SOUTH AFRICAN TERTIARY MATHEMATICS OLYMPIAD

26 August 2017

Solutions

1. Determine the value of $2^{\ln e} + e^{\ln 2}$.

   Solution:
   $$2^{\ln e} + e^{\ln 2} = 2 + 2 = 4.$$

2. Find all real solutions to the equation $x^2 + 2|x| = 8$.

   Solution: Completing the square, we get
   $$x^2 + 2|x| + 1 = |x|^2 + 2|x| + 1 = (|x| + 1)^2 = 9,$$
   thus $|x| + 1 = \pm 3$. However, $|x|$ cannot be negative, so $|x| = 2$, which finally gives us the solutions $x = 2$ and $x = -2$.

3. Determine all values of a for which the lines $y = (3a+4)x - 5$ and $y = (\frac{3}{a} - 4)x + 5$ are parallel.

   Solution: For the two lines to be parallel, the gradients need to be the same:
   $$3a + 4 = \frac{3}{a} - 4.$$
   Writing the equation with a common denominator, we get
   $$\frac{3a^2 + 8a - 3}{a} = \frac{(3a-1)(a+3)}{a} = 0.$$
   Thus the two possible values of a are $\frac{1}{3}$ and -3.

4. Professor Linear accidentally spilled coffee over the system of equations
   $$\begin{aligned} x + 2y + ??z &= ?? \\ 2x + 3y + ??z &= ?? \\ 3x + 4y + ??z &= ?? \end{aligned}$$
   so that the six coefficients indicated by question marks are no longer visible. Fortunately, she remembers that the general solution was $x = t$, $y = -2t + 1$, $z = t + 2$. What is the sum of the six missing numbers?

   Solution: Plugging in the general solution, the first equation becomes (writing A and B for the unknown coefficients)
   $$t + 2(-2t + 1) + A(t + 2) = B,$$
   or equivalently
   $$(A - 3)t = B - 2A - 2.$$

Since this has to hold for all t, we have $A = 3$, $B = 8$. In the same way, we obtain the other two equations:

$$2x + 3y + 4z = 11$$
$$3x + 4y + 5z = 14$$

Thus the sum of the six missing numbers is $3 + 4 + 5 + 8 + 11 + 14 = 45$.

5. Find the smallest possible value of the function

$$f(x) = \cos^2 x + \operatorname{cosec}^2 x + \sec^2 x + \sin^2 x.$$

Solution: We simplify the expression first:

$$\begin{aligned} f(x) = \cos^2 x + \operatorname{cosec}^2 x + \sec^2 x + \sin^2 x &= (\cos^2 x + \sin^2 x) + \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} \\ = 1 + \frac{\cos^2 x + \sin^2 x}{\sin^2 x \cos^2 x} &= 1 + \frac{1}{(\sin x \cos x)^2} \\ = 1 + \frac{1}{((\sin 2x)/2)^2} &= 1 + \frac{4}{\sin^2 2x}. \end{aligned}$$

Since the maximum value of $\sin^2 2x$ is 1, the minimum of $f(x)$ is $1 + 4 = 5$.

6. If $f(x) = 2^{3^{4^x}}$ and $g(x) = 4^{3^{2^x}}$, what is $\dfrac{g'(0)}{f'(0)}$?

Solution: The chain rule gives us

$$f'(x) = \ln 2 \cdot 2^{3^{4^x}} \cdot \ln 3 \cdot 3^{4^x} \cdot \ln 4 \cdot 4^x$$

and

$$g'(x) = \ln 4 \cdot 4^{3^{2^x}} \cdot \ln 3 \cdot 3^{2^x} \cdot \ln 2 \cdot 2^x,$$

thus $f'(0) = \ln 2 \cdot 8 \cdot \ln 3 \cdot 3 \cdot \ln 4 = 24 \ln 2 \ln 3 \ln 4$ and $g'(0) = \ln 4 \cdot 64 \cdot \ln 3 \cdot 3 \cdot \ln 2 = 192 \ln 2 \ln 3 \ln 4$. Finally,

$$\frac{g'(0)}{f'(0)} = \frac{192 \ln 2 \ln 3 \ln 4}{24 \ln 2 \ln 3 \ln 4} = 8.$$

7. The product

$$17! \cdot \left(1 + \frac{1}{1}\right)^1 \cdot \left(1 + \frac{1}{2}\right)^2 \cdot \left(1 + \frac{1}{3}\right)^3 \cdots \left(1 + \frac{1}{17}\right)^{17}$$

can be written as a^b for certain integers $a, b > 1$. What are these two integers?

Solution: We can rewrite the expression as follows:

$$\begin{aligned} 17! \cdot \left(1 + \frac{1}{1}\right)^1 \cdot \left(1 + \frac{1}{2}\right)^2 \cdot \left(1 + \frac{1}{3}\right)^3 \cdot \left(1 + \frac{1}{17}\right)^{17} &= 17! \cdot \frac{2^1}{1^1} \cdot \frac{3^2}{2^2} \cdot \frac{4^3}{3^3} \cdots \frac{18^{17}}{17^{17}} \\ &= 17! \cdot \frac{18^{17}}{1 \cdot 2 \cdot 3 \cdots 17} = 18^{17}. \end{aligned}$$

Thus $a = 18$ and $b = 17$.

8. Cheslin writes the powers of 2 on a chessboard: 1 on the first square, 2 on the second square, 4 on the third square, and so forth, until he reaches the last (64th) square. What is the last digit that he writes on that last square?

   Solution: The first powers of 2 are $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$. The final digits now repeat with a period of 4: $2, 4, 8, 6, 2, 4, 8, 6, \ldots$. In particular, the final digit of 2^{63} is the same as the final digit of $2^3 = 8$. Since this is the number that Cheslin writes on the last square, the last digit he writes is 8.

9. Find the missing entries a and b in the following matrix equation:

$$\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}.$$

   Solution:

$$\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}^2 = \begin{bmatrix} a^2+b & a+1 \\ ab+b & b+1 \end{bmatrix}$$

   Looking at the second column, we see immediately that $a = -2$ and $b = -1$. The entries in the first column agree as well in this case, so this is the only solution.

10. Find the equation of the parabola which passes through $(0, 1)$ and is tangent to the lines $y = x$ and $y = -x$.

    Solution: The parabola must be symmetric with respect to the y-axis, so its equation is of the form $y = ax^2 + b$. Since it passes through the point $(0, 1)$, we must have $b = 1$. At the point x_0 where it is tangent to the line $y = x$, the value and the derivative of the line and the parabola must agree:

$$ax_0^2 + 1 = x_0 \qquad \text{and} \qquad 2ax_0 = 1.$$

    The second equation gives us $x_0 = \frac{1}{2a}$, which we plug into the first equation to obtain

$$\frac{1}{4a} + 1 = \frac{1}{2a}.$$

    This yields $a = \frac{1}{4}$, so the equation of the parabola must be $y = \frac{x^2}{4} + 1$.

11. There are sixteen football clubs in the South African Premier Division. Each club plays each of the others twice. Three points are awarded for a match win, one point for a draw, and no points for a loss. What is the lowest possible score of a club winning the league if no tiebreakers are needed to determine the champion (i.e., the winning club has more points that any other club)?

    Solution: There are $2\binom{16}{2} = 2 \cdot \frac{16 \cdot 15}{2} = 240$ matches, so the total number of points of all clubs combined is at least $2 \cdot 240 = 480$. A club with more points than any of the other clubs must thus have more than $\frac{480}{16} = 30$ points. Suppose that club A has exactly 31 points. At least two of this club's matches ended with a win for either side, for otherwise its total score would be $29 \cdot 1 + 3 = 32$ or $29 \cdot 1 + 0 = 29$. However, this means that the total number of points of all teams increases to at least 482, so either there are at least two clubs with 31 points, or there is a club with 32 points. Either way, club A cannot be the sole winner. A club can be the sole league champion with only 32 points though, if all matches of the entire league are drawn except for exactly one match won by this club.

12. A polynomial $P(x)$ has nonnegative integer coefficients and satisfies $P(1) = 7$ as well as $P(10) = 124$. Determine all possibilities for P.

