# SOUTH AFRICAN TERTIARY MATHEMATICS OLYMPIAD

5 October 2013

Solutions

1. For which real value of $x$ is $\log_2(\log_3 x) = 1$?

   **Solution:** We have to have $\log_3 x = 2$ and thus $x = 9$.

2. Determine the value of the limit

$$\lim_{x \to 2013} \frac{\sin(\pi x)}{x - 2013}.$$

   **Solution:** L'Hospital's rule yields

$$\lim_{x \to 2013} \frac{\sin(\pi x)}{x - 2013} = \lim_{x \to 2013} \frac{\pi \cos(\pi x)}{1} = \pi \cos(2013\pi) = -\pi.$$

3. At the university where Peter studies, every student is friends on Facebook with precisely 13 other students. How many students are there at most who are either Peter's Facebook friends or friends with one of his friends?

   **Solution:** Each of Peter's 13 friends has 12 friends other than Peter. Thus there are at most $13 \cdot 12$ students who are friends with one of his friends. Together with his 13 friends, we get $13 \cdot 13 = 169$ people.

4. If

$$f(x) = \sin(\sin(\sin(\sin(\sin(x))))),$$

   what is $f'(0)$?

   **Solution:** By the chain rule,

$$f'(x) = \cos(\sin(\sin(\sin(\sin(x))))) \cos(\sin(\sin(\sin(x)))) \cos(\sin(\sin(x))) \cos(\sin(x)) \cos x$$

   and thus $f'(0) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$.

5. For which value of $a$ does the following system of equations have no solution?

$$\begin{aligned} x + 2y &= 5, \\ -3x + ay &= 1. \end{aligned}$$

   **Solution:** By Cramer's rule (or other methods), we get

$$x = \frac{5a - 2}{a + 6} \qquad \text{and} \qquad y = \frac{16}{a + 6}.$$

   Thus there is a solution in all cases except $a = -6$.

6. At a recent mathematics competition, 252 participants solved the first question, but only 36 solved the last one. Surprisingly, only half of the participants who solved the last question could also solve the first one, but one third of the participants who did not solve the first question could solve the last one. How many people took part in this competition?

   **Solution:** Half the participants who solved the last question (18 people) did not solve the first question. This is exactly one third of the participants who did not solve the first question. Thus there were 252 people who solved the first question, and 54 who did not, which means there were 306 participants.

7. Thomas notices that in four hours, the clock's hour hand will be in precisely the same position where the minute hand is at the moment. How long will it take until the minute hand is in the exact same position where the hour hand is now?

   **Solution:** Four hours correspond to an angle of $120°$ between the two hands. Thus the minute hand has to perform a $240°$ rotation, which takes 40 minutes.

8. The complex number $z$ satisfies $\operatorname{Re} z = 2$. Determine the largest possible value of $\operatorname{Re}(z^2) + \operatorname{Im}(z^2)$.

   **Solution:** Write $z = 2 + iy$. Then we have $z^2 = 4 + 4iy - y^2$ und thus

   $$\operatorname{Re}(z^2) + \operatorname{Im}(z^2) = 4 - y^2 + 4y = 8 - (y-2)^2,$$

   which shows that the maximum is 8 (obtained for $y = 2$).

9. A positive integer $a$ has precisely three positive divisors, while another positive integer $b$ has precisely four positive divisors. What is the minimum possible number of positive divisors of $ab$?

   **Solution:** Let $d$ be the divisor of $a$ that is not 1 or $a$ itself. We find that $ab$ has at least six divisors: all four divisors of $b$ are divisors of $ab$, as are $db$ and $ab$. The example $a = 4$, $b = 8$ shows that the number of divisors of $ab$ can indeed be 6.

