SOUTH AFRICAN TERTIARY MATHEMATICS OLYMPIAD

22 August 2015

Solutions

1. If $\ln x + \ln y = 0$ for positive real numbers x and y, what is the minimum value of $x + y$?

   Solution: From $\ln x + \ln y = 0$, we obtain $xy = 1$, so $x + y = x + \frac{1}{x}$. The derivative of this function is $1 - \frac{1}{x^2}$, which shows that it is decreasing for $0 < x < 1$ and increasing for $x > 1$, with a minimum at $x = 1$ and minimum value 2.

2. Find the value of the limit $\lim_{x\to 0} \frac{\frac{d}{dx}\left(\frac{\sin x}{x}\right)}{x}$.

   Solution:

$$\lim_{x\to 0} \frac{\frac{d}{dx}\left(\frac{\sin x}{x}\right)}{x} = \lim_{x\to 0} \frac{\frac{x\cos x - \sin x}{x^2}}{x} = \lim_{x\to 0} \frac{x\cos x - \sin x}{x^3}$$
$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{3x^2} = -\lim_{x\to 0} \frac{\sin x}{3x} = -\frac{1}{3}.$$

3. If
$$x \cdot \binom{5}{5} \cdot \binom{10}{5} \cdot \binom{15}{5} \cdots \binom{2015}{5} = \binom{6}{5} \cdot \binom{11}{5} \cdot \binom{16}{5} \cdots \binom{2016}{5},$$
   what is the value of x?

   Solution: We have
$$\binom{5}{5} \cdot \binom{10}{5} \cdot \binom{15}{5} \cdots \binom{2015}{5} = \frac{5!}{5!\cdot 0!} \cdot \frac{10!}{5!\cdot 5!} \cdot \frac{15!}{5!\cdot 10!} \cdots \frac{2015!}{5!\cdot 2010!} = \frac{2015!}{5!\cdot 5!\cdots 5!}$$
   after cancellation. Likewise,
$$\binom{6}{5} \cdot \binom{11}{5} \cdot \binom{16}{5} \cdots \binom{2016}{5} = \frac{6!}{5!\cdot 1!} \cdot \frac{11!}{5!\cdot 6!} \cdot \frac{16!}{5!\cdot 11!} \cdots \frac{2016!}{5!\cdot 2011!} = \frac{2016!}{5!\cdot 5!\cdots 5!}.$$
   Therefore,
$$x = \frac{\frac{2016!}{5!^{403}}}{\frac{2015!}{5!^{403}}} = \frac{2016!}{2015!} = 2016.$$

4. Determine the definite integral $\int_0^1 e^{\sqrt{x}}\,dx$.

   Solution: Substituting $\sqrt{x} = u$ (so that $\frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u}$) yields
$$\int_0^1 e^{\sqrt{x}}\,dx = \int_0^1 e^u \cdot 2u\,du = 2ue^u\Big|_0^1 - \int_0^1 2e^u\,du = 2e - 2e^u\Big|_0^1 = 2e - 2e + 2 = 2.$$

5. The solution set of the equation

$$\tan(\arctan x) = \arctan(\tan x)$$

is an interval. How long is this interval?

Solution: Note that $\tan(\arctan x) = x$ for all real x, while $\arctan(\tan x)$ is always a number in the interval $(-\pi/2, \pi/2)$. Indeed, $\arctan(\tan x) = x$ for all x in this interval, so $(-\pi/2, \pi/2)$ is the solution set, and its length is π.

6. In the popular number-placement puzzle Sudoku, one has to fill a 9×9-grid with numbers from 1 to 9 in such a way that each row, each column and each of nine 3×3 sub-grids contains all the numbers from 1 to 9. After having completed a Sudoku, Suzy erases two of the numbers. The average of the remaining numbers is $7/79$ greater than the average of all numbers in the completed Sudoku grid. Which two numbers did Suzy erase?

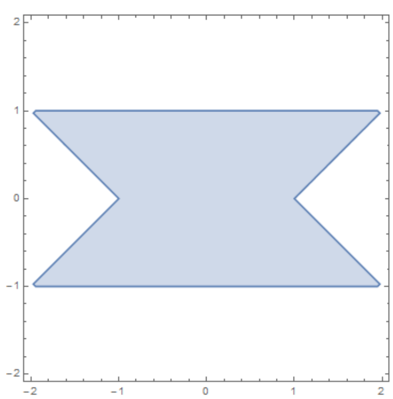
Solution: The total of the numbers in the Sudoku grid is

$$9 \cdot (1 + 2 + \cdots + 9) = 9 \cdot \frac{9 \cdot 10}{2} = 405,$$

and the average of the numbers is thus $405/81 = 5$. It follows that the average after removal is $5 + 7/79 = 402/79$. Thus the total must have decreased by 3 to 402. This is only possible if the numbers erased by Suzy are 1 and 2.

