SOUTH AFRICAN TERTIARY MATHEMATICS OLYMPIAD

6 October 2012

Solutions

1. Determine the limit
$$\lim_{x\to 0} \frac{1}{x} e^{-1/x^2}.$$

   Solution: By L'Hospital's rule, we get
$$\lim_{x\to 0} \frac{1}{x} e^{-1/x^2} = \lim_{y\to\infty} y e^{-y^2} = \lim_{y\to\infty} \frac{y}{e^{y^2}} = \lim_{y\to\infty} \frac{1}{2ye^{y^2}} = 0.$$

2. Max has a curious habit: when he is asked questions about his favourite number, he lies and tells the truth alternatingly. His first answer can be either the truth or a lie. His friend Bob asks him the following questions:

   - Is your favourite number less than 100?
   - Is your favourite number a prime?
   - Does your favourite number contain the digit 4?
   - Is your favourite number even?
   - Is your favourite number a square?

   His answer is always yes. What is Max's favourite number?

   Solution: Note that Max's first answer has to be the truth. If not, his second and fourth statement have to be true, implying that his favourite number is 2 (the only even prime). But then his first answer would not have been a lie, contradiction. This means that his favourite number is less than 100, a square, odd (since his fourth answer was a lie), and it contains the digit 4. Of the squares < 100 (1, 4, 9, 16, 25, 36, 49, 64, 81, 100), only 49 is both odd and contains the digit 4.

3. All the sides of a box are rectangular. Three of the faces have diagonals of length $\sqrt{34}$, $\sqrt{58}$ and $\sqrt{74}$. What is the volume of the box?

   Solution: Let the side lengths of the box be a, b and c. By the Pythagorean Theorem, we get
$$\begin{aligned} a^2 + b^2 &= 34, \\ a^2 + c^2 &= 58, \\ b^2 + c^2 &= 74. \end{aligned}$$

   Add the first two equations and subtract the last one. This yields $2a^2 = 34 + 58 - 74 = 18$, thus $a = 3$, from which we also get $b = 5$ and $c = 7$. Hence the volume of the box is $3 \cdot 5 \cdot 7 = 105$.

4. Let the function $f(x)$ be defined by

$$f(x) = x^{x^{x^{x^{\cdots}}}}$$

for positive real x. Determine $f'(a)$ if $f(a) = 2$.

Solution: Note that the function satisfies the functional equation

$$f(x) = x^{f(x)}.$$

Hence if $f(a) = 2$, then $f(a) = 2 = a^2 = a^{f(a)}$, which implies $a = \sqrt{2}$. Moreover, implicit differentiation yields

$$f'(x) = \frac{d}{dx}x^{f(x)} = \frac{d}{dx}e^{f(x)\ln x} = e^{f(x)\ln x}\left(f'(x)\ln x + \frac{f(x)}{x}\right) = f(x)\left(f'(x)\ln x + \frac{f(x)}{x}\right).$$

For $x = a = \sqrt{2}$, we get

$$f'(a) = f(a)\left(f'(a)\ln a + \frac{f(a)}{a}\right) = 2\left(f'(a)\frac{\ln 2}{2} + \frac{2}{\sqrt{2}}\right) = f'(a)\ln 2 + 2\sqrt{2}.$$

Solving for $f'(a)$ yields

$$f'(a) = \frac{2\sqrt{2}}{1 - \ln 2}.$$

5. A sequence $a_1, a_2, a_3, \ldots$ of real numbers is defined by the following rule: $a_1 = 2$, $a_2 = 12$, and

$$a_{n+1} = \frac{a_n + 1}{a_{n-1}}$$

for $n > 1$. Determine a_{2012}.

Solution: Note that $a_3 = \frac{13}{2}$, $a_4 = \frac{5}{8}$, $a_5 = \frac{1}{4}$, $a_6 = 2$ and $a_7 = 12$. Hence the sequence is periodic with period 5, which means that $a_{2012} = a_2 = 12$.

6. Six political leaders must have their photographs taken sitting in a row. Helen and Patricia insist on sitting next to each other. Jacob and Julius refuse to sit next to each other. Tokyo and Trevor don't mind where they sit. In how many ways can they be seated?