10. Let $\|x\|$ be the distance of $x$ from the nearest integer, e.g. $\|1.7\| = 0.3$ or $\|3.2\| = 0.2$. Determine

    $$\int_0^{100} \|x\|\, dx.$$

    **Solution:** Note that

    $$\int_n^{n+1} \|x\|\, dx = \int_n^{n+1/2} (x-n)\, dx + \int_{n+1/2}^{n+1} (n+1-x)\, dx = \int_0^{1/2} u\, du + \int_{1/2}^{1} (1-u)\, du$$
    $$= \frac{u^2}{2}\Big|_0^{1/2} - \frac{(1-u)^2}{2}\Big|_{1/2}^{1} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

    for every integer $n$. Thus

    $$\int_0^{100} \|x\|\, dx = 100 \cdot \frac{1}{4} = 25.$$

11. Determine the sum

$$\sum_{n=1}^{1000} \ln\left(1+\frac{1}{n}\right).$$

**Solution:** The sum is a telescoping sum:

$$\begin{aligned}\sum_{n=1}^{1000} \ln\left(1+\frac{1}{n}\right) &= \sum_{n=1}^{1000}(\ln(n+1)-\ln n)\\ &= (\ln 2-\ln 1)+(\ln 3-\ln 2)+(\ln 4-\ln 3)+\cdot+(\ln 1001-\ln 1000)\\ &= \ln 1001-\ln 1=\ln 1001.\end{aligned}$$

12. Each of the squares of an $8\times 8$-board is coloured randomly black or white (each with probability $\frac{1}{2}$). An "isolated" square is a square whose colour differs from all its horizontally or vertically adjacent squares. What is the expected number of isolated squares?

**Solution:** For the four corners, the probability of being isolated is $\frac{1}{2^2}=\frac{1}{4}$. For the 24 other squares on the edge of the board, the probability is $\frac{1}{2^3}=\frac{1}{8}$, and for the 36 central squares it is $\frac{1}{2^4}=\frac{1}{16}$. Therefore, the expected number of isolated squares is

$$\frac{4}{4}+\frac{24}{8}+\frac{36}{16}=\frac{25}{4}.$$

13. Given that

$$\begin{bmatrix}3 & 3 & x\\ -2 & -3 & -3\\ -1 & -2 & 0\end{bmatrix}^{2013}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

determine $x$.

**Solution:** Taking the determinant on both sides, we find

$$\begin{vmatrix}3 & 3 & x\\ -2 & -3 & -3\\ -1 & -2 & 0\end{vmatrix}^{2013}=0,$$

thus

$$\begin{vmatrix}3 & 3 & x\\ -2 & -3 & -3\\ -1 & -2 & 0\end{vmatrix}=x-9=0,$$

which yields $x=9$.

14. Determine all differentiable functions $f:(0,\infty)\to\mathbb{R}$ such that

$$f'\left(\frac{x}{y}\right)=\frac{f(y)}{f(x)}$$

for all $x,y>0$.

**Solution:** We set $y = 1$. This gives us

$$f'(x) = \frac{f(1)}{f(x)},$$

which is a separable differential equation that can be solved explicitly:

$$\frac{d}{dx}\frac{f(x)^2}{2} = f(x)f'(x) = f(1),$$

thus

$$\frac{f(x)^2}{2} = f(1)x + C$$

for some constant $C$. This means that $f(x) = \pm\sqrt{Ax+B}$, with $A = 2f(1)$ and $B = 2C$. We plug this back into the original equation:

$$\pm\frac{A}{2\sqrt{Ax/y+B}} = \frac{\sqrt{Ay+B}}{\sqrt{Ax+B}}$$

for all $x, y > 0$. Squaring and multiplying out yields

$$A^3x+A^2B = A^2(Ax+B) = 4(Ay+B)(Ax/y+B) = 4A^2x+4ABy+4ABx/y+4B^2.$$

Since this must hold for all $y > 0$, we have $B = 0$. This leaves us with $A^3 = 4A^2$ and consequently $A = 4$ ($A = 0$ yields $f(x) = 0$ for all $x$, in which case the right hand side of the original equation is undefined). $f(x) = -\sqrt{4x}$ is not a solution (the signs do not match), but $f(x) = \sqrt{4x} = 2\sqrt{x}$ is.

