SOUTH AFRICAN TERTIARY MATHEMATICS OLYMPIAD

23 August 2014

Solutions

1. Determine the limit

$$\lim_{x\to 1} \frac{\sqrt{\sqrt{\sqrt{x}}}-1}{\sqrt{x}-1}.$$

Solution: By L'Hospital's rule, we have

$$\lim_{x\to 1} \frac{\sqrt{\sqrt{\sqrt{x}}}-1}{\sqrt{x}-1} = \lim_{x\to 1} \frac{x^{1/8}-1}{\sqrt{x}-1} = \lim_{x\to 1} \frac{\frac{1}{8}x^{-7/8}}{\frac{1}{2}x^{-1/2}} = \frac{1}{4}.$$

2. For which real value of x is $\ln(3x) = 3\ln x$?

Solution: Since $\ln(3x) = \ln x + \ln 3$, we get $2\ln x = \ln 3$, thus $\ln x = \frac{1}{2}\ln 3 = \ln(3^{1/2})$. We conclude that $x = 3^{1/2} = \sqrt{3}$.

3. This year, August has five Fridays, five Saturdays and five Sundays. According to a recent internet hoax, this situation only occurs every 823 years, but of course this claim is wrong. What is the next year in which August will actually have five Fridays, five Saturdays and five Sundays again?

Solution: Note that this situation can only occur if the 1st of August is a Friday (and the 31st a Sunday). A year has 365 days, which equals 52 weeks and one day, so the day of the week on which the 1st of August falls moves backwards one step each year, and two steps in leap years. Since the 1st of August is a Friday this year, we obtain the following table:

14	15	16	17	18	19	20	21	22	23	24	25
Fr	Th	Tu	Mo	So	Sa	Th	We	Tu	Mo	Sa	Fr

So we see that the situation will next occur in 2025.

4. If

$$\frac{1}{(x-1)(x-2)(x-3)(x-4)(x-5)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} + \frac{D}{x-4} + \frac{E}{x-5}$$

for all real numbers $x \neq 1,2,3,4,5$, what is C?

Solution: Multiply by the denominator $(x-1)(x-2)(x-3)(x-4)(x-5)$ and take the limit as $x \to 3$: then all but the middle term on the right hand side tend to 0, and we get

$$1 = C(3-1)(3-2)(3-4)(3-5),$$

which gives us $C = \frac{1}{4}$.

5. Suppose that x, y are two real numbers in the interval $(-\pi/2, \pi/2)$ such that $\tan x = \cos y$. What is the greatest possible value of x?

Solution: Since $\cos y \leq 1$ for all possible values of y, x can at most be equal to $\frac{\pi}{4}$ (for which $\tan x = 1$).

6. The function $f(x)$ satisfies $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $\ldots$, $f(10) = 11$ and $f'(1) = \sqrt{1}$, $f'(2) = \sqrt{2}$, $f'(3) = \sqrt{3}$, $\ldots$, $f'(10) = \sqrt{10}$. If $g(x) = f(f(f(f(x))))$, what is $g'(1)$?

Solution: By the chain rule,

$$\begin{aligned} g'(1) &= f'(f(f(f(1))))f'(f(f(1)))f'(f(1))f'(1) = f'(4)f'(3)f'(2)f'(1) \\ &= \sqrt{4} \cdot \sqrt{3} \cdot \sqrt{2} \cdot \sqrt{1} = \sqrt{24} = 2\sqrt{6}. \end{aligned}$$

7. Each of the two players in a game of chess has a set of 16 pieces. The pieces in a set are: one king, one queen, two bishops, two knights, two rooks, and eight pawns. One player has white pieces, the other player has black pieces. Kenny chooses two of the 32 pieces from a chess set – how many possible combinations are there? For example, "one white pawn and one black knight" is a possible combination, "two black bishops" another one. The order in which Kenny chooses the pieces does not matter.

Solution: Kenny can either choose two distinct pieces, for which there are $\binom{12}{2}$ possibilities (since there are twelve different types of pieces), or two equal pieces, which is possible for bishops, knights, rooks or pawns (of either colour). This adds another eight possibilities, so the total number of combinations is

$$\binom{12}{2} + 8 = \frac{12 \cdot 11}{2} + 8 = 74.$$

8. 2^{29} is a 9-digit number. As it happens it has no repeated digit, and so has 9 distinct digits. Which of the ten digits $0, 1, 2, \ldots, 9$ is missing?