7. Let R be the region consisting of the points (x, y) of the Cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Find the area of R.

Solution: The region is shown in the following figure:



It consists of a 2×2-square and four right-angled isosceles triangles whose legs have length 1. Thus the total area is

$$4 + 4 \cdot \frac{1}{2} = 6.$$

8. How many complex numbers z satisfy the equation $\overline{z} = z^{2015}$?

Solution: If we take absolute values on both sides of the equation, we get

$$|z| = |\overline{z}| = |z^{2015}| = |z|^{2015},$$

which implies that $|z|$ is either 0 or 1. In the former case, we obtain the obvious solution $z = 0$, and in the latter we multiply by z:

$$1 = |z|^2 = z \cdot \overline{z} = z^{2016},$$

so z is one of the 2016 roots of unity. Thus there are 2017 solutions in total.

9. Given a regular 100-gon, how many ways are there to draw a rectangle whose vertices are vertices of the 100-gon?

   Solution: Each rectangle is determined uniquely by its two diagonals, and each of those diagonals has to be a diameter of the circumcircle. There are 50 potential diagonals, which gives us $\binom{50}{2} = 1225$ different choices.

10. Point P lies in the Cartesian plane, but not on the x-axis, and three straight lines ℓ_1, ℓ_2 and ℓ_3 pass through it. The x-axis forms a triangle with the lines ℓ_1 and ℓ_2 that is divided into two smaller triangles of equal area by ℓ_3. If the gradients of ℓ_1 and ℓ_2 are 1 and -2 respectively, what is the gradient of ℓ_3?

    Solution: Suppose that the coordinates of P are (a, b). Then the equation of ℓ_1 is $y = x - a + b$, while the equation of ℓ_2 is $y = -2x + 2a + b$. The lines cut the x-axis at $(a - b, 0)$ and $(a + b/2, 0)$ respectively. Now note that ℓ_3 has to pass through the midpoint of these two to cut the triangle into parts of equal area (which follows from the familiar formula for the area of a triangle as base times height divided by 2). The midpoint is $(a - b/4, 0)$, so the equation of ℓ_3 must be $y = 4x - 4a + b$. This means that the gradient of ℓ_3 is 4.

11. Two random numbers are chosen (independently of each other) from the interval $[0, 1]$. What is the probability that they differ by more than their average?

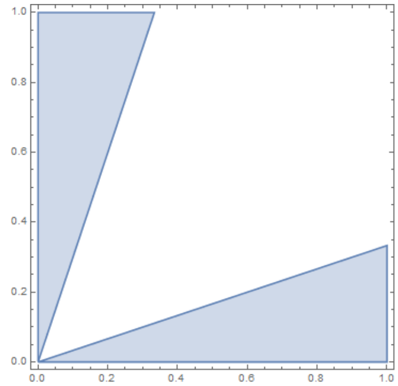
    Solution: If x and y are the two numbers, then the given condition can be written as

    $$|x - y| > \frac{x + y}{2}$$

    or equivalently one of

    $$x - y > \frac{x + y}{2} \qquad \text{or} \qquad y - x > \frac{x + y}{2},$$

    i.e. $x > 3y$ or $y > 3x$. The part of the unit square defined by these two inequalities consists of two right-angled triangles, each with side lengths $\frac{1}{3}$ and 1 (as shown in the figure). Their total area is $\frac{1}{3}$, so the probability of falling into one of the regions is $\frac{1}{3}$.



12. In the solution to the system of equations

$$\begin{aligned} 20x_1 + x_2 + x_3 + \cdots + x_{14} + x_{15} &= -7, \\ x_1 + 20x_2 + x_3 + \cdots + x_{14} + x_{15} &= -6, \\ x_1 + x_2 + 20x_3 + \cdots + x_{14} + x_{15} &= -5, \\ \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\ x_1 + x_2 + x_3 + \cdots + 20x_{14} + x_{15} &= 6, \\ x_1 + x_2 + x_3 + \cdots + x_{14} + 20x_{15} &= 7, \end{aligned}$$

what is the value of x_{15}?

