Analytic Combinatorics – Part I: Laplace's method

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Laplace's method in a nutshell

Typical situation: we have an integral representation

$$f(t) = \int_a^b e^{g(t,x)}\, dx$$

of some function of interest (combinatorial or otherwise) and would like to know its asymptotic behaviour as $t \to \infty$.

This is done in the following steps:

- Identify the "peak", where $g(t,x)$ reaches its maximum.
- Approximate $g(t,x)$ in the "central" region around the peak.
- Estimate the contribution of the "tails" (remaining intervals).

Heuristics

Suppose the function $g(t, x)$ is differentiable sufficiently often. The peak occurs at a point $x_0 = x_0(t)$ where

$$g_x(t, x_0) = 0 \quad \text{and} \quad g_{xx}(t, x_0) < 0.$$

Around x_0, we have the Taylor expansion

$$g(t, x) = g(t, x_0) + \frac{g_{xx}(t, x_0)}{2}(x - x_0)^2 + \cdots$$

Heuristics

The integral

$$f(t) = \int_a^b e^{g(t,x)}\,dx$$

is now approximated by

$$\int_{-\infty}^{\infty} e^{g(t,x_0)+\frac{g_{xx}(t,x_0)}{2}(x-x_0)^2}\,dx,$$

which can be evaluated explicitly: it gives

$$\sqrt{\frac{2\pi}{-g_{xx}(t,x_0)}} \cdot e^{g(t,x_0)}.$$

Requirements

For this approach to work, we need two things:

- The error term in Taylor's approximation must be small in a suitable region $[x_0 - A, x_0 + A]$ around the peak.
  This usually means that A needs to be chosen in such a way that
  $$g_{xxx}(t, x_0)A^3 \to 0.$$
- On the other hand, the contribution of the "tails" (parts outside the central region $[x_0 - A, x_0 + A]$) needs to be negligible.
  For this, it is generally required that
  $$g_{xx}(t, x_0)A^2 \to \infty.$$

The right choice of A is therefore crucial.

The Gamma function

The Gamma function is commonly defined by the integral

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

Integration by parts yields the well-known relation

$$\Gamma(\alpha+1) = \int_0^\infty x^{\alpha} e^{-x}\, dx = \alpha\Gamma(\alpha).$$

We write this as

$$\alpha\Gamma(\alpha) = \int_0^\infty e^{\alpha \log x - x}\, dx.$$

The Gamma function

The peak of the function $g(\alpha, x) = \alpha \log x - x$ is easily identified:

$$g_x(\alpha, x) = \frac{\partial}{\partial x} g(\alpha, x) = \frac{\alpha}{x} - 1,$$

so the maximum is attained for $x = \alpha$.
We have

$$g_{xx}(\alpha, \alpha) = -\frac{1}{\alpha}$$

and

$$g_{xxx}(\alpha, \alpha) = \frac{2}{\alpha^2},$$

so we need to choose A in such a way that $\frac{A^2}{\alpha} \to \infty$ and $\frac{A^3}{\alpha^2} \to 0$.

Stirling's formula

Going through all the technicalities, we obtain

$$\alpha\Gamma(\alpha) \sim \frac{\alpha^\alpha}{e^\alpha} \cdot \sqrt{2\pi\alpha}$$

as $\alpha \to \infty$, a version of Stirling's formula.

More precise approximations

It is often possible to make an asymptotic formula obtained from Laplace's method more precise by including more terms of the Taylor expansion around the peak.

For the Gamma function, this yields an asymptotic expansion

$$\Gamma(\alpha) = \frac{\alpha^\alpha}{e^\alpha} \cdot \sqrt{2\pi\alpha}\Big(\sum_{k=0}^{K} \frac{c_k}{\alpha^k} + O(\alpha^{-K-1})\Big).$$

The first few coefficients are $c_0 = 1$, $c_1 = \frac{1}{12}$, $c_2 = \frac{1}{288}$, ...

Another example

The number of ways to divide the set $\{1, 2, \ldots, n\}$ into two sets with the same sum is given by

$$\frac{2^{n-2}}{\pi} \int_{-\pi}^{\pi} \prod_{j=1}^{n} \cos(jx)\, dx.$$

Laplace's method can be used to show that this is asymptotically equal to

$$\sqrt{\frac{3}{2\pi n^3}} \cdot 2^n$$

if $n \equiv 0, -1 \mod 4$ (and 0 if $n \equiv 1, 2 \mod 4$). Analogous statements are known for more general sets.