SOUTH AFRICAN TERTIARY MATHEMATICS OLYMPIAD

23 August 2014

Rules:

- Time allowed: 120 minutes.
- No calculators allowed.
- 1 mark for each correct answer.

1. Determine the limit
$$\lim_{x\to 1} \frac{\sqrt{\sqrt{\sqrt{x}}}-1}{\sqrt{x}-1}.$$

2. For which real value of x is $\ln(3x) = 3\ln x$?

3. This year, August has five Fridays, five Saturdays and five Sundays. According to a recent internet hoax, this situation only occurs every 823 years, but of course this claim is wrong. What is the next year in which August will actually have five Fridays, five Saturdays and five Sundays again?

4. If
$$\frac{1}{(x-1)(x-2)(x-3)(x-4)(x-5)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} + \frac{D}{x-4} + \frac{E}{x-5}$$
for all real numbers $x \neq 1,2,3,4,5$, what is C?

5. Suppose that x, y are two real numbers in the interval $(-\pi/2, \pi/2)$ such that $\tan x = \cos y$. What is the greatest possible value of x?

6. The function $f(x)$ satisfies $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $\ldots$, $f(10) = 11$ and $f'(1) = \sqrt{1}$, $f'(2) = \sqrt{2}$, $f'(3) = \sqrt{3}$, $\ldots$, $f'(10) = \sqrt{10}$. If $g(x) = f(f(f(f(x))))$, what is $g'(1)$?

7. Each of the two players in a game of chess has a set of 16 pieces. The pieces in a set are: one king, one queen, two bishops, two knights, two rooks, and eight pawns. One player has white pieces, the other player has black pieces. Kenny chooses two of the 32 pieces from a chess set – how many possible combinations are there? For example, "one white pawn and one black knight" is a possible combination, "two black bishops" another one. The order in which Kenny chooses the pieces does not matter.

8. 2^{29} is a 9-digit number. As it happens it has no repeated digit, and so has 9 distinct digits. Which of the ten digits $0, 1, 2, \ldots, 9$ is missing?

9. Suppose that the three real numbers x, y, z satisfy the system of equations

$$\begin{aligned} 2^x \cdot 4^y \cdot 16^z &= 1, \\ 4^x \cdot 16^y \cdot 2^z &= 2, \\ 16^x \cdot 2^y \cdot 4^z &= 4. \end{aligned}$$

What is the value of x?

10. A and B are both 2×2-matrices with real entries. Determine the missing entry in the following equation:

$$AB - BA = \begin{bmatrix} 3 & 5 \\ -4 & ? \end{bmatrix}.$$

11. Suppose that n and m are positive integers such that

$$\frac{n}{20} + \frac{m}{14} = \frac{2014}{20 \cdot 14}.$$

Determine the greatest possible value of n.

12. What is the maximum value of the function

$$f(x) = \sqrt{8x - x^2} - \sqrt{14x - x^2 - 48} \ ?$$

13. Determine all real numbers $x \in [-1, 1]$ for which

$$(\arcsin x)^2 + (\arccos x)^2 = \frac{5\pi^2}{36}.$$

14. Find all pairs (n, m) of (not necessarily positive) integers that satisfy the equation

$$n(m^2 + 9) + m(n^2 - 9) + n^2(n - 6) = 0.$$

15. Suppose that $P(x)$ is a polynomial that satisfies

$$xP(x-1) = (x-5)P(x).$$

for all real values of x, and $P(-1) = 1$. Determine the value of $P(\frac{1}{2})$.

16. How should the number r be chosen in the interval $[1, 2]$ so that the maximum value of the function

$$f(x) = \left| \frac{r - x}{x} \right|$$

on the interval $[1, 2]$ is as small as possible?

17. Three players play a game. Each of them has a black or a white hat on his or her head, and each can see all the other players' hats except his/her own. Each player must write on a piece of paper what they think their own hat's colour is, or write "pass" if they do not want to participate. They all win if nobody gets their hat colour wrong and at least one person guesses right (they cannot all write "pass"). What is the maximum probability of winning if each hat is white with a probability of exactly $\frac{1}{2}$, and this is independent of the colours of the two other hats? The three players may agree on a strategy beforehand, but not communicate during the game.

18. Determine the integral

$$\int_0^{\pi} (\cos x + \cos 2x + \cos 3x + \cdots + \cos 100x)^2 \, dx.$$

19. Consider a triangle ABC with vertices $A = (-1, 0)$, $B = (1, 0)$ and $C = (0, y)$. The lines through A and B perpendicular to BC and AC respectively meet at H (the *orthocentre* of ABC). The angle bisectors of $\angle BAC$ and $\angle ABC$ meet at I (the *incentre* of ABC). Finally, let M and N be the midpoints of AC and BC respectively. The lines AN and BM meet at S (the *centroid* of ABC). Determine the limit of the quotient HI/IS as $y \to \sqrt{3}$ (so that ABC becomes an equilateral triangle).

20. Let S be the set $\{1, 2, \ldots, 20\}$. We want to define a function v that assigns to every subset A of S a positive integer $v(A)$, which we call its *value*. This function should satisfy $v(\emptyset) = 1$, $v(S) = 14$ and

$$v(A \cup B)v(A \cap B) = v(A)v(B)$$

for all subsets A and B of S. How many possible ways are there to define such a function v?