    Solution: If the degree of the polynomial is 3 or greater, then it must contain a term ax^k with $a \geq 1$ and $k \geq 3$. Since all terms are nonnegative when we plug in 10, this means that $P(10) \geq a \cdot 10^k \geq 1000 > 124$, a contradiction. So the polynomial is at most quadratic, and we can write it as $P(x) = ax^2 + bx + c$. The condition $P(1) = 7$ gives us $a + b + c = 7$. If $a = 0$, then $P(10) = 10b + c \leq 10(a + b + c) = 70 < 124$, a contradiction. If $a \geq 2$, then $P(10) = 100a + 10b + c \geq 100a \geq 200 > 124$, which is also impossible. Thus $a = 1$. Finally, we are left with the equations $b + c = 7 - a = 6$ and $10b + c = 124 - 100a = 24$, which give us $b = 2$ and $c = 4$. Thus $P(x) = x^2 + 2x + 4$.

13. A contest involves a prize locked in a chest. There are 10 keys (indistinguishable to the naked eye) exactly one of which will open the chest. The winner draws three keys and tries them all. If he opens the chest he wins the prize, if not the keys are replaced and shuffled. The second place winner now draws two keys and tries them. If he opens the chest he wins the prize, if not the keys are reshuffled and the third place finisher gets to draw one key and try it. Find the probability that someone wins the prize.

    Solution: The winner has a chance of $\frac{3}{10}$ to win the prize. The second place winner gets to draw keys with a probability of $\frac{7}{10}$ and wins with a probability of $\frac{2}{10}$ if he does. Finally, the third place winner gets to draw keys with a probability of $\frac{7}{10} \cdot \frac{8}{10}$ and wins with a probability of $\frac{1}{10}$ if he does. Thus the total probability that someone wins the prize is

$$\frac{3}{10} + \frac{7}{10} \cdot \frac{2}{10} + \frac{7}{10} \cdot \frac{8}{10} \cdot \frac{1}{10} = \frac{496}{1000} = \frac{62}{125} = 0.496.$$

14. A complex number z satisfies $z^4 + \dfrac{1}{z^4} = 6$. Determine the value of $\left(z + \dfrac{i}{z}\right)^8$.

    Solution: By the binomial theorem, we have

$$\left(z + \frac{i}{z}\right)^4 = z^4 + 4iz^2 - 6 - \frac{4i}{z^2} + \frac{1}{z^4}.$$

    The condition $z^4 + \frac{1}{z^4} = 6$ gives us

$$\left(z + \frac{i}{z}\right)^4 = 4iz^2 - \frac{4i}{z^2} = 4i\left(z^2 - \frac{1}{z^2}\right),$$

    thus

$$\left(z + \frac{i}{z}\right)^8 = -16\left(z^4 - 2 + \frac{1}{z^4}\right) = -16(6 - 2) = -64.$$

15. Determine the following limit:

$$\lim_{n\to\infty} n^{3/2}\Big(\sqrt{n+1}-2\sqrt{n}+\sqrt{n-1}\Big).$$

Solution:

$$\begin{aligned}
n^{3/2}\Big(\sqrt{n+1}-2\sqrt{n}+\sqrt{n-1}\Big) &= n^{3/2}\frac{(\sqrt{n+1}+\sqrt{n-1})^2-(2\sqrt{n})^2}{\sqrt{n+1}+\sqrt{n-1}+2\sqrt{n}}\\
&= n^{3/2}\frac{n+1+2\sqrt{n^2-1}+n-1-4n}{\sqrt{n+1}+\sqrt{n-1}+2\sqrt{n}}\\
&= 2n^{3/2}\frac{\sqrt{n^2-1}-n}{\sqrt{n+1}+\sqrt{n-1}+2\sqrt{n}}\\
&= 2n^{3/2}\frac{n^2-1-n^2}{(\sqrt{n+1}+\sqrt{n-1}+2\sqrt{n})(\sqrt{n^2-1}+n)}\\
&= -\frac{2n^{3/2}}{(\sqrt{n+1}+\sqrt{n-1}+2\sqrt{n})(\sqrt{n^2-1}+n)}\\
&= -\frac{2}{(\sqrt{1+1/n}+\sqrt{1-1/n}+2)(\sqrt{1-1/n^2}+1)}.
\end{aligned}$$

Hence the limit is

$$\begin{aligned}
\lim_{n\to\infty} n^{3/2}\Big(\sqrt{n+1}-2\sqrt{n}+\sqrt{n-1}\Big) &= \lim_{n\to\infty} -\frac{2}{(\sqrt{1+1/n}+\sqrt{1-1/n}+2)(\sqrt{1-1/n^2}+1)}\\
&= \frac{-2}{4\cdot 2} = -\frac{1}{4}.
\end{aligned}$$

16. What is the smallest positive integer that can be expressed as a fraction $\frac{b!}{a!}$ with $1 \le a < b$ in at least three different ways?