Solution: Since Helen and Patricia want to sit next to each other, we consider them as one person for the moment, which gives us $5! = 120$ possible permutations. For each of these permutations, there are two choices for Patricia and Helen (Patricia first or Helen first). However, we need to subtract all arrangements in which Jacob and Julius sit next to each other. To count these, we consider Jacob and Julius as one person as well, giving us $4! = 24$ permutations. For each of these, we still have $2 \cdot 2 = 4$ choices for Patricia/Helen and Jacob/Julius. Hence we end up with

$$2 \cdot 5! - 2 \cdot 2 \cdot 4! = 240 - 96 = 144$$

possible arrangements.

7. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$, the integrals

$$\int_{-\infty}^{x} f(t)\,dt \qquad \text{and} \qquad \int_{-\infty}^{x} f(t)^2\,dt$$

are both convergent and

$$\left(\int_{-\infty}^{x} f(t)\,dt\right)^2 = \int_{-\infty}^{x} f(t)^2\,dt$$

for all $x \in \mathbb{R}$.

Solution: Set

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

Differentiating both sides of the given equation, we get

$$2F(x)F'(x) = 2F(x)f(x) = f(x)^2$$

by the Fundamental Theorem of Calculus. Hence either $f(x) = 0$ or

$$2F(x) = f(x) = F'(x).$$

The solution of the latter differential equation is $F(x) = Ce^{2x}$, which means that $f(x) = F'(x) = 2Ce^{2x}$. Using the fact that $f(0) = 1$, we end up with $f(x) = e^{2x}$.

8. A clock's minute hand has length 4 and its hour hand length 3. What is the distance between the tips at the moment when this distance is increasing most rapidly?

Solution: Consider the triangle ABC that is formed by the two tips A and B and the centre C. If $a = 4$ is the length of the minute hand and $b = 3$ the length of the hour hand, then the distance x between the tips satisfies

$$x^2 = a^2 + b^2 - 2ab\cos C$$

by the law of cosines. Since the angle at C changes at a constant speed, it suffices to maximize the derivative dx/dC. Implicit differentiation yields

$$2x\frac{dx}{dC} = 2ab\sin C, \text{ thus } \frac{dx}{dC} = \frac{ab\sin C}{x} = b\sin A$$

by the law of sines. This is obviously maximal when the angle at A is a right angle, and we get $x = \sqrt{a^2 - b^2} = \sqrt{7}$.

9. Let the Fibonacci sequence be defined by $a_1 = 1$, $a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for all $n > 1$. Find the sum

$$\sum_{n=1}^{\infty} \frac{a_n}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \frac{13}{128} + \cdots$$

Solution: Note that

$$\begin{aligned} S = \sum_{n=1}^{\infty} \frac{a_n}{2^n} &= \frac{a_1}{2} + \frac{a_2}{4} + \sum_{n=3}^{\infty} \frac{a_n}{2^n} \\ &= \frac{1}{2} + \frac{1}{4} + \sum_{n=3}^{\infty} \frac{a_{n-1}}{2^n} + \sum_{n=3}^{\infty} \frac{a_{n-2}}{2^n} \\ &= \frac{3}{4} + \sum_{n=2}^{\infty} \frac{a_n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{a_n}{2^{n+2}} \\ &= \frac{3}{4} + \frac{1}{2}\left(S - \frac{a_1}{2}\right) + \frac{1}{4}S \\ &= \frac{1}{2} + \frac{3}{4}S. \end{aligned}$$

Solving for the sum S, we get $S = 2$.

10. Find all triples (a, b, c) of positive integers with $a < b < c$ such that a, b, c are the sides of a right-angled triangle whose area is twice its circumference.

Solution: The given condition translates to the equation

$$\frac{ab}{2} = 2(a + b + c),$$

thus

$$4c = ab - 4a - 4b.$$

Squaring both sides yields

$$16c^2 = 16(a^2 + b^2) = a^2b^2 - 8a^2b - 8ab^2 + 32ab + 16a^2 + 16b^2$$

or

$$a^2b^2 - 8a^2b - 8ab^2 + 32ab = ab(ab - 8a - 8b + 32) = 0.$$

Since a and b are positive, we must have

$$ab - 8a - 8b + 32 = (a - 8)(b - 8) - 32 = 0.$$

Since a and b are integers, $a - 8$ has to be a divisor of 32. Checking all cases, we find the only solutions $(a, b, c) = (9, 40, 41)$, $(a, b, c) = (10, 24, 26)$ and $(a, b, c) = (12, 16, 20)$.