Solution: If we add all fifteen equations, we obtain

$$34(x_1 + x_2 + \cdots + x_{15}) = 0,$$

so the sum of all variables is 0. Subtracting this from the last equation gives us

$$19x_{15} = 7,$$

so $x_{15} = \frac{7}{19}$.

13. Which pairs (a, b) of positive integers satisfy the equation $(-2)^a + 228 = b^2$?

Solution: If a is odd, then $(-2)^a$ is negative. In this case, a cannot be greater than 7, since otherwise $(-2)^a + 228 \leq (-2)^9 + 228 < 0 \leq b^2$. Among the remaining cases $a = 1, 3, 5, 7$, only $a = 5$ and $a = 7$ yield a solution ($b = 14$ and $b = 10$, respectively). If a is even, we can write $a = 2c$ and factorise the resulting equation $228 = b^2 - (-2)^{2c} = b^2 - 2^{2c}$:

$$228 = (b - 2^c)(b + 2^c).$$

The number 228 factorises as $228 = 1 \cdot 228 = 2 \cdot 114 = 3 \cdot 76 = 4 \cdot 57 = 6 \cdot 38 = 12 \cdot 19$. The difference of the two factors $b - 2^c$ and $b + 2^c$ must be a power of 2, namely 2^{c+1}. This is only the case for the factorisation $6 \cdot 38$, which gives us the final solution $b = 22$ and $c = 4$ (thus $a = 8$). So we have three possible triples: $(5, 14)$, $(7, 10)$ and $(8, 22)$.

14. The sum of k consecutive squares is equal to the sum of the following $k-1$ consecutive squares. The last of these $2k-1$ squares is 2015^2. What is the first one?

Solution: Let x^2 be the square in the middle; then the first k squares are $(x-k+1)^2, (x-k+2)^2, \dots, x^2$, and the following $k-1$ squares are $(x+1)^2, (x+2)^2, \dots, (x+k-1)^2$. We obtain the equation

$$\sum_{j=0}^{k-1}(x-j)^2 = \sum_{j=1}^{k-1}(x+j)^2,$$

which simplifies to

$$kx^2 - 2x\sum_{j=0}^{k-1} j + \sum_{j=0}^{k-1} j^2 = (k-1)x^2 + 2x\sum_{j=0}^{k-1} j + \sum_{j=0}^{k-1} j^2,$$

so

$$x^2 = 4x\sum_{j=0}^{k-1} j = 2k(k-1)x.$$

Since x cannot be 0, this gives us $x = 2k(k-1)$, which means that the last square is $(2k(k-1)+(k-1))^2 = ((k-1)(2k+1))^2$. So $(k-1)(2k+1) = 2015$ and thus $k = 32$. Finally, we obtain that the first of the $2k-1$ squares is $(2015 - 2(k-1))^2 = 1953^2$.

15. How many 2×2-matrices with determinant 1 are there whose entries are (not necessarily distinct) elements of the set $\{1, 2, 3, 4\}$?

Solution: If the matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then we must have $ad = bc + 1$. Note that the product ad and the product bc admit one of the following values: $1, 2, 3, 4, 6, 8, 9, 12, 16$. Thus we have the following possible combinations for the pair (ad, bc):

$$(2, 1), (3, 2), (4, 3), (9, 8).$$

We can summarise the possibilities in the following table:

value of ad	2	3	4	9
value of bc	1	2	3	8
number of possibilities for (a, d)	2	2	3	1
number of possibilities for (b, c)	1	2	2	2
number of possibilities combined	2	4	6	2

Thus the overall number of possible matrices is 14.

16. How many polynomials $P(x)$ of degree 4 with real coefficients satisfy the equation $P(x^2) = P(x)P(-x)$ for all x?

Solution: Suppose the leading coefficient of $P(x)$ is a, i.e. $P(x) = ax^4 + \cdots$. In this case, $P(x^2) = ax^8 + \cdots$ and $P(x)P(-x) = a^2x^8 + \cdots$, so the two can only coincide if $a = 1$ (if $a = 0$, then the degree is no longer 4). Now suppose that $P(x)$ factors as follows:

$$P(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4).$$

Then

$$P(x^2) = (x^2 - \alpha_1)(x^2 - \alpha_2)(x^2 - \alpha_3)(x^2 - \alpha_4),$$

and on the other hand

$$P(x)P(-x) = (x^2 - \alpha_1^2)(x^2 - \alpha_2^2)(x^2 - \alpha_3^2)(x^2 - \alpha_4^2).$$

This means that $\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2$ form a permutation of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. In particular, if α is one of the zeros of $P(x)$, then so is α^2, thus also α^4, α^8, ... Since there are only four (and in particular not infinitely many) zeros, this means that some of these powers must coincide, so α is either 0 or a root of unity.

