SOUTH AFRICAN TERTIARY MATHEMATICS OLYMPIAD

6 October 2012

Rules:

- Time allowed: 120 minutes.
- No calculators allowed.
- 1 mark for each correct answer.

1. Determine the limit
$$\lim_{x\to 0} \frac{1}{x} e^{-1/x^2}.$$

2. Max has a curious habit: when he is asked questions about his favourite number, he lies and tells the truth alternatingly. His first answer can be either the truth or a lie. His friend Bob asks him the following questions:

   - Is your favourite number less than 100?
   - Is your favourite number a prime?
   - Does your favourite number contain the digit 4?
   - Is your favourite number even?
   - Is your favourite number a square?

   His answer is always yes. What is Max's favourite number?

3. All the sides of a box are rectangular. Three of the faces have diagonals of length $\sqrt{34}$, $\sqrt{58}$ and $\sqrt{74}$. What is the volume of the box?

4. Let the function $f(x)$ be defined by
$$f(x) = x^{x^{x^{x^{\cdots}}}}$$
   for positive real x. Determine $f'(a)$ if $f(a) = 2$.

5. A sequence $a_1, a_2, a_3, \ldots$ of real numbers is defined by the following rule: $a_1 = 2$, $a_2 = 12$, and
$$a_{n+1} = \frac{a_n + 1}{a_{n-1}}$$
   for $n > 1$. Determine a_{2012}.

6. Six political leaders must have their photographs taken sitting in a row. Helen and Patricia insist on sitting next to each other. Jacob and Julius refuse to sit next to each other. Tokyo and Trevor don't mind where they sit. In how many ways can they be seated?

7. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$, the integrals

$$\int_{-\infty}^{x} f(t)\,dt \qquad \text{and} \qquad \int_{-\infty}^{x} f(t)^2\,dt$$

are both convergent and

$$\left(\int_{-\infty}^{x} f(t)\,dt\right)^2 = \int_{-\infty}^{x} f(t)^2\,dt$$

for all $x \in \mathbb{R}$.

8. A clock's minute hand has length 4 and its hour hand length 3. What is the distance between the tips at the moment when this distance is increasing most rapidly?

9. Let the Fibonacci sequence be defined by $a_1 = 1$, $a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for all $n > 1$. Find the sum

$$\sum_{n=1}^{\infty} \frac{a_n}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \frac{13}{128} + \cdots$$

10. Find all triples (a, b, c) of positive integers with $a < b < c$ such that a, b, c are the sides of a right-angled triangle whose area is twice its circumference.

11. Let f and g be continuous and differentiable on $\mathbb{R}$, and assume that $f(x)$ and $f'(x)$ are never equal to zero. Moreover, suppose that for all $x \in \mathbb{R}$,

$$\frac{g'(x)}{f'(x)} = 2 \cdot \frac{g(x)}{f(x)}.$$

Given that $f(1) = 1$, $g(1) = 2$ and $f(2) = 3$, determine $g(2)$.

12. A point P lies on the positive half of the x-axis and the point Q lies in the first quadrant on the graph of $y = x^2$ such that $OP = OQ$ (O being the origin). The straight line through P and Q cuts the y-axis in R. Determine $\lim_{OP \to 0} OR$.

13. How many different planes are there, which pass through three or more vertices of a given cube?

14. In a mathematics class, there are three times as many girls as boys. It turns out that the number of ways to select a team of two girls from this class is six times the number of ways to select a team of three boys. How many students are attending this class?

15. Find the value of the determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 99 & 100 \\ 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 101 & 102 \\ 0 & 0 & 7 & 8 & 9 & 10 & \cdots & 103 & 104 \\ 0 & 0 & 9 & 10 & 11 & 12 & \cdots & 105 & 106 \\ 0 & 0 & 0 & 0 & 13 & 14 & \cdots & 107 & 108 \\ 0 & 0 & 0 & 0 & 15 & 16 & \cdots & 109 & 110 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 295 & 296 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 297 & 298 \end{vmatrix}.$$

16. Compute the integral

$$\int_{-\pi}^{\pi} \frac{1}{1+e^{\sin x}}\, dx.$$

17. Two players are playing Tic-Tac-Toe on a 3×3 board, but instead of placing Xs and Os, they place 1s and 0s (1 begins). Once the board has been filled completely, the determinant of the resulting 3×3-matrix is calculated. Find all possible values of this determinant.

18. If $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(m+n) = f(m) + f(n) + mn$ for all positive integers m and n, and $f(1) = 3$, calculate $f(13)$.

19. Determine the greatest possible number of elements of a subset A of $\{1, 2, \ldots, 30\}$ with the property that any product of one or more distinct elements of A is not a square.

20. For sets A and B, define the operation $A \Delta B$ by

$$A \Delta B = (A \cup B) \setminus (A \cap B).$$

Moreover, let A_i be the set of all multiples of i in $\{1, 2, \ldots, 1000\}$. Determine the number of elements in the set

$$A_1 \Delta (A_2 \Delta (A_3 \Delta \cdots \Delta (A_{999} \Delta A_{1000}) \cdots)).$$