11. Let f and g be continuous and differentiable on $\mathbb{R}$, and assume that $f(x)$ and $f'(x)$ are never equal to zero. Moreover, suppose that for all $x \in \mathbb{R}$,

$$\frac{g'(x)}{f'(x)} = 2 \cdot \frac{g(x)}{f(x)}.$$

Given that $f(1) = 1$, $g(1) = 2$ and $f(2) = 3$, determine $g(2)$.

Solution: Note that the statement is equivalent to

$$\frac{d}{dx} \ln|g(x)| = \frac{g'(x)}{g(x)} = 2 \cdot \frac{f'(x)}{f(x)} = 2 \cdot \frac{d}{dx} \ln|f(x)|.$$

It follows that

$$\ln|g(x)| = 2\ln|f(x)| + C$$

or $g(x) = Kf(x)^2$ for some real constant K. From the given values for $f(1)$ and $g(1)$, we immediately deduce $K = 2$, thus $g(2) = 2f(2)^2 = 18$.

12. A point P lies on the positive half of the x-axis and the point Q lies in the first quadrant on the graph of $y = x^2$ such that $OP = OQ$ (O being the origin). The straight line through P and Q cuts the y-axis in R. Determine $\lim_{OP\to 0} OR$.

    Solution: Let Q have coordinates (a, a^2). Then P has coordinates $(\sqrt{a^2+a^4}, 0)$. The line through P and Q has gradient

$$\frac{a^2}{a-\sqrt{a^2+a^4}} = \frac{a}{1-\sqrt{1+a^2}}$$

    and thus equation

$$y = \frac{a}{1-\sqrt{1+a^2}}(x-\sqrt{a^2+a^4}).$$

    Thus it intersects the y-axis at $(0, \frac{a^2\sqrt{1+a^2}}{\sqrt{1+a^2}-1})$. We get

$$\lim_{OP\to 0} OR = \lim_{a\to 0} \frac{a^2\sqrt{1+a^2}}{\sqrt{1+a^2}-1} = \lim_{a\to 0} \sqrt{1+a^2}(\sqrt{1+a^2}+1) = 2.$$

13. How many different planes are there, which pass through three or more vertices of a given cube?

    Solution: There are only three types of such planes: the six planes through a face of the cube, the six planes through two (diagonally) opposite edges of the cube and the eight planes through three pairwise nonadjacent vertices. This gives us a total of 20 planes.

14. In a mathematics class, there are three times as many girls as boys. It turns out that the number of ways to select a team of two girls from this class is six times the number of ways to select a team of three boys. How many students are attending this class?

    Solution: Let b be the number of boys and $g = 3b$ the number of girls. It follows from the given statement that

$$\binom{g}{2} = \binom{3b}{2} = 6\binom{b}{3}.$$

    Hence we have

$$\frac{3b(3b-1)}{2} = 6\cdot\frac{b(b-1)(b-2)}{6} = b(b-1)(b-2),$$

    which is equivalent to

$$2b(b-1)(b-2) - 3b(3b-1) = 2b^3 - 15b^2 + 7b = b(2b-1)(b-7) = 0.$$

    The only feasible solution is $b = 7$, thus $g = 21$, which means that the total number of students is 28.

15. Find the value of the determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 99 & 100 \\ 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 101 & 102 \\ 0 & 0 & 7 & 8 & 9 & 10 & \cdots & 103 & 104 \\ 0 & 0 & 9 & 10 & 11 & 12 & \cdots & 105 & 106 \\ 0 & 0 & 0 & 0 & 13 & 14 & \cdots & 107 & 108 \\ 0 & 0 & 0 & 0 & 15 & 16 & \cdots & 109 & 110 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 295 & 296 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 297 & 298 \end{vmatrix}.$$

Solution: We apply elementary row operations to eliminate the entries below the diagonal. To this end, consider the blocks formed around the diagonal in the k-th and $(k+1)$-th row (for odd k):

$$\begin{bmatrix} 3k-2 & 3k-1 \\ 3k & 3k+1 \end{bmatrix}.$$

Subtract $\frac{3k}{3k-2}$ times the k-th row from the $(k+1)$-th row for every odd k, which turns the diagonal entry in the $(2k)$-th row into

$$3k+1-\frac{3k}{3k-2}\cdot(3k-1) = -\frac{2}{3k-2}.$$

We obtain an upper triangular matrix of the form

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 99 & 100 \\ 0 & -2 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 7 & 8 & 9 & 10 & \cdots & 103 & 104 \\ 0 & 0 & 0 & -\frac{2}{7} & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 13 & 14 & \cdots & 107 & 108 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{13} & \cdots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 295 & 296 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{2}{295} \end{vmatrix}.$$