Now we check which roots of unity can occur. The zeros have to form cycles $\alpha \to \alpha^2 \to \cdots \to \alpha^{2^r} = \alpha$, where r is at most 4. Thus $\alpha = 0$ or one of $\alpha, \alpha^3, \alpha^7, \alpha^{15}$ is 1. The 15th roots of unity $e^{2\pi i/15}, e^{4\pi i/15}, e^{8\pi i/15}, e^{16\pi i/15}$ or their conjugates form a cycle of the required form, but the polynomial $(x - e^{2\pi i/15})(x - e^{4\pi i/15})(x - e^{8\pi i/15})(x - e^{16\pi i/15})$ does not have real coefficients: the coefficient of x^3 is

$$-e^{2\pi i/15} - e^{4\pi i/15} - e^{8\pi i/15} - e^{16\pi i/15},$$

whose imaginary part is $-\sin(2\pi/15) - \sin(4\pi/15) - \sin(8\pi/15) - \sin(16\pi/15) < 0$. For the same reason, seventh roots of unity can be excluded.

This leaves us with 0, 1, the third roots of unity $e^{\pm 2\pi i/3}$, which give a factor

$$(x - e^{2\pi i/3})(x - e^{-2\pi i/3}) = \frac{x^3 - 1}{x - 1} = x^2 + x + 1,$$

and the fifth roots of unity $e^{2j\pi i/5}$, $j = 1, 2, 3, 4$ (which are also 15th roots of unity, so they form the required cycle). They give us a factor

$$(x - e^{2\pi i/5})(x - e^{4\pi i/5})(x - e^{6\pi i/5})(x - e^{8\pi i/5}) = \frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1.$$

Now there are ten possible ways to combine the feasible factors x, $x - 1$, $x^2 + x + 1$ and $x^4 + x^3 + x^2 + x + 1$ to a polynomial of degree 4:

$$\begin{aligned}
P(x) &= x^4 + x^3 + x^2 + x + 1,\\
P(x) &= (x^2 + x + 1)^2 = x^4 + 2x^3 + 3x^2 + 2x + 1,\\
P(x) &= (x - 1)^2(x^2 + x + 1) = x^4 - x^3 - x + 1,\\
P(x) &= x(x - 1)(x^2 + x + 1) = x^4 - x,\\
P(x) &= x^2(x^2 + x + 1) = x^4 + x^3 + x^2,\\
P(x) &= (x - 1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1,\\
P(x) &= x(x - 1)^3 = x^4 - 3x^3 + 3x^2 - x,\\
P(x) &= x^2(x - 1)^2 = x^4 - 2x^3 + x^2,\\
P(x) &= x^3(x - 1) = x^4 - x^3,\\
P(x) &= x^4.
\end{aligned}$$

17. Using the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find the value of $\sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3}$.

Solution: Making use of the partial fraction decomposition $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, we simplify the sum as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3} &= \sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)}\right)^3 = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)^3 \\
&= \sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{\infty} \frac{3}{n^2(n+1)} + \sum_{n=1}^{\infty} \frac{3}{n(n+1)^2} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^3} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=2}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{\infty} \frac{3}{n(n+1)} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \sum_{n=1}^{\infty} 3\left(\frac{1}{n} - \frac{1}{n+1}\right)^2 \\
&= 1 - \sum_{n=1}^{\infty} \frac{3}{n^2} + \sum_{n=1}^{\infty} \frac{6}{n(n+1)} - \sum_{n=1}^{\infty} \frac{3}{(n+1)^2} \\
&= 1 - \sum_{n=1}^{\infty} \frac{3}{n^2} - \sum_{n=2}^{\infty} \frac{3}{n^2} + \sum_{n=1}^{\infty} 6\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - 3 \cdot \frac{\pi^2}{6} - 3 \cdot \left(\frac{\pi^2}{6} - 1\right) + 6\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} \cdots\right) \\
&= 1 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + 3 + 6 = 10 - \pi^2.
\end{aligned}$$

18. A mathdie has the following six matrices on its faces:

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Gus the gambler throws a mathdie three times and multiplies the matrices that show up on top in the order thrown. What is the probability that the final product is the matrix M_2?