Solution: Every integer $n > 1$ has one such representation as $n = \frac{n!}{(n-1)!}$. The following numbers can be written as $\frac{a!}{(a-2)!} = a(a-1)$:

$$6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, \ldots,$$

the following numbers can be written as $\frac{a!}{(a-3)!} = a(a-1)(a-2)$:

$$24, 60, 120, 210, 336, \ldots,$$

and the following numbers can be written as $\frac{a!}{(a-4)!} = a(a-1)(a-2)(a-3)$:

$$120, 360, 840, \ldots.$$

The smallest positive integer that occurs twice in these lists is 120, which thus has three representations: $120 = \frac{5!}{1!}$, $120 = \frac{6!}{3!}$ and $120 = \frac{120!}{119!}$. All fractions $\frac{a!}{b!}$ with $b < a - 4$ are the product of at least five consecutive integers greater than 1, thus greater or equal to $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$, hence there are no smaller numbers with three or more representations.

17. We have an unlimited supply of building blocks in two colours (red and blue) and sizes $1\times1\times2$, $1\times1\times3$, $1\times1\times4$, ... How many possible ways are there to arrange one or more of these blocks to form a $1\times1\times10$ shape?

Solution: Let generally a_n be the number of ways to form a $1\times1\times n$ shape. The final block can have any size $1\times1\times k$ with $2\le k\le n$, leaving us to cover the remaining $1\times1\times(n-k)$, and it can be either red or blue. Thus

$$a_n = 2a_{n-2} + 2a_{n-3} + \cdots + 2a_0.$$

Subtracting the equations

$$a_n = 2a_{n-2} + 2a_{n-3} + \cdots + 2a_0.$$

and

$$a_{n+1} = 2a_{n-1} + 2a_{n-2} + \cdots + 2a_0,$$

we get

$$a_{n+1} - a_n = 2a_{n-1}.$$

The first two values are $a_1 = 0$ (since there are no $1\times1\times1$ blocks) and $a_2 = 2$ (since the only block can be either red or blue in this case). We can now show by induction that $a_n = \frac{2^n+2(-1)^n}{3}$: this is true for $n=1$ and $n=2$, and for the induction step we have

$$\begin{aligned} a_{n+1} = a_n + 2a_{n-1} &= \frac{2^n+2(-1)^n}{3} + 2\cdot\frac{2^{n-1}+2(-1)^{n-1}}{3} \\ &= \frac{2^n+2^n+2(-1)^n-4(-1)^n}{3} = \frac{2^{n+1}+2(-1)^{n+1}}{3}. \end{aligned}$$

In particular, the number of ways to form a $1\times1\times10$ shape is $\frac{2^{10}+2}{3} = 342$.

18. The sequence of functions f_n is defined by $f_0(x) = \sqrt{x}$ and $f_{n+1}(x) = \sqrt{x+f_n(x)}$ for $n\ge 0$. Determine

$$\int_0^6 \lim_{n\to\infty} f_n(x)\,dx.$$

Solution: The functions in this sequence are

$$f_0(x) = \sqrt{x}, \quad f_1(x) = \sqrt{x+\sqrt{x}}, \quad f_2(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}, \ldots$$

We see that $f_n(x)$ is an increasing sequence for every fixed x, and it is also bounded above by 3: This is clearly the case for f_0, since $f_0(x) = \sqrt{x} \le \sqrt{6} < 3$, and we can continue by induction:

$$f_{n+1}(x) = \sqrt{x+f_n(x)} \le \sqrt{6+3} = 3.$$

Therefore, the limit $f_\infty(x) = \lim_{n\to\infty} f_n(x)$ exisits, and it must satisfy

$$f_\infty(x) = \sqrt{x+f_\infty(x)}.$$

This yields the quadratic equation

$$f_\infty^2(x) - f_\infty(x) - x = 0,$$

whose solution is given by $f_\infty(x) = \frac{1+\sqrt{1+4x}}{2}$ (the other solution is negative and therefore not applicable). So we get

$$\int_0^6 \lim_{n\to\infty} f_n(x)\,dx = \int_0^6 f_\infty(x)\,dx = \int_0^6 \frac{1+\sqrt{1+4x}}{2}\,dx.$$

Thus

$$\int_0^6 \lim_{n\to\infty} f_n(x)\,dx = \frac{x}{2} + \frac{(1+4x)^{3/2}}{12}\Big|_0^6 = 3 + \frac{124}{12} = \frac{40}{3}.$$

19. The function $f(x) = e^{1/x^2-6/x-2x}$ has three local extrema (minima/maxima). Determine the product of the three values of f at these points.