Solution: The sum of digits of a number leaves the same remainder upon division by 9 as the number itself. Now since

$$2^{29} = (2^6)^4 \cdot 2^5 = 64^4 \cdot 32 \equiv 1^4 \cdot 5 \equiv 5 \bmod 9,$$

we know that the sum of digits must be $\equiv 5 \bmod 9$, and since $0+1+2+\cdots+9 = 45$ is divisible by 9, the digit that is left out must be 4. Of course, the number can also just be calculated explicitly: $2^{29} = 536\,870\,912$.

9. Suppose that the three real numbers x, y, z satisfy the system of equations

$$\begin{aligned} 2^x \cdot 4^y \cdot 16^z &= 1, \\ 4^x \cdot 16^y \cdot 2^z &= 2, \\ 16^x \cdot 2^y \cdot 4^z &= 4. \end{aligned}$$

What is the value of x?

Solution: If we divide the first equation by the square of the third one, we get

$$\frac{2^x \cdot 4^y \cdot 16^z}{(16^x \cdot 2^y \cdot 4^z)^2} = \frac{2^x}{2^{8x}} = \frac{1}{16},$$

thus $2^{7x} = 16 = 2^4$, which gives us $x = \frac{4}{7}$.

10. A and B are both 2×2-matrices with real entries. Determine the missing entry in the following equation:

$$AB - BA = \begin{bmatrix} 3 & 5 \\ -4 & ? \end{bmatrix}.$$

Solution: Note that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, so $\operatorname{tr}(AB - BA) = 0$, which means that the missing entry has to be -3.

11. Suppose that n and m are positive integers such that

$$\frac{n}{20} + \frac{m}{14} = \frac{2014}{20 \cdot 14}.$$

Determine the greatest possible value of n.

Solution: The equation is equivalent to $14n + 20m = 2014$, or

$$7n + 10m = 1007.$$

We need to find the greatest integer n such that $1007 - 7n$ is still positive and a multiple of 10. This means that the last digit of n must be 1, and $n \leq \frac{1007}{7} = 143 + \frac{6}{7}$. Thus $n = 141$ is the greatest possible value.

12. What is the maximum value of the function

$$f(x) = \sqrt{8x - x^2} - \sqrt{14x - x^2 - 48}\ ?$$

Solution: The two parts can be interpreted geometrically as two half-circles over the diameters $[0, 8]$ and $[6, 8]$ respectively. Therefore, the domain of the function is $[6, 8]$, and the maximum is attained for $x = 6$, where $\sqrt{8x - x^2}$ reaches its maximum of $\sqrt{12} = 2\sqrt{3}$ on this interval, and $\sqrt{14x - x^2 - 48}$ its minimum of 0. The maximum value of $f(x)$ is thus $\sqrt{12} = 2\sqrt{3}$.

13. Determine all real numbers $x \in [-1, 1]$ for which

$$(\arcsin x)^2 + (\arccos x)^2 = \frac{5\pi^2}{36}.$$

Solution: Note that $\arcsin x + \arccos x = \frac{\pi}{2}$. Set $\arcsin x = u$, so that $\arccos x = \frac{\pi}{2} - u$ and thus

$$u^2 + \left(\frac{\pi}{2} - u\right)^2 = \frac{5\pi^2}{36},$$

which simplifies to

$$2u^2 - \pi u + \frac{\pi^2}{9} = \left(2u - \frac{\pi}{3}\right)\left(u - \frac{\pi}{3}\right) = 0.$$

Thus $\arcsin x = \frac{\pi}{6}$ or $\arcsin x = \frac{\pi}{3}$, which yields $x = \frac{1}{2}$ or $x = \frac{\sqrt{3}}{2}$.

14. Find all pairs (n, m) of (not necessarily positive) integers that satisfy the equation

$$n(m^2 + 9) + m(n^2 - 9) + n^2(n - 6) = 0.$$

Solution: This is a quadratic equation for m, which we can rewrite as

$$nm^2 + (n^2 - 9)m + (n^3 - 6n^2 + 9n) = 0.$$

The solution is given by

$$m = \frac{-(n^2 - 9) \pm \sqrt{(n^2 - 9)^2 - 4n(n^3 - 6n^2 + 9n)}}{2n}.$$

Thus the expression under the square root must be a perfect square. It factorises as

$$\begin{aligned}(n^2 - 9)^2 - 4n(n^3 - 6n^2 + 9n) &= (n - 3)^2(n + 3)^2 - 4n^2(n - 3)^2 \\ &= (n - 3)^2((n + 3)^2 - 4n^2) \\ &= (n - 3)^2(3 + 3n)(3 - n).\end{aligned}$$

However, this is negative unless $-1 \leq n \leq 3$. For $n = 1$, we do not get a perfect square (the expression evaluates to 48), and the remaining cases $n = -1$, $n = 0$, $n = 2$ and $n = 3$ yield $m = -4$, $m = 0$, $m = 2$ and $m = 0$ respectively.