15. If $\dfrac{\cos 3x}{\cos x} = \dfrac{1}{3}$ and $0 < x < \frac{\pi}{2}$, determine $\dfrac{\sin 3x}{\sin x}$.

    **Solution:** We use the addition formula for the sine function: Since

    $$\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = \frac{\sin 3x\cos x - \cos 3x \sin x}{\sin x\cos x} = \frac{\sin 2x}{\sin x\cos x} = 2,$$

    we have

    $$\frac{\sin 3x}{\sin x} = \frac{7}{3}.$$

16. What is the expected maximum of two independent uniformly distributed random numbers on $[0, 1]$?

    **Solution:** If the first of the two numbers is $x$, then the second one will be smaller with probability $x$ (so that the maximum is $x$) and larger with probability $1-x$, yielding an average maximum of $\frac{1+x}{2}$. Thus the expected value is

    $$x^2 + (1-x)\frac{1+x}{2} = \frac{1+x^2}{2}.$$

    The average over all $x$ is now

    $$\int_0^1 \frac{1+x^2}{2}\,dx = \frac{x}{2} + \frac{x^3}{6}\Big|_0^1 = \frac{2}{3}.$$

17. Let $R$ and $S$ be points on the sides $AB$ and $AC$, respectively, of triangle $ABC$, and let $P$ be the point of intersection of $BS$ and $CR$. If the areas of triangles $BPR$, $BPC$ and $CPS$ are 5, 6, and 7, respectively, find the area of triangle $ABC$.

**Solution:** Let $\alpha = \angle BPC$. Since $\angle BPR = \angle CPS = 180^\circ - \alpha$ and $\angle RPS = \alpha$, we have

$$\mathsf{Area}(BPR) = \frac{BP \cdot RP}{2}\sin(180^\circ - \alpha) = \frac{BP \cdot RP}{2}\sin\alpha,\ \mathsf{Area}(BPC) = \frac{BP \cdot CP}{2}\sin\alpha,$$
$$\mathsf{Area}(CPS) = \frac{CP \cdot SP}{2}\sin(180^\circ - \alpha) = \frac{CP \cdot SP}{2}\sin\alpha,\ \mathsf{Area}(RPS) = \frac{RP \cdot SP}{2}\sin\alpha.$$

It follows that $\mathsf{Area}(BPR) \cdot \mathsf{Area}(CPS) = \mathsf{Area}(BPC) \cdot \mathsf{Area}(RPS)$, which gives us $\mathsf{Area}(RPS) = \frac{35}{6}$. In the same way, let $\beta = \angle BAC$, and note that

$$\mathsf{Area}(RAS) = \frac{AS \cdot AR}{2}\sin\beta,\ \mathsf{Area}(RAC) = \frac{AR \cdot AC}{2}\sin\beta,$$
$$\mathsf{Area}(BAS) = \frac{AB \cdot AS}{2}\sin\beta,\ \mathsf{Area}(BAC) = \frac{AB \cdot AC}{2}\sin\beta,$$

from which we obtain $\mathsf{Area}(RAS) \cdot \mathsf{Area}(BAC) = \mathsf{Area}(RAC) \cdot \mathsf{Area}(BAS)$. Now let $x$ be the area of triangle $ABC$. We obtain

$$\left(x - 5 - 6 - 7 - \frac{35}{6}\right) \cdot x = (x - 5 - 6)\,(x - 6 - 7)\,,$$

which reduces to $x/6 = 143$ or $x = 858$.

18. Determine the shortest distance between a point on the parabola $y = 8x - x^2$ and a point on the parabola $y = x^2 + 15$.

**Solution:** Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be the points on the two parabolas that are closest to each other ($y_1 = 8x_1 - x_1^2$ and $y_2 = x_2^2 + 15$). The circle centred at $P_1$ that passes through $P_2$ has to touch the second parabola at that point with a common tangent (if they were to intersect instead, one could replace $P_2$ by a point that is closer to $P_1$). Thus the line that connects $P_1$ and $P_2$ is perpendicular to the parabola tangent at $P_2$. The same applies to $P_1$, so the tangents are parallel.

