SOUTH AFRICAN TERTIARY MATHEMATICS OLYMPIAD

27 August 2016

Solutions

1. Determine the following value: $\Big||2-\pi|-|2+\pi|\Big|$.

   Solution: Since $2-\pi<0$, we have

   $$\Big||2-\pi|-|2+\pi|\Big| = \Big|(\pi-2)-(\pi+2)\Big| = |-4| = 4.$$

2. Determine the positive real number x that satisfies the equation

   $$\ln(x^x) = \ln(20^x) + \ln(16^x).$$

   Solution: We have

   $$x\ln x = \ln(x^x) = \ln(20^x) + \ln(16^x) = x\ln 20 + x\ln 16 = x(\ln 20 + \ln 16) = x\ln 320.$$

   Since x is positive, we can cancel the factor x and end up with $x = 320$.

3. Determine the positive real number a for which

   $$\sqrt{\int_0^a x\,dx} = \int_0^a \sqrt{x}\,dx.$$

   Solution: We have

   $$\sqrt{\int_0^a x\,dx} = \sqrt{\frac{x^2}{2}\Big|_0^a} = \sqrt{\frac{a^2}{2}}$$

   and

   $$\int_0^a \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^a = \frac{2}{3}a^{3/2}.$$

   So we must have

   $$\frac{a}{\sqrt{2}} = \sqrt{\frac{a^2}{2}} = \frac{2}{3}a^{3/2},$$

   which gives us $\sqrt{a} = \frac{3}{2}\cdot\frac{1}{\sqrt{2}}$ and finally $a = \frac{9}{8}$.

4. Which real numbers α in the interval $[0, 2\pi]$ satisfy the equation

   $$\cos\alpha = \sin 2\alpha = \cos 3\alpha = \sin 4\alpha = \cos 5\alpha = \sin 6\alpha = \cos 7\alpha = \sin 8\alpha = \cos 9\alpha = \sin 10\alpha?$$

   Solution: Consider only the first two expressions:

   $$\cos\alpha = \sin 2\alpha = 2\sin\alpha\cos\alpha,$$

   so either $\cos\alpha = 0$ or $\sin\alpha = \frac{1}{2}$. In the former case, we get $\alpha = \frac{\pi}{2}$ or $\alpha = \frac{3\pi}{2}$, and indeed all expressions evaluate to 0 for those two values of α. If $\sin\alpha = \frac{1}{2}$, however, then we must have $\alpha = \frac{\pi}{6}$ or $\alpha = \frac{5\pi}{6}$, and (e.g.) $\sin(6\alpha) = 0$ in either case, while $\cos\alpha \neq 0$. Thus there are only the two solutions $\alpha = \frac{\pi}{2}$ and $\alpha = \frac{3\pi}{2}$.

5. Find the smallest positive integer with the following property: it is divisible by 7, the sum of its digits is divisible by 7, but none of its digits is 7.

   Solution: Clearly, the only one-digit multiple of 7 does not satisfy the last condition. There is also no two-digit number with this property: if a and b are the digits, then $10a + b$ and $a + b$ must both be multiples of 7. So $9a$ is a multiple of 7 as well, which means that $a = 7$ (since a is a leading digit). This contradicts the third condition again.

   However, checking the first few three-digit multiples of 7 $(105, 112, 119, 126, 133, \ldots)$, we see that 133 is the smallest positive integer that has all three required properties.

6. Caster and Wayde run ten practice laps in an athletics stadium. They start at the same point and at the same time, and they run in the same direction. They also complete their ten laps at exactly the same time. Caster runs the first five laps twice as fast as the last five laps, and Wayde runs the last five laps twice as fast as the first five laps. How often does it happen during this practice session that one of the two overtakes the other?

   Solution: After two laps, Caster overtakes Wayde a first time, then a second time after four laps. Once she has completed the fifth lap, they run at the same speed for a while (so none of them overtakes the other) until Wayde completes the fifth lap. He then overtakes Caster when he completes his sixth and eighth lap. Thus it happens exactly four times.

7. A collection of 2016 balls is arranged in a triangle, with one ball in the first row, two balls in the second, three balls in the third, etc. Now all rows with an even number of balls are removed. How many balls are still left after this?

   Solution: Let n be the number of rows. Then we must have

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} = 2016.$$

   Solving the quadratic equation $n(n + 1) = 2 \cdot 2016 = 4032$ gives us $n = 63$. When the rows with an even number of balls are left out, we have

$$1 + 3 + 5 + \cdots + 63 = 32^2 = 1024$$

   balls remaining.