15. Suppose that $P(x)$ is a polynomial that satisfies

$$xP(x - 1) = (x - 5)P(x).$$

for all real values of x, and $P(-1) = 1$. Determine the value of $P(\frac{1}{2})$.

Solution: Since x occurs as a factor on the left hand side, it must also be a factor of $P(x)$, so $P(x) = xQ(x)$, and we must have $x(x - 1)Q(x - 1) = (x - 5)xQ(x)$, hence

$$(x - 1)Q(x - 1) = (x - 5)Q(x).$$

Repeating the argument, we find that $Q(x)$ must have a factor $x - 1$, etc. After five such steps, we find $P(x) = x(x - 1)(x - 2)(x - 3)(x - 4)R(x)$, where

$$x(x-1)(x-2)(x-3)(x-4)(x-5)R(x-1) = x(x-1)(x-2)(x-3)(x-4)(x-5)R(x).$$

So $R(x-1) = R(x)$, which means that R is a periodic polynomial and thus constant. It is given that $P(-1) = -120R(-1) = 1$, so the constant is $-\frac{1}{120}$, i.e.,

$$P(x) = -\frac{x(x - 1)(x - 2)(x - 3)(x - 4)}{120},$$

which finally yields $P(\frac{1}{2}) = -\frac{7}{256}$.

16. How should the number r be chosen in the interval $[1,2]$ so that the maximum value of the function

$$f(x) = \left| \frac{r-x}{x} \right|$$

on the interval $[1,2]$ is as small as possible?

Solution: We have

$$f(x) = \begin{cases} \frac{r-x}{x} = \frac{r}{x} - 1 & \text{if } x \le r, \\ \frac{x-r}{x} = 1 - \frac{r}{x} & \text{if } x > r, \end{cases}$$

so $f(x)$ is decreasing for $1 \le x \le r$ and increasing for $r \le x \le 2$. This means that the maximum is either attained at $x = 1$ (where it would be equal to $r - 1$) or at $x = 2$ (where it would be equal to $1 - \frac{r}{2}$). We find that the maximum is

$$\max_{1 \le x \le 2} f(x) = \begin{cases} r - 1 & \text{if } r \ge \frac{4}{3}, \\ 1 - \frac{r}{2} & \text{if } r < \frac{4}{3}, \end{cases}$$

which attains its minimum of $\frac{1}{3}$ at $r = \frac{4}{3}$.

17. Three players play a game. Each of them has a black or a white hat on his or her head, and each can see all the other players' hats except his/her own. Each player must write on a piece of paper what they think their own hat's colour is, or write "pass" if they do not want to participate. They all win if nobody gets their hat colour wrong and at least one person guesses right (they cannot all write "pass"). What is the maximum probability of winning if each hat is white with a probability of exactly $\frac{1}{2}$, and this is independent of the colours of the two other hats? The three players may agree on a strategy beforehand, but not communicate during the game.

Solution: The following strategy gives the players a $\frac{3}{4}$ chance of winning: a player who sees that the two others wear hats of the same colour will always guess that his hat colour is different. A player who sees that the two others wear hats of distinct colours will always pass. This guarantees them a win in six out of the $2^3 = 8$ possible cases (all except those where all three hat colours are the same).

There is no way to improve on this strategy: the expected number of correct guesses is always equal to the expected number of incorrect guesses (since a player who guesses is right with probability $\frac{1}{2}$ no matter what the strategy). Let p be the probability of winning. If the players win, they must have made at least one correct guess, so the expected number of correct guesses is at least p. On the other hand, the number of incorrect guesses is 0 in case of a win and at most 3 otherwise. Hence the expected number of incorrect guesses is at most $3(1-p)$. This gives us the inequality

$$p \le 3(1-p),$$

so $p \le \frac{3}{4}$, which completes our proof.

