SOUTH AFRICAN TERTIARY MATHEMATICS OLYMPIAD

5 October 2013

Rules:

- Time allowed: 120 minutes.
- No calculators allowed.
- 1 mark for each correct answer.

1. For which real value of x is $\log_2(\log_3 x) = 1$?

2. Determine the value of the limit
$$\lim_{x\to 2013} \frac{\sin(\pi x)}{x - 2013}.$$

3. At the university where Peter studies, every student is friends on Facebook with precisely 13 other students. How many students are there at most who are either Peter's Facebook friends or friends with one of his friends?

4. If
$$f(x) = \sin(\sin(\sin(\sin(\sin(x))))),$$
what is $f'(0)$?

5. For which value of a does the following system of equations have no solution?
$$x + 2y = 5,$$
$$-3x + ay = 1.$$

6. At a recent mathematics competition, 252 participants solved the first question, but only 36 solved the last one. Surprisingly, only half of the participants who solved the last question could also solve the first one, but one third of the participants who did not solve the first question could solve the last one. How many people took part in this competition?

7. Thomas notices that in four hours, the clock's hour hand will be in precisely the same position where the minute hand is at the moment. How long will it take until the minute hand is in the exact same position where the hour hand is now?

8. The complex number z satisfies $\operatorname{Re} z = 2$. Determine the largest possible value of $\operatorname{Re}(z^2) + \operatorname{Im}(z^2)$.

9. A positive integer a has precisely three positive divisors, while another positive integer b has precisely four positive divisors. What is the minimum possible number of positive divisors of ab?

10. Let $\|x\|$ be the distance of x from the nearest integer, e.g. $\|1.7\| = 0.3$ or $\|3.2\| = 0.2$. Determine

$$\int_0^{100} \|x\| \, dx.$$

11. Determine the sum

$$\sum_{n=1}^{1000} \ln\left(1 + \frac{1}{n}\right).$$

12. Each of the squares of an 8×8-board is coloured randomly black or white (each with probability $\frac{1}{2}$). An "isolated" square is a square whose colour differs from all its horizontally or vertically adjacent squares. What is the expected number of isolated squares?

13. Given that

$$\begin{bmatrix} 3 & 3 & x \\ -2 & -3 & -3 \\ -1 & -2 & 0 \end{bmatrix}^{2013} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

determine x.

14. Determine all differentiable functions $f : (0, \infty) \to \mathbb{R}$ such that

$$f'\left(\frac{x}{y}\right) = \frac{f(y)}{f(x)}$$

for all $x, y > 0$.

15. If $\dfrac{\cos 3x}{\cos x} = \dfrac{1}{3}$ and $0 < x < \frac{\pi}{2}$, determine $\dfrac{\sin 3x}{\sin x}$.

16. What is the expected maximum of two independent uniformly distributed random numbers on $[0, 1]$?

17. Let R and S be points on the sides AB and AC, respectively, of triangle ABC, and let P be the point of intersection of BS and CR. If the areas of triangles BPR, BPC and CPS are 5, 6, and 7, respectively, find the area of triangle ABC.

18. Determine the shortest distance between a point on the parabola $y = 8x - x^2$ and a point on the parabola $y = x^2 + 15$.

19. Let the sum of ten positive numbers $x_1, x_2, \ldots, x_{10}$ be equal to 1, and let z denote the largest number in the sequence

$$\frac{x_1}{1 + x_1}, \frac{x_2}{1 + x_1 + x_2}, \frac{x_3}{1 + x_1 + x_2 + x_3}, \ldots, \frac{x_{10}}{1 + x_1 + x_2 + \cdots + x_{10}}.$$

What is the smallest possible value of z?

20. Determine

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan x) dx.$$