It follows that the determinant of the matrix (and thus also of the original matrix) is

$$(1\cdot(-2))\left(7\cdot\left(-\tfrac{2}{7}\right)\right)\left(13\cdot\left(-\tfrac{2}{13}\right)\right)\cdots = (-2)^{50} = 2^{50}.$$

16. Compute the integral

$$\int_{-\pi}^{\pi}\frac{1}{1+e^{\sin x}}\,dx.$$

Solution: The substitution $x=-y$ yields

$$\int_{-\pi}^{0}\frac{1}{1+e^{\sin x}}\,dx = \int_{\pi}^{0}\frac{1}{1+e^{\sin(-y)}}\,(-dy) = \int_{0}^{\pi}\frac{1}{1+e^{-\sin y}}\,dy = \int_{0}^{\pi}\frac{e^{\sin y}}{1+e^{\sin y}}\,dy.$$

It follows that

$$\int_{-\pi}^{\pi} \frac{1}{1+e^{\sin x}}\,dx = \int_0^{\pi} \frac{1}{1+e^{\sin x}}\,dx + \int_0^{\pi} \frac{e^{\sin x}}{1+e^{\sin x}}\,dx = \int_0^{\pi} 1\,dx = \pi.$$

17. Two players are playing Tic-Tac-Toe on a 3×3 board, but instead of placing Xs and Os, they place 1s and 0s (1 begins). Once the board has been filled completely, the determinant of the resulting 3×3-matrix is calculated. Find all possible values of this determinant.

    Solution: We apply the rule of Sarrus: since there are only five 1s, at most of the NW-SE diagonals does not contain a 0. Likewise, at most one of the NE-SW diagonals does not contain a 0. Thus there are at most two diagonals (at most one in each direction) for which the product of the entries is not 0. The products along these diagonals can only be equal to 1. Hence the determinant is either 1 or -1 (one nonzero product) or 0 (no or two nonzero products).

18. If $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(m+n) = f(m) + f(n) + mn$ for all positive integers m and n, and $f(1) = 3$, calculate $f(13)$.

    Solution: Plugging in $m = 1$, we find $f(n+1) = f(n) + n + f(1) = f(n) + n + 3$. It follows that

    $$f(13) = f(12)+15 = f(11)+15+14 = \cdots = f(1)+15+14+\cdots+4 = 3+4+\cdots+15 = 117.$$

    Generally, induction shows that $f(n) = \frac{n(n+5)}{2}$ (which indeed satisfies the given functional equation).

19. Determine the greatest possible number of elements of a subset A of $\{1, 2, \ldots, 30\}$ with the property that any product of one or more distinct elements of A is not a square.

    Solution: The set P of all primes ≤ 30 satisfies the given condition (it is clear that a product of distinct primes can never be a square), and it has 10 elements:

    $$P = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}.$$

    This is best possible: Consider the exponents in the prime factor decompositions of all elements in A modulo 2. Since only at most 10 primes (namely those in P) can occur in these decompositions, we can regard them as vectors of length 10 over the finite field $\mathbb{F}_2$. The condition that no product of distinct elements in A is a square is equivalent to the statement that the vectors are linearly independent. This is impossible if there are more than 10 such vectors.

20. For sets A and B, define the operation $A \Delta B$ by

    $$A \Delta B = (A \cup B) \setminus (A \cap B).$$

    Moreover, let A_i be the set of all multiples of i in $\{1, 2, \ldots, 1000\}$. Determine the number of elements in the set

    $$A_1 \Delta (A_2 \Delta (A_3 \Delta \cdots \Delta (A_{999} \Delta A_{1000}) \cdots)).$$

Solution: Note that

$$A_1 \Delta (A_2 \Delta (A_3 \Delta \cdots \Delta (A_{999} \Delta A_{1000}) \cdots))$$

consists precisely of the numbers in $\{1, 2, \ldots, 1000\}$ that are contained in an odd number of sets A_i: whenever x is an element of A_i and the symmetric difference $A_i \Delta X$ is formed, x is included if it is not in X and excluded if it is already in X. Hence elements are alternatingly included and excluded, so they are elements of the final set if and only if they are elements of an odd number of sets A_i.

An element n that is included in an odd number of A_i has an odd number of divisors and is thus a square (divisors of n come in pairs $(d, n/d)$, and singletons only occur when n is a square). There are 31 squares ≤ 1000, so the given set has 31 elements.