Analytic Combinatorics – Part III: Singularity analysis 2

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Singularity analysis: O-version

Now we deal with more general singularities of the form $(1 - x/s)^{-\alpha}$.

Theorem

Suppose that the function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *is analytic and satisfies*

$$A(x) = O(|1 - x/s|^{-\alpha})$$

in a domain of the form

$$\{x \in \mathbb{C} : |x| < |s| + \epsilon,\ \mathrm{Arg}(x/s - 1) > \delta\},$$

where $\epsilon > 0$ *and* $0 \leq \delta < \pi/2$.
Then the coefficients of $A(x)$ *satisfy*

$$a_n = O(|s|^{-n} n^{\alpha - 1}).$$

Singularity analysis: asymptotic version

Theorem

Suppose that the function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *is analytic and satisfies*

$$A(x) = C(1 - x/s)^{-\alpha} + O(|1 - x/s|^{-\beta}) \quad (\alpha \notin \{0, -1, -2, \ldots\},\ \beta < \alpha)$$

in a domain of the form

$$\{x \in \mathbb{C} \,:\, |x| < |s| + \epsilon,\ \operatorname{Arg}(x/s - 1) > \delta\},$$

where $\epsilon > 0$ *and* $0 \leq \delta < \pi/2$.
Then the coefficients of $A(x)$ *satisfy*

$$a_n = \frac{C}{\Gamma(\alpha)} s^{-n} n^{\alpha - 1} + O\big(|s|^{-n} n^{\max(\alpha-2, \beta-1)}\big).$$

Example: central binomial coeffcients

The central binomial coefficients $c_n = \binom{2n}{n}$ have the generating function

$$\sum_{n=0}^{\infty} c_n x^n = \frac{1}{\sqrt{1-4x}},$$

from which we readily deduce that

$$\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}}\big(1 + O(n^{-1})\big).$$