Analytic Combinatorics – Part IV: Singularity analysis 3

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An excursion: the symbolic method

The symbolic method provides us with a way to derive generating functions from symbolic descriptions.
For a class $\mathcal{A}$ of discrete objects with a weight function $|\cdot| : \mathcal{A} \to \{0, 1, 2, \ldots\}$ that has the property that there are only finitely many objects of any given weight, the associated generating function is

$$A(x) = \sum_{\alpha \in \mathcal{A}} x^{|\alpha|} = \sum_{n=0}^{\infty} a_n x^n,$$

where a_n is the number of elements of weight n.

An excursion: the symbolic method

The main operations are the disjoint union of classes

$$\mathcal{A} \uplus \mathcal{B} \mapsto A(x) + B(x)$$

and the Cartesian product

$$\mathcal{A} \times \mathcal{B} \mapsto A(x) \cdot B(x).$$

Several further operations can be derived, such as k-tuples

$$\mathcal{A}^k = \mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A} \mapsto A(x)^k$$

and sequences

$$\mathsf{Seq}(\mathcal{A}) \mapsto \frac{1}{1 - A(x)}.$$

An excursion: the symbolic method

The same also works if the objects are labelled, i.e. the "atoms" of an object are numbered $1, 2, \ldots, n$, if exponential generating functions are used instead:

$$A(x) = \sum_{\alpha \in \mathcal{A}} \frac{x^{|\alpha|}}{|\alpha|!} = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

We still have the operations of disjoint union and Cartesian product:

$$\mathcal{A} \uplus \mathcal{B} \mapsto A(x) + B(x), \quad \mathcal{A} \times \mathcal{B} \mapsto A(x) \cdot B(x)$$

as well as further operations such as sequences (ordered!) and sets (unordered!):

$$\mathsf{Seq}(\mathcal{A}) \mapsto \frac{1}{1 - A(x)}, \quad \mathsf{Set}(\mathcal{A}) \mapsto e^{A(x)}.$$

Examples

The central Delannoy numbers D_n count the number of ways to move from $(0,0)$ to (n,n) using three types of steps: $(0,1)$, $(1,0)$ and $(1,1)$. They can be expressed as

$$D_n = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}.$$

One can derive the generating function

$$D(x) = \sum_{n=0}^{\infty} D_n x^n = \frac{1}{\sqrt{1-6x+x^2}}$$

from this formula, or one can apply the symbolic method.

Examples

Let $\mathcal{D}$ be the class of all Delannoy paths and $\mathcal{B}$ the class of those that never move above the diagonal ($x = y$). The associated generating functions $D(x)$ and $B(x)$ satisfy

$$B(x) = xB(x) + xB(x)^2$$

and

$$D(x) = \frac{1}{1 - (x + 2xB(x))}.$$

Examples

Solving these equations, we obtain

$$D(x) = \sum_{n=0}^{\infty} D_n x^n = \frac{1}{\sqrt{1-6x+x^2}}.$$

The dominant singularity is at $3-2\sqrt{2}$, and singularity analysis gives us

$$D_n \sim \frac{1}{\sqrt{(12\sqrt{2}-16)\pi n}} \cdot (3+2\sqrt{2})^n.$$

Likewise, if B_n is the number of paths that never move above the diagonal, we have

$$B_n \sim \frac{1}{\sqrt{(6\sqrt{2}-8)\pi n^3}} \cdot (3+2\sqrt{2})^n.$$

Examples

The number $t_n = n^{n-1}$ of rooted labelled trees has the exponential generating function

$$T(x) = \sum_{n=1}^{\infty} \frac{t_n}{n!} x^n = x e^{T(x)},$$

the so-called "tree function" which is closely related to the Lambert W-function. It has a dominant square root singularity at $x = \frac{1}{e}$ with expansion

$$T(x) = 1 - \sqrt{2(1 - ex)} + O(|1 - ex|).$$

Examples

Singularity analysis of the tree function gives us

$$\frac{t_n}{n!} = \frac{n^{n-1}}{n!} \sim \frac{1}{\sqrt{2\pi n^3}} e^n,$$

in agreement with Stirling's formula.
However, we can also use this for the more complicated enumeration of unlabelled trees.

Examples

The generating function for rooted unlabelled trees satisfies the functional equation

$$R(x) = x \exp\Big(\sum_{k=1}^{\infty} \frac{1}{k} R(x^k)\Big).$$

Writing

$$S(x) = \sum_{k=2}^{\infty} \frac{1}{k} R(x^k),$$

this gives us

$$R(x) = T(xe^{S(x)}),$$

and the singularity carries over from T since S has greater radius of convergence than R.

Examples

$R(x)$ has a singularity of square root type at $\rho \approx 0.33832$, giving us an asymptotic formula for the number r_n of rooted unlabelled trees with n vertices:

$$r_n \sim A \cdot n^{-3/2} \cdot \rho^{-n},$$

where $A \approx 0.43992$.

The generating function for unrooted unlabelled trees is given by

$$U(x) = R(x) - \frac{1}{2}\Big(R(x)^2 - R(x^2)\Big),$$

from which one can infer that the number u_n of unrooted unlabelled trees with n vertices satisfies

$$u_n \sim B \cdot n^{-5/2} \cdot \rho^{-n},$$

where $B \approx 0.53495$.