdie three times and multiplies the matrices that show up on top in the order thrown. What is the probability that the final product is the matrix M_2?

19. Suppose that f is twice differentiable on the interval $[0, 1]$, and that $f(0) = f(1) = 0$ as well as $f''(x) \geq -1$ on the entire interval. Determine the greatest possible value of $\int_0^1 f(x)\,dx$.

20. Let $f(x)$ be the function

$$f(x) = \sum_{k=0}^{1000} \binom{2015}{k} x^k (1-x)^{2015-k}.$$

Determine $\dfrac{f''(\frac{1}{2})}{f'(\frac{1}{2})}$.