Solution: If the first of the three factors is one of M_1, M_4, M_5, M_6, then the first and last row of the product must be equal. Since this is not the case, the first factor is M_2 or M_3. The same argument (with columns instead of rows) shows that the last factor is M_2 or M_3. So the product is equal to the middle factor, possibly with the middle row and/or column replaced by zeros (if M_2 is the first or last factor). This also leaves only M_2 or M_3 as middle factor. Of the eight remaining possibilities, all except for $M_3 \cdot M_3 \cdot M_3 = M_3$ give the product M_2, so the probability is $\frac{7}{6^3} = \frac{7}{216}$.

19. Suppose that f is twice differentiable on the interval $[0, 1]$, and that $f(0) = f(1) = 0$ as well as $f''(x) \geq -1$ on the entire interval. Determine the greatest possible value of $\int_0^1 f(x)\,dx$.

Solution: We integrate by parts to obtain

$$\begin{aligned}\int_0^1 f(x)\,dx &= (x+a)f(x)\Big|_0^1 - \int_0^1 (x+a)f'(x)\,dx \\ &= (a+1)f(1) - af(0) - \Big(\frac{x^2}{2} + ax + b\Big)f'(x)\Big|_0^1 + \int_0^1 \Big(\frac{x^2}{2} + ax + b\Big)f''(x)\,dx \\ &= -\Big(\frac{1}{2} + a + b\Big)f'(1) + bf'(0) + \int_0^1 \Big(\frac{x^2}{2} + ax + b\Big)f''(x)\,dx.\end{aligned}$$

Here, a and b can be arbitrary. We choose them as $a = -\frac{1}{2}$ and $b = 0$, so that the first two terms vanish:

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{x^2 - x}{2} f''(x)\,dx = \int_0^1 \frac{x(1-x)}{2}(-f''(x))\,dx.$$

The factor $x(1-x)/2$ is positive on the entire interval $(0, 1)$, and we are given that $f''(x) \geq -1$, so $-f''(x) \leq 1$ on the entire interval. This yields

$$\int_0^1 f(x)\,dx \leq \int_0^1 \frac{x(1-x)}{2}\,dx = \frac{x^2}{4} - \frac{x^3}{6}\Big|_0^1 = \frac{1}{12}.$$

Equality holds if $f''(x) = -1$ for all x, which is the case for $f(x) = x(1-x)/2$.

20. Let $f(x)$ be the function

$$f(x) = \sum_{k=0}^{1000} \binom{2015}{k} x^k (1-x)^{2015-k}.$$

Determine $f''(\frac{1}{2})/f'(\frac{1}{2})$.

Solution: The product rule gives us

$$\begin{aligned}
f'(x) &= \sum_{k=0}^{1000} \binom{2015}{k} \left(kx^{k-1}(1-x)^{2015-k} - (2015-k)x^k(1-x)^{2014-k}\right) \\
&= \sum_{k=0}^{1000} k\binom{2015}{k} x^{k-1}(1-x)^{2015-k} - \sum_{k=0}^{1000}(2015-k)\binom{2015}{k} x^k(1-x)^{2014-k} \\
&= \sum_{k=0}^{1000} k \cdot \frac{2015!}{k!(2015-k)!} x^{k-1}(1-x)^{2015-k} - \sum_{k=0}^{1000}(2015-k) \cdot \frac{2015!}{k!(2015-k)!} x^k(1-x)^{2014-k} \\
&= \sum_{k=1}^{1000} \frac{2015!}{(k-1)!(2015-k)!} x^{k-1}(1-x)^{2015-k} - \sum_{k=0}^{1000} \frac{2015!}{k!(2014-k)!} x^k(1-x)^{2014-k} \\
&= \sum_{k=1}^{1000} 2015\binom{2014}{k-1} x^{k-1}(1-x)^{2015-k} - \sum_{k=0}^{1000} 2015\binom{2014}{k} x^k(1-x)^{2014-k} \\
&= \sum_{k=0}^{999} 2015\binom{2014}{k} x^k(1-x)^{2014-k} - \sum_{k=0}^{1000} 2015\binom{2014}{k} x^k(1-x)^{2014-k} \\
&= -2015\binom{2014}{1000} x^{1000}(1-x)^{1014}.
\end{aligned}$$

Therefore,

$$\frac{f''(x)}{f'(x)} = \frac{-2015\binom{2014}{1000}\left(1000x^{999}(1-x)^{1014} - 1014x^{1000}(1-x)^{1013}\right)}{-2015\binom{2014}{1000}x^{1000}(1-x)^{1014}} = \frac{1000(1-x) - 1014x}{x(1-x)}.$$

Plugging in $x = \frac{1}{2}$ now gives us $f''(\frac{1}{2})/f'(\frac{1}{2}) = -28$.