Solution: We have

$$f'(x) = e^{1/x^2-6/x-2x}\Big(-\frac{2}{x^3} + \frac{6}{x^2} - 2\Big) = -2x^{-3}e^{1/x^2-6/x-2x}(x^3 - 3x + 1).$$

So the three local extrema occur at the three solutions x_1, x_2, x_3 of the equation $x^3 - 3x + 1 = 0$. Since we have

$$\begin{aligned}(x-x_1)(x-x_2)(x-x_3) &= x^3 - (x_1+x_2+x_3)x^2 + (x_1x_2+x_2x_3+x_3x_1)x - x_1x_2x_3\\ &= x^3 - 3x + 1,\end{aligned}$$

we find that $x_1+x_2+x_3 = 0$, $x_1x_2+x_2x_3+x_3x_1 = -3$ and $x_1x_2x_3 = -1$. The product we are interested in is given by

$$f(x_1)f(x_2)f(x_3) = e^{1/x_1^2+1/x_2^2+1/x_3^2-6(1/x_1+1/x_2+1/x_3)-2(x_1+x_2+x_3)}.$$

Noting that

$$\begin{aligned}\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{1}{x_3^2} &= \frac{x_2^2x_3^2+x_1^2x_3^2+x_1^2x_2^2}{x_1^2x_2^2x_3^2} = \frac{(x_1x_2+x_2x_3+x_3x_1)^2 - 2x_1x_2x_3(x_1+x_2+x_3)}{(x_1x_2x_3)^2}\\ &= \frac{(-3)^2 - 2\cdot(-1)\cdot 0}{(-1)^2} = 9\end{aligned}$$

and

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3} = \frac{x_2x_3+x_1x_3+x_1x_2}{x_1x_2x_3} = \frac{-3}{-1} = 3,$$

we get

$$f(x_1)f(x_2)f(x_3) = e^{9-6\cdot 3-2\cdot 0} = e^{-9}.$$

20. A legal set is a set that can be constructed in finitely many steps from the following definition:

- the empty set is legal,

- a finite set whose elements are previously constructed legal sets is again a legal set.

The power set $\mathcal{P}(A)$ is the set of all subsets of A. A legal set A is called magic if $A \subseteq \mathcal{P}(A)$. How many magic sets with four elements are there?

Solution: If a 1-element set $A = \{a\}$ is magic, then a must be one of the two subsets $\emptyset = \{\}$ and $A = \{a\}$. We clearly cannot have $a = \{a\}$, so $a = \emptyset$, which gives us the only magic 1-element set: $\{\emptyset\}$.

Generally, we note that every legal set A has an element that is not itself an element of any of the others. Otherwise, we could choose an element $a_1 \in A$, which is an element of $a_2 \in A$, which is an element of $a_3 \in A$, etc. Since there are only finitely many elements in A, we must eventually encounter an element again, creating an infinite loop that contradicts the definition.

In particular, every magic set A contains at least one element a that is not an element of any of the others. When we remove it, the remaining set $A \setminus \{a\}$ is still magic, since $\mathcal{P}(A \setminus \{a\})$ contains all subsets of A that do not contain a as an element. Since none of the elements of $A \setminus \{a\}$ contains a, we must have $A \setminus \{a\} \subseteq \mathcal{P}(A \setminus \{a\})$.

Conversely, we see that all magic sets can be built by repeatedly adding an element to a magic set that is contained in its power set (but is not already an element of the set itself). We find the only 2-element magic set:

$$\{\emptyset, \{\emptyset\}\},$$

as well as the only two 3-element magic sets:

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \quad \text{and} \quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\},$$

and finally nine 4-element magic sets: we can add five possible elements to either of the two (the eight elements of the powerset, minus the three that are already present), but adding $\{\{\emptyset\}\}$ to the first set is equivalent to adding $\{\emptyset, \{\emptyset\}\}$ to the second set, so this case is counted twice and needs to be subtracted. So the final answer is 9.