8. It is known that the three straight lines in the Cartesian plane that are given by the equations $2x + 3y = a$, $3x + ay = 2$ and $ax + 2y = 3$ pass through a common point. What is the value of a?

   Solution: We first determine the point of intersection of the two lines $2x + 3y = a$ and $3x + ay = 2$. We obtain (e.g. by Cramer's rule)

$$x = \frac{a^2 - 6}{2a - 9} \qquad \text{and} \qquad y = \frac{4 - 3a}{2a - 9}.$$

   Plugging these into the third equation gives us

$$\frac{a(a^2 - 6)}{2a - 9} + \frac{2(4 - 3a)}{2a - 9} = 3,$$

so

$$a^3 - 6a + 8 - 6a = 6a - 27$$

or

$$a^3 - 18a + 35 = 0.$$

This can be factorised as

$$(a+5)(a^2 - 5a + 7),$$

and since the second factor has no real zeros, we get $a = -5$.

9. Determine the limit

$$\lim_{x\to 3} \frac{x^x - x^3}{x^x - 3^x}.$$

Solution: L'Hospital's rule gives us

$$\lim_{x\to 3} \frac{x^x - x^3}{x^x - 3^x} = \lim_{x\to 3} \frac{e^{x\ln x} - x^3}{e^{x\ln x} - 3^x} = \lim_{x\to 3} \frac{e^{x\ln x}(\ln x + 1) - 3x^2}{e^{x\ln x}(\ln x + 1) - 3^x \ln 3}$$
$$= \frac{3^3(\ln 3 + 1) - 3^3}{3^3(\ln 3 + 1) - 3^3 \ln 3} = \frac{3^3 \ln 3}{3^3} = \ln 3.$$

10. Positive integers (not necessarily distinct) are written on the faces of a cube in such a way that the difference of the numbers on adjacent faces is never a multiple of 3. The sum of all six numbers is S. Determine the number of possible values of S that are less than 100.

Solution: Numbers on adjacent faces cannot have the same remainder when divided by 3. This leaves only two possible remainders for the four faces adjacent to a given face, which must alternate. So we find that the numbers on opposite faces must always leave the same remainder, and that two of the six numbers must have a remainder of 0, two of them a remainder of 1 and two a remainder of 2. Adding everything up yields a number that must be a multiple of 3, and it must be at least equal to $1+1+2+2+3+3 = 12$. Every number from $12 = 3\cdot 4$ to $99 = 3\cdot 33$ can be attained (for instance, one can use the numbers $1, 1, 2, 2, 3, 3k$, where k ranges from 1 to 30), so there are 30 possible values of S.

11. Determine all complex numbers z that satisfy the equation $4|z| - 2z = 2 - i$.

Solution: Since $4|z|$ is real, the imaginary part of $-2z$ must be $-i$, so $z = x + \frac{i}{2}$ for some real number x. This gives us

$$4\sqrt{x^2 + \frac{1}{4}} - 2\Big(x + \frac{i}{2}\Big) = 2 - i,$$

which simplifies to

$$2\sqrt{x^2 + \frac{1}{4}} = x + 1.$$

Squaring this equation yields $3x^2 - 2x = 0$, so $x = 0$ or $x = \frac{2}{3}$. Both $z = \frac{i}{2}$ and $z = \frac{2}{3} + \frac{i}{2}$ are indeed solutions.

12. How many triplets (A, B, C) of three sets are there for which the two properties $A \cup B \cup C = \{1, 2, \ldots, 100\}$ and $A \cap B \cap C = \emptyset$ hold simultaneously?

Solution: For each of the 100 numbers $1, 2, \ldots, 100$, there are exactly six possibilities: it can belong to exactly one of the three sets (only A, only B, or only C), or it can belong to exactly two of the three sets (A and B, A and C, or B and C). If one of the numbers belongs to all three, the second condition is violated. If one of the numbers belongs to none of the three, or some other number occurs in one of the sets, the first condition is violated. The choices we make for the 100 numbers are independent of each other, so there are 6^{100} possibilities.

13. Suppose that A is a 3×3-matrix such that $\det(A) = 1$, $\det(A + I) = 3$ and $\det(A + 2I) = 5$, where I is the 3×3 identity matrix. Find $\det(A + 3I)$.

Solution: We generally consider the function $f(x) = \det(A + xI)$, which is a polynomial of degree 3 in x when the determinant is expanded:

$$f(x) = \det(A + xI) = x^3 + ax^2 + bx + c.$$

We know that $f(0) = 1$, $f(1) = 3$ and $f(2) = 5$ respectively. Solving the resulting system of equations gives us $a = -3$, $b = 4$ and $c = 1$. So $f(x) = x^3 - 3x^2 + 4x + 1$ and consequently

$$\det(A + 3I) = f(3) = 13.$$

14. A sequence $f_0, f_1, f_2, \ldots$ of real functions is defined as follows: $f_0(x) = x$, and

$$f_{n+1}(x) = \begin{cases} x f_n'(x) & n \text{ even}, \\ \int_0^x f_n(t)\, dt & n \text{ odd}. \end{cases}$$

Determine $f_{2016}(1)$.

Solution: We prove by induction that

$$f_n(x) = \begin{cases} \frac{x^{k+1}}{k+1} & n = 2k \text{ even}, \\ x^{k+1} & n = 2k+1 \text{ odd}. \end{cases}$$

Indeed this is true for $n = 0$. For even $n = 2k$, the induction hypothesis gives us

$$f_{n+1}(x) = x \frac{d}{dx} \frac{x^{k+1}}{k+1} = x^{k+1},$$

while for odd $n = 2k + 1$, we have

$$f_{n+1}(x) = \int_0^x t^{k+1}\, dt = \frac{x^{k+2}}{k+2}.$$

This completes the induction, so we have

$$f_{2016}(x) = \frac{x^{1009}}{1009}$$

and thus $f_{2016}(1) = \frac{1}{1009}$.

15. The sides of a die are numbered from 1 to 6. It is rolled three times, and the outcomes are multiplied. What is the probability that the product is not divisible by 6?

    Solution: The product is not divisible by 6 if we either never roll an even number, or if we never roll a multiple of 3. There are 3^3 possibilities for the former, and 4^3 for the latter. However, the case that we only roll 1 or 5 in all three attempts is counted twice. Hence we find that $3^3 + 4^3 - 2^3 = 83$ of the $6^3 = 216$ outcomes are favourable, which gives us a probability of $\frac{83}{216}$.

16. Determine the smallest prime that does not occur as a factor of any of the numbers $1! + 2016$, $2! + 2016$, $3! + 2016$, ...

    Solution: We notice that $2, 3$, and 7 are all factors of 2016, so they also occur e.g. as factors in $7! + 2016$. Moreover, 5 is a factor of $4! + 2016 = 2040$. However, 11 does not occur as a prime factor of any of these numbers (and is therefore the smallest such prime). Note first that 2016 leaves a remainder of 3 when divided by 11 (and is in particular not divisible by 11). From $11! + 2016$ onwards, the factorial is divisible by 11, so the sum is not. This leaves us with $1! + 2016$, $2! + 2016$, ..., $10! + 2016$. We find that these have remainders of $4, 5, 9, 5, 2, 8, 5, 8, 4, 2$, respectively, so in none of these cases 11 is a factor.

17. Find the maximum value of the expression

    $$xy + x\sqrt{1-y^2} + y\sqrt{1-x^2} - \sqrt{(1-x^2)(1-y^2)},$$

    where x and y are two real numbers in the interval $[-1, 1]$.

    Solution: We substitute $x = \sin\alpha$ and $y = \sin\beta$ (where $\alpha, \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$) and obtain

    $$\begin{aligned} &xy + x\sqrt{1-y^2} + y\sqrt{1-x^2} - \sqrt{(1-x^2)(1-y^2)} \\ &= \sin\alpha\sin\beta + \sin\alpha\cos\beta + \sin\beta\cos\alpha - \cos\alpha\cos\beta = \sin(\alpha+\beta) - \cos(\alpha+\beta). \end{aligned}$$

    Setting $\alpha + \beta = \theta$, it remains to determine the maximum of $f(\theta) = \sin\theta - \cos\theta$, which can be done in different ways: e.g. by differentiating, or by observing that

    $$\sin\theta - \cos\theta = \sqrt{2}\Big(\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}\Big) = \sqrt{2}\sin\Big(\theta - \frac{\pi}{4}\Big).$$

    The maximum of $\sqrt{2}$ is attained when $\theta = \alpha + \beta = \frac{3\pi}{4}$, e.g. for $x = 1$ and $y = \frac{1}{\sqrt{2}}$.