18. Determine the integral

$$\int_0^{\pi} (\cos x + \cos 2x + \cos 3x + \cdots + \cos 100x)^2 \, dx.$$

Solution: The integral is equal to

$$\begin{aligned} I &= \int_0^\pi \left(\sum_{n=1}^{100} \cos nx \right)^2 dx \\ &= \int_0^\pi \sum_{n=1}^{100} \sum_{m=1}^{100} (\cos nx)(\cos mx)\, dx \\ &= \frac{1}{2} \int_0^\pi \sum_{n=1}^{100} \sum_{m=1}^{100} \left(\cos(nx+mx) + \cos(nx-mx) \right) dx \\ &= \frac{1}{2} \sum_{n=1}^{100} \sum_{m=1}^{100} \int_0^\pi \cos((n+m)x)\, dx + \frac{1}{2} \sum_{n=1}^{100} \sum_{m=1}^{100} \int_0^\pi \cos((n-m)x)\, dx. \end{aligned}$$

Now note that

$$\int_0^\pi \cos(kx)\, dx = \frac{\sin kx}{k}\bigg|_0^\pi = \frac{\sin(k\pi) - \sin 0}{k} = 0$$

if $k \neq 0$, and

$$\int_0^\pi \cos(0x)\, dx = \int_0^\pi 1\, dx = \pi.$$

So almost all the integrals vanish, except those in the second term for which $n = m$. Since there are exactly 100 such integrals in the sum, the value of the original integral is

$$\frac{1}{2} \cdot 100 \cdot \pi = 50\pi.$$

19. Consider a triangle ABC with vertices $A = (-1, 0)$, $B = (1, 0)$ and $C = (0, y)$. The lines through A and B perpendicular to BC and AC respectively meet at H (the *orthocentre* of ABC). The angle bisectors of $\angle BAC$ and $\angle ABC$ meet at I (the *incentre* of ABC). Finally, let M and N be the midpoints of AC and BC respectively. The lines AN and BM meet at S (the *centroid* of ABC). Determine the limit of the quotient HI/IS as $y \to \sqrt{3}$ (so that ABC becomes an equilateral triangle).

    Solution: Let α be the angle $\angle CAB = \angle CBA$, which tends to $\frac{\pi}{3}$ as $y \to \sqrt{3}$. Then $\angle HAB = \frac{\pi}{2} - \alpha$ and $\angle IAB = \frac{\alpha}{2}$. Moreover, it is well known that S divides the segment from the midpoint of AB to the vertex C in a $1:2$ ratio. Therefore, the y-coordinates of H, I and S are $\tan(\frac{\pi}{2} - \alpha)$, $\tan(\frac{\alpha}{2})$ and $\frac{\tan\alpha}{3}$ respectively (their x-coordinates are all 0). Thus L'Hospital's rule gives us

$$\begin{aligned} \lim_{y\to\sqrt{3}} \frac{HI}{IS} &= \lim_{\alpha\to\pi/3} \frac{\tan(\pi/2-\alpha) - \tan(\alpha/2)}{\tan(\alpha/2) - (\tan\alpha)/3} = \lim_{\alpha\to\pi/3} \frac{-\sec^2(\pi/2-\alpha) - \sec^2(\alpha/2)/2}{\sec^2(\alpha/2)/2 - (\sec^2\alpha)/3} \\ &= \frac{-4/3 - 2/3}{2/3 - 4/3} = 3. \end{aligned}$$

20. Let S be the set $\{1, 2, \ldots, 20\}$. We want to define a function v that assigns to every subset A of S a positive integer $v(A)$, which we call its *value*. This function should satisfy $v(\emptyset) = 1$, $v(S) = 14$ and

$$v(A \cup B)v(A \cap B) = v(A)v(B)$$

for all subsets A and B of S. How many possible ways are there to define such a function v?

Solution: If we apply the given equation to two disjoint sets A and B, we get

$$v(A \cup B)v(\emptyset) = v(A)v(B),$$

thus $v(A \cup B) = v(A)v(B)$. A simple induction now shows that

$$v(\{x_1, x_2, \ldots, x_n\}) = v(\{x_1\})v(\{x_2\}) \cdots v(\{x_n\}) \tag{1}$$

for any set $\{x_1, x_2, \ldots, x_n\} \subseteq S$. This means that the values of the singleton sets $\{1\}$, $\{2\}$, ..., $\{20\}$ define the function v completely. Conversely, it is easy to check that any function for which (1) holds will also satisfy the original condition.

The values of the singleton sets have to be integers, and their product needs to be $v(S) = 14$. This gives us two possible cases: one of them has value 14, the others value 1 (20 possibilities), or one of them has value 7, one has value 2, the others value 1 ($20 \cdot 19$ possibilities). This gives us a total number of $20 + 20 \cdot 19 = 400$ possible ways to define v.