The gradients of the two tangents are $8 - 2x_1$ and $2x_2$ respectively, so we get $x_2 = 4 - x_1$. Since they are also perpendicular to the line between $P_1$ and $P_2$, we also have

$$-\frac{x_2 - x_1}{y_2 - y_1} = 8 - 2x_1 = 2x_2.$$

We plug in $y_1$ and $y_2$:

$$-\frac{(4 - x_1) - x_1}{(4 - x_1)^2 + 15 - 8x_1 + x_1^2} = 8 - 2x_1,$$

which gives

$$(2x_1 - 4) = (8 - 2x_1)(2x_1^2 - 16x_1 + 31)$$

or

$$4x_1^3 - 48x_1^2 + 192x_1 - 252 = 4(x_1 - 3)(x_1^2 - 9x_1 + 21) = 0.$$

The second factor does not have any real zeros, hence $x_1 = 3$, $x_2 = 1$, $y_1 = 15$, $y_2 = 16$, and the shortest distance is $\sqrt{5}$.

19. Let the sum of ten positive numbers $x_1, x_2, \ldots, x_{10}$ be equal to 1, and let $z$ denote the largest number in the sequence

$$\frac{x_1}{1+x_1}, \frac{x_2}{1+x_1+x_2}, \frac{x_3}{1+x_1+x_2+x_3}, \ldots, \frac{x_{10}}{1+x_1+x_2+\cdots+x_{10}}.$$

What is the smallest possible value of $z$?

**Solution:** Set $y_k = 1 + x_1 + x_2 + \cdots + x_k$ (and $y_0 = 1$). Then $x_k = y_k - y_{k-1}$ for all $k$, and the terms of the sequence can be rewritten as

$$\frac{y_k - y_{k-1}}{y_k} = 1 - \frac{y_{k-1}}{y_k}.$$

All elements of the sequence are $\leq z$, thus

$$\frac{y_{k-1}}{y_k} \geq 1 - z,$$

which implies

$$\frac{y_0}{y_{10}} = \frac{y_0}{y_1} \cdot \frac{y_1}{y_2} \cdot \frac{y_2}{y_3} \cdot \cdots \cdot \frac{y_9}{y_{10}} \geq (1-z)^{10}.$$

We are given that $x_1 + x_2 + \cdots + x_{10} = 1$, thus $y_{10} = 2$ and consequently

$$\frac{1}{2} \geq (1-z)^{10},$$

from which we finally deduce $z \geq 1 - \frac{1}{\sqrt[10]{2}}$. We get equality if $y_k = 2^{k/10}$ and thus $x_k = 2^{k/10} - 2^{(k-1)/10}$. So $1 - \frac{1}{\sqrt[10]{2}}$ is the smallest possible value of $z$.

20. Determine

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan x) dx.$$

**Solution:** Note that

$$\begin{aligned} 1 + \tan x &= \frac{\cos x + \sin x}{\cos x} = \frac{\cos x + \cos(\pi/2 - x)}{\cos x} \\ &= \frac{2\cos(\pi/4)\cos(x - \pi/4)}{\cos x} = \frac{\sqrt{2}\cos(x - \pi/4)}{\cos x}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \ln(1+\tan x) dx &= \int_0^{\frac{\pi}{4}} \ln\sqrt{2}\, dx + \int_0^{\frac{\pi}{4}} \ln\cos(x - \pi/4)\, dx - \int_0^{\frac{\pi}{4}} \ln\cos x\, dx \\ &= \frac{\pi \ln 2}{8} + \int_{-\frac{\pi}{4}}^{0} \ln\cos x\, dx - \int_0^{\frac{\pi}{4}} \ln\cos x\, dx \\ &= \frac{\pi \ln 2}{8}. \end{aligned}$$

The two integrals cancel, since $\ln\cos x$ is an even function.