18. Determine all matrices X that satisfy the equation

    $$X^2 + X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

    Solution: Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If we multiply the matrix product out, we obtain

    $$\begin{bmatrix} a^2 + bc + a & ab + bd + b \\ ac + cd + c & d^2 + bc + d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

So we must have $b(a+d+1) = c(a+d+1) = 1$, which gives us $b = c = \frac{1}{a+d+1}$. Subtracting the equations for the two diagonal entries gives us

$$a^2 + a - d^2 - d = (a-d)(a+d+1) = 0.$$

Since $a+d+1$ cannot be 0 (this would violate the equation $b(a+d+1) = c(a+d+1) = 1$), we also find that $a = d$. So we are left with $a^2 + a + b^2 = 1$ and $b(2a+1) = 1$. Substituting $b = \frac{1}{2a+1}$ into the first equation gives us

$$a^2 + a + \frac{1}{(2a+1)^2} = 1.$$

We complete the square and set $u = (2a+1)^2$:

$$4a^2 + 4a + 1 + \frac{4}{(2a+1)^2} = 5,$$

so

$$u + \frac{4}{u} = 5,$$

which gives us $u = 1$ or $u = 4$. Hence we have four possible values for a ($a = 0$, $a = -1$, $a = \frac{1}{2}$ and $a = -\frac{3}{2}$), which give us the four possible matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}.$$

19. A circle of radius 1 rolls along the x-axis until it has made one complete revolution after one minute. A point on the circle is marked and its position in the plane is recorded as the circle moves. What is the maximum speed (in units per minute) at which this point moves along its journey?

    Solution: Without loss of generality we assume that the point A we are considering is the one at the bottom of the circle at the start. After $\frac{\theta}{2\pi}$ minutes ($0 \le \theta \le 2\pi$), the circle touches the point θ on the x-axis, and the angle between the point now at the bottom of the circle and A is θ radians. Thus A is $\sin\theta$ units to the left of this point, and $1 - \cos\theta$ units further up. Thus, its coordinates are $(\theta - \sin\theta, 1 - \cos\theta)$.

    Let t be the time in minutes, $0 \le t \le 1$. At time t, we have $\theta = 2\pi t$, so $\frac{d\theta}{dt} = 2\pi$. The speed of A at time t is

$$v(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

    We have $\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = (1-\cos\theta)2\pi$ and $\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = (\sin\theta)2\pi$. So

$$v(t)^2 = (2\pi)^2((1-\cos\theta)^2 + (\sin\theta)^2) = (2\pi)^2(2 - 2\cos\theta),$$

    which is maximized when $\cos\theta = -1$. In that case $v(t) = 4\pi$.

20. A horizontal line cuts the parabola that is given by the equation $y = x^2$. The area of the finite region enclosed by the line and the parabola is denoted by A. A circle is inscribed in this region in such a way that it touches the parabola in two points and the horizontal line in one point. If B is the area of this circle, determine the maximum of the ratio B/A.

Solution: Suppose that the circle touches the parabola at $(-a, a^2)$ and (a, a^2). Since the gradient of the tangent at (a, a^2) is $2a$, the line from the centre of the circle to the point (a, a^2) must have a gradient of $-\frac{1}{2a}$. Its equation is therefore $y = -\frac{x-a}{2a} + a^2$. This line intersects the y-axis at $(0, a^2 + \frac{1}{2})$. We find that the radius of the circle is

$$r = \sqrt{(0-a)^2 + \left(a^2 + \frac{1}{2} - a^2\right)^2} = \sqrt{a^2 + \frac{1}{4}}.$$

The equation of the horizontal line in the problem statement is therefore

$$y = a^2 + \frac{1}{2} + \sqrt{a^2 + \frac{1}{4}} = \left(\sqrt{a^2 + \frac{1}{4}} + \frac{1}{2}\right)^2 = \left(r + \frac{1}{2}\right)^2,$$

which intersects the parabola at $(-r - 1/2, (r + 1/2)^2)$ and $(r + 1/2, (r + 1/2)^2)$. Thus the area enclosed by the line and the parabola is

$$A = 2\left(r + \frac{1}{2}\right)^3 - \int_{-(r+1/2)}^{r+1/2} x^2\,dx = \frac{4}{3}\left(r + \frac{1}{2}\right)^3.$$

The area of the circle is simply $B = \pi r^2$, so it remains to maximise the quotient

$$\frac{B}{A} = \frac{3\pi r^2}{4(r + 1/2)^3} = \frac{6\pi r^2}{(2r+1)^3}.$$

The derivative with respect to r is

$$\frac{d}{dr}\frac{6\pi r^2}{(2r+1)^3} = \frac{12\pi r(1-r)}{(2r+1)^4}.$$

Thus the maximum is obtained for $r = 1$, and it is equal to $\frac{2\pi}{9}$.