Analytic Combinatorics – Part II: Singularity analysis 1

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A consequence of Stirling's formula

An important consequence of Stirling's formula is an asymptotic formula for the coefficients of the binomial series:

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} (-1)^n \binom{-\alpha}{n} x^n = \sum_{n=0}^{\infty} \binom{\alpha+n-1}{n} x^n.$$

Now

$$\binom{\alpha+n-1}{n} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)} \sim \frac{1}{\Gamma(\alpha)} n^{\alpha-1}.$$

We will see that functions that "behave like" $(1-x)^{-\alpha}$ have coefficients that "behave like" $\frac{1}{\Gamma(\alpha)} n^{\alpha-1}$.

Singularities and coefficients

The location of the singularities of a function

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

is intimately connected to the behaviour of its coefficients.
It is well known that the radius of convergence is given by

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}},$$

and there is always at least one singularity on the circle of convergence.

Singularities and coefficients

Theorem

If the function

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

is (complex) analytic for $|x| < R$*, then* $a_n = O((R-\epsilon)^{-n})$ *for every* $\epsilon > 0$*.*

Simply put:

Location of singularities $\longleftrightarrow$ exponential growth of coefficients

Pringsheim's Theorem

For functions with nonnegative coefficients, we can say even more:

Theorem

Let the radius of convergence of

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

be R*. If the coefficients* a_n *are all nonnegative, then* $A(x)$ *has a singularity at* R*.*

Singularity analysis: meromorphic version

Theorem

Suppose that

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

is meromorphic in the domain $\{x \in \mathbb{C} : |x| < R\}$, *i.e. all singularities are poles. If the poles are located at* $s_1, s_2, \ldots, s_k$ *and their orders are* $\mu_1, \mu_2, \ldots, \mu_k$ *respectively, then there exist polynomials* $p_1(n), p_2(n), \ldots, p_k(n)$ *such that the degree of* p_i *is* $\mu_i - 1$, *and*

$$a_n = p_1(n) s_1^{-n} + p_2(n) s_2^{-n} + \cdots + p_k(n) s_k^{-n} + O((R - \epsilon)^{-n}),$$

for every $\epsilon > 0$.

Examples

The Fibonacci numbers F_n satisfy

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}.$$

From this, one can deduce Binet's formula:

$$F_n = \frac{1}{\sqrt{5}}\Big(\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big).$$

Examples

The Bernoulli numbers are commonly defined by the power series

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1}.$$

We have

$$B_n \sim 2 \cdot (-1)^{n/2+1} \cdot (2\pi)^{-n} \cdot n!$$

if n is even ($B_n = 0$ if n is odd and $n > 1$).

Examples

Let d_n be the number of acyclic digraphs (DAGs) with n labelled vertices. We have the "special generating function"

$$D(x) = \sum_{n=0}^{\infty} \frac{d_n}{n!2^{\binom{n}{2}}} x^n = \frac{1}{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^{\binom{n}{2}}} x^n}.$$

The denominator has its first zero at $x_0 \approx 1.48808$. It follows that the number of DAGs with n vertices is asymptotically given by

$$d_n \sim A \cdot x_0^{-n} \cdot n! \cdot 2^{\binom{n}{2}},$$

where $A \approx 1.74106$.