Analytic Combinatorics – Part V: Saddle point method

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The saddle point method

For functions that are entire (e.g. $A(x) = e^{e^x - 1}$) or have essential singularities (e.g. $A(x) = e^{x/(1-x)}$), the saddle point method is often the method of choice.
It can be considered a complex version of Laplace's method. The main steps are:

- In Cauchy's integral formula

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{A(x)}{x^{n+1}} \, dx,$$

  identify the so-called "saddle point" of the integrand $\frac{A(x)}{x^{n+1}}$.
- Choose a suitable contour $\mathcal{C}$ (typically a circle) passing (at least approximately) through the saddle point.
- Split the integral into a "central" part and the "tails".

Examples

The function $A(x) = e^{x/(1-x)}$ is the exponential generating function for partitions of $\{1, 2, \dots, n\}$ into ordered lists.

It has an essential singularity at 1, so singularity analysis does not apply.

Substitute $x = Re^{iu}$ (for R to be determined), and rewrite the coefficient-extracting integral as

$$\frac{a_n}{n!} = [x^n]A(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\Big(\frac{Re^{iu}}{1 - Re^{iu}} - n \log R - inu \Big)\, du.$$

Examples

The radius R needs to be chosen in such a way that the derivative

$$\frac{\partial}{\partial u}\Big(\frac{Re^{iu}}{1-Re^{iu}} - n\log R - inu\Big)\Big|_{u=0}$$

vanishes, at least approximately. A suitable choice is $R = 1 - 1/\sqrt{n}$.

The integral is now split into a central part (u close to 0), where it is approximated by a Gaussian integral, and the rest.
This procedure yields

$$\frac{a_n}{n!} \sim \frac{1}{2\sqrt{\pi}} n^{-3/4} e^{2\sqrt{n}-1/2},$$

or

$$a_n \sim \frac{1}{\sqrt{2}} n^{n-1/4} e^{-n+2\sqrt{n}-1/2}.$$

Examples

The function $B(x) = e^{e^x - 1}$ is the exponential generating function for the Bell numbers, which count set partitions (partitions of $\{1, 2, \ldots, n\}$ into unordered lists):

$$B(x) = e^{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

This function is entire (no singularities), so again singularity analysis cannot apply.

Setting $x = Re^{iu}$, we obtain a saddle point equation that cannot be solved explicitly (or only by means of special functions, specifically the Lambert W-function): $Re^R = n$.

Examples

However, if we let R be this implicitly defined quantity, then the remaining steps of the saddle point method can be carried out again, and we obtain

$$\frac{B_n}{n!} \sim \frac{1}{\sqrt{2\pi n(1+R)}} e^{nR - n\log n + n/R - 1}.$$

Since $R = \log n - \log\log n + O(\frac{\log\log n}{\log n})$, this yields

$$\log B_n = n\log n - n\log\log n - n + O\Big(\frac{n\log\log n}{\log n}\Big).$$

Examples

An example where identifying the saddle point and evaluating the derivatives is more complicated is the generating function of integer partitions:

$$P(x) = \prod_{j=1}^{\infty} (1 - x^j)^{-1}.$$

Here, the saddle point method can be used to show that the number p_n of partitions of n (ways to write n as an unordered sum of positive integers) satisfies

$$p_n \sim \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n}.$$

There are even more precise formulas, as well as generalisations to different types of partitions.

Literature

For more information, consult the "bible" of analytic combinatorics: