COMBINATORICS EXERCISES – SOLUTIONS **Stephan Wagner**

1. There are $8^5 = 32768$ such words, of which $\frac{8!}{3!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$ consist of distinct letters.

2. There are $26^2 \cdot 10^5 = 67600000$ possible number plates.

3. There are six possible colours for the first stripe, then five for the second one (since we may not choose the same colour again), and finally five possible colours for the last stripe (we may not choose the colour of the middle stripe); hence there are $6 \cdot 5 \cdot 5 = 150$ possible flags.

4. There are $\binom{12}{2} = \frac{12 \cdot 11}{1 \cdot 2} = 66$ different pairs of vertices (and thus lines that can be drawn between vertices), of which 12 are the outer sides of the dodecagon. The remaining $66 - 12 = 54$ lines are the diagonals.

5. If the three digits are all distinct, then their order is uniquely determined by the condition $a \le b \le c$. Therefore, each of the $\binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$ possible choices of three digits between 1 and 9 corresponds to exactly one number that satisfies the condition;

   Likewise, if $a = b$ but $a \neq c$, then we have $\binom{9}{2} = \frac{9 \cdot 8}{1 \cdot 2} = 36$ possible choices for the digits (the order follows automatically). We obtain the same number of possibilities if $b = c$ but $a \neq b$.

   Finally, there are 9 possible three digit numbers abc for which $a = b = c$; hence we obtain a total of
$$84 + 36 + 36 + 9 = 165$$
   different numbers.

   An alternative approach is to note that any multiset of three elements taken from the set of possible digits $\{1, 2, \ldots, 9\}$ corresponds to exactly one feasible number (the order is uniquely determined by the condition $a \le b \le c$). The formula for multisets (Theorem 1.6 in your notes) yields
$$\binom{9+3-1}{3} = \binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3} = 165.$$
   Yet another possibility would be to consider all possibilities with $c = 9$, then $c = 8$, etc.

6. The number 420 factorises as $2 \cdot 2 \cdot 3 \cdot 5 \cdot 7$; a factor 7 can only occur if the digit 7 is present (since $2 \cdot 7 = 14$ is already too large for a digit), and for the same reason 5 must also be one of the four digits. This leaves us with two possibilities for the digits: $2, 5, 6, 7$ or $3, 4, 5, 7$; in both cases, we have $4!$ possible arrangements for the digits. Hence there are $2 \cdot 4! = 48$ such numbers.

7. If five mathematicians and three physicists are chosen for the committee, there are
$$\binom{15}{5} \cdot \binom{20}{3} = 3423420$$
   possible committees. If six mathematicians and two physicists are chosen, there are
$$\binom{15}{6} \cdot \binom{20}{2} = 950950$$

possible committees. Finally, if seven mathematicians and one physicist are chosen, there are

$$\binom{15}{7} \cdot \binom{20}{1} = 128700$$

possible committees. Summing the three, we find that there are altogether 4503070 possible ways for the dean to select a committee.

8. A palindrome of length 9 is already uniquely determined by its first five letters (the others are simply the mirror image of the first four). Therefore, we have $26^5 = 11881376$ 9-letter palindromes.

9. We are counting sequences (of men) without repetitions (since no man can be married by two women). Hence the number of possibilities is $\frac{6!}{2!} = 6 \cdot 5 \cdot 4 \cdot 3 = 360$.

10. Since there are only five even numbers among the set $\{1, 2, \ldots, 10\}$, there is only one five-element subset that does not contain at least one odd element. Hence there are $\binom{10}{5} - 1 = 251$ such sets.

11. If we substitute $w = y^2$ in (a), we see that the coefficient of x^3y^4z in $(x + y^2 + z)^6$ is the same as the coefficient of x^3w^2z in $(x + w + z)^6$, which is

$$\binom{6}{3, 2, 1} = \frac{6!}{3!2!1!} = 60.$$

In (b), the coefficient is

$$2^3 \cdot (-1)^4 \cdot (-3) \cdot \binom{8}{3, 4, 1} = 8 \cdot 1 \cdot (-3) \cdot \frac{8!}{3!4!1!} = -6720.$$

12. The word MATHEMATICS consists of two A's, two M's, two T's, one H, one E, one I, one C, and one S. First of all, we can choose the positions of the two A's in $\binom{11}{2}$ ways (there are 11 letters in total); then there are nine positions left, of which we can choose two for the M's in $\binom{9}{2}$ ways. Likewise, we have $\binom{7}{2}$ choices for the two T's after that. Once the positions of the A's, M's and T's are fixed, the remaining letters can be arranged in any of the 5! possible ways among the remaining positions. Hence we have a total of

$$\binom{11}{2} \cdot \binom{9}{2} \cdot \binom{7}{2} \cdot 5! = 4989600$$

different arrangements.

13. If the boys and the girls are numbered from 1 to 20, then each possible way to form pairs corresponds exactly to a permutation of $1, 2, \ldots, 20$. Therefore, 20! is the answer to the first question. If we do not distinguish between boys and girls, then one can argue as follows: the first person can choose 39 different partners; the next person still has 37 choices (everyone except for the first pair); the next person has 35 choices, etc. Therefore, the answer to the second question is

$$39 \cdot 37 \cdot 35 \cdot \ldots \cdot 3 \cdot 1,$$

which is also known as the *double-factorial* (written 39!!).

14. There are n^n sequences in total and $(n-1)^n$ sequences that do not contain the number 1. Therefore, the proportion is
$$\frac{(n-1)^n}{n^n} = \left(1-\frac{1}{n}\right)^n.$$
If we let $n \to \infty$, we find
$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\cdot\ln(1-\frac{1}{n})},$$
and
$$\lim_{n\to\infty} n\cdot\ln\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln(1-\frac{1}{n})}{1/n} = \lim_{n\to\infty}\frac{\frac{1}{n^2(1-1/n)}}{-1/n^2} = \lim_{n\to\infty} -\frac{1}{1-1/n} = -1.$$
Therefore, the limit of the proportion is e^{-1}. Likewise, if we consider sequences that do not contain the numbers $1, 2, \ldots, k$, then the proportion is $\frac{(n-k)^n}{n^n}$, which tends to e^{-k} as $n \to \infty$.

15. Suppose that we use the letter A k times; then there are $\binom{n}{k}$ possibilities for the positions of the A's and 2^{n-k} possible choices for the remaining positions (two possibilities for each of the positions); hence the number is
$$\sum_{\substack{k=0\\k \text{ even}}}^{n}\binom{n}{k}2^{n-k}.$$
Now we use the same trick as in the proof of Theorem 2.4: since
$$\frac{1^k+(-1)^k}{2} = \begin{cases}1 & n \text{ even,}\\ 0 & n \text{ odd,}\end{cases}$$
we find that
$$\sum_{\substack{k=0\\k \text{ even}}}^{n}\binom{n}{k}2^{n-k} = \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k}1^k2^{n-k} + \sum_{k=0}^{n}\binom{n}{k}(-1)^k2^{n-k}\right)$$
$$= \frac{1}{2}\left((1+2)^n + (-1+2)^n\right) = \frac{3^n+1}{2}$$
by the binomial theorem.

16. For each element of $1, 2, \ldots, n$, we have three options: it can be included in A and B or in B only or in neither of the two sets. Hence there are 3^n such pairs.

17.
    - The statement is clear if $n = m$, since both sides of the equation are 1 in this case (if $n < m$, both sides are 0). Now assume that the equation holds for a certain n. Then we have
    $$\sum_{k=m}^{n+1}\binom{k}{m} = \binom{n+1}{m} + \sum_{k=m}^{n}\binom{k}{m} = \binom{n+1}{m} + \binom{n+1}{m+1} = \binom{n+2}{m+1}$$
    by the induction hypothesis and the recursion for binomial coefficients, which completes the induction.

- If we want to choose a subset of $m+1$ elements from the set $\{0,1,\dots,n\}$ such that the maximum of the subset is k, then it remains to choose m elements from the set $\{0,1,\dots,k-1\}$, for which there are $\binom{k}{m}$ possibilities. Summing over all possible values of k, we must obtain the total number of $(m+1)$-element subsets of $\{0,1,\dots,n\}$, which is $\binom{n+1}{m+1}$.

18. In the binomial theorem
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$
we differentiate both sides of the equation to obtain
$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} x^{k-1}.$$
It remains to plug in $x=1$:
$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$
The average number of elements in a random subset of $\{1,2,\dots,n\}$ is thus
$$\frac{\sum_{k=0}^{n} k\binom{n}{k}}{\sum_{k=0}^{n}\binom{n}{k}} = \frac{n2^{n-1}}{2^n} = \frac{n}{2},$$
which is not surprising, since the number of subsets of size k is exactly the same as the number of subsets of size $n-k$ for any k.

19. We argue as follows: if the maximum is k, then the remaining $r-1$ elements are to be chosen from the set $\{1,2,\dots,k-1\}$ that contains $k-1$ elements. Hence there are $\binom{k-1}{r-1}$ subsets whose maximum is k, which proves the stated formula (note that the maximum has to be at least r). Next we have
$$k\binom{k-1}{r-1} = k\cdot\frac{(k-1)!}{(r-1)!(k-r)!} = \frac{k!}{(r-1)!(k-r)!} = r\cdot\frac{k!}{r!(k-r)!} = r\cdot\binom{k}{r},$$
and so we obtain
$$\sum_{k=r}^{n} k\binom{k-1}{r-1} = r\sum_{k=r}^{n}\binom{k}{r} = r\binom{n+1}{r+1}$$
by the formula that was stated above in Problem 17. So the average maximum is
$$\frac{r\binom{n+1}{r+1}}{\binom{n}{r}} = \frac{r\cdot\frac{(n+1)!}{(r+1)!(n-r)!}}{\frac{n!}{r!(n-r)!}} = \frac{r}{r+1}(n+1)$$
after some cancellations.

20. Suppose that l_1 is the first number, that l_2 is the difference between the first and second number, l_3 the difference between the second and third number, ... Finally, l_6 is defined as the difference between the sixth number and 21. The sum of all these numbers (all of which are positive integers) must be 21 (think of l_1 as the distance from 0 to the first number, then l_2 is the distance from the second to the third, etc.), and we have

the additional restriction that the differences l_2, l_3, l_4, l_5 have to be at least 2 (to avoid consecutive numbers). This explains the stated equation. Substituting as described yields

$$l_1 + m_2 + m_3 + m_4 + m_5 + l_6 = 17.$$

The dots-and-bars argument shows that there are $\binom{16}{5} = 4368$ solutions to this equation (where $l_1, m_2, m_3, m_4, m_5, l_6$ can be arbitrary positive integers). This must also be the number of solutions to the original problem.

Generally, the same argument shows that there are $\binom{n-k+1}{k}$ possible ways to choose k elements from the set $\{1, 2, \ldots, n\}$ if no two consecutive elements are allowed.

21. In the binomial theorem,

$$(1+x)^{4n} = \sum_{l=0}^{4n} \binom{4n}{l} x^l,$$

plug in $x = 1$, $x = -1$, $x = i$ and $x = -i$ respectively, and add the four resulting formulas. Then one finds

$$2^{4n} + (1+i)^{4n} + (1-i)^{4n} = \sum_{l=0}^{4n} \binom{4n}{l} (1 + (-1)^l + i^l + (-i)^l).$$

Now one distinguishes the cases $l = 4k$, $l = 4k+1$, $l = 4k+2$, $l = 4k+3$: since $(-1)^4 = i^4 = (-i)^4 = 1$, the equations

$$\begin{aligned} 1 + (-1)^{4k} + i^{4k} + (-i)^{4k} &= 1 + 1 + 1 + 1 = 4, \\ 1 + (-1)^{4k+1} + i^{4k+1} + (-i)^{4k+1} &= 1 - 1 + i - i = 0, \\ 1 + (-1)^{4k+2} + i^{4k+2} + (-i)^{4k+2} &= 1 + 1 - 1 - 1 = 0, \\ 1 + (-1)^{4k+3} + i^{4k+3} + (-i)^{4k+3} &= 1 - 1 - i + i = 0, \end{aligned}$$

hold. Therefore,

$$2^{4n} + (1+i)^{4n} + (1-i)^{4n} = 4 \sum_{k=0}^{n} \binom{4n}{4k},$$

and since $(1+i)^4 = (1-i)^4 = -4$, we have

$$\sum_{k=0}^{n} \binom{4n}{4k} = \frac{1}{4} \left(2^{4n} + 2 \cdot (-4)^n \right) = 2^{4n-2} + (-1)^n \cdot 2^{2n-1}.$$

22. Suppose that the man has n friends; then he has $\binom{n}{3}$ possibilities to invite three friends. We thus have to find the smallest value of n such that $\binom{n}{3} = \frac{n(n-1)(n-2)}{6} \geq 365$. Since $\frac{n^3}{6} \geq \binom{n}{3} \geq 365$, one must have $n \geq \sqrt[3]{6 \cdot 365} = 12.9862$. However, one finds that $\binom{13}{3} = 286$ and $\binom{14}{3} = 364$ are still too small. Therefore, the man has to have at least 15 friends.

23. Seven of the moves have to be horizontal, the other seven vertical. No matter what their order is, the piece will then reach the opposite corner in 14 moves. Therefore, there are $\binom{14}{7} = 3432$ possibilities, corresponding to the number of ways to decide which of the 14 moves are horizontal moves.

24. In the identity

$$\begin{aligned}(1+x)^{n+2} &= (1+x)^2(1+x)^n = (1+x)^2\sum_{k=0}^{n}\binom{n}{k}x^k \\ &= (1+2x+x^2)\sum_{k=0}^{n}\binom{n}{k}x^k \\ &= \sum_{k=0}^{n}\binom{n}{k}(x^k+2x^{k+1}+x^{k+2}),\end{aligned}$$

the coefficient of x^{m+1} on the left hand side is $\binom{n+2}{m+1}$, by the binomial theorem. The coefficient of x^{m+1} on the right hand side, on the other hand, is $\binom{n}{m+1}$ (corresponding to $k = m+1$) plus $2\binom{n}{m}$ (corresponding to $k+1 = m+1$) plus $\binom{n}{m-1}$ (corresponding to $k+2 = m+1$). The identity

$$\binom{n+2}{m+1} = \binom{n}{m+1} + 2\binom{n}{m} + \binom{n}{m-1}$$

follows as a result.

25. By the inclusion-exclusion principle, this number is

(number of all subsets) − (number of those that do not contain a multiple of 2)
− (number of those that do not contain a multiple of 5)
+ (number of those that do not contain any multiples of 2 or 5)

There are 50 multiples of 2 (thus also 50 non-multiples), 20 multiples of 5 (thus 80 non-multiples), and $100-50-20+10 = 40$ numbers that are neither divisible by 2 nor divisible by 5. Hence we obtain a total of

$$\binom{100}{3} - \binom{50}{3} - \binom{80}{3} + \binom{40}{3} = 69820$$

subsets that satisfy the conditions.

26. Since all 2^9 colourings are equally likely, we have to determine the number of those colourings for which one of the 2×2-squares is completely red. For each of the squares, there are 2^5 such colourings. To apply the inclusion-exclusion principle, we also have to consider the intersections. The number of colourings for which two given 2×2-squares are both entirely red is 2^2 if they are diagonally opposite, and otherwise 2^3. Finally, for any three 2×2-squares, there are two possible colourings leaving all of them completely red. Finally, if every 2×2-square is entirely red, then the whole 3×3-square is red, and there is obviously just one possibility for that. We obtain the total number of

$$2^9 - 4\cdot 2^5 + 4\cdot 2^3 + 2\cdot 2^2 - 4\cdot 2 + 1 = 512 - 128 + 32 + 8 - 8 + 1 = 417$$

colourings, i.e., the probability is $417/512 \approx 0.814453$.

27. By the inclusion-exclusion principle, the percentage of students that do not play any of these sports is

$$100 - 60 - 50 - 70 + 30 + 35 + 30 - 20 = -5,$$

which is clearly impossible. Therefore, the claim (or any other part of the information) must be incorrect.

28. There are 1000 squares between 1 and $1000000 = 1000^2$ as well as 100 cubes; if we subtract these from the total number, we overcount all sixth powers, of which there are 10. Hence we find that there are

$$1000000 - 1000 - 100 + 10 = 998910$$

numbers between 1 and 1000000 that are neither squares nor cubes.

29. (a) Every possible distribution of the seats among the three parties corresponds to a composition of 400 into three (nonnegative) summands:

$$x_1 + x_2 + x_3 = 400.$$

We know that there are $\binom{402}{2}$ solutions to this equation (see Theorem 1.6 in your notes). We have to exclude those solutions for which one of the x_i is greater than 200. Suppose, for instance, that the first party gets at least 201 seats; then the remaining 199 seats have to be divided among the three parties, which corresponds to a solution of

$$x_1 + x_2 + x_3 = 199,$$

where x_1, x_2, x_3 are nonnegative integers again. There are $\binom{201}{2}$ possible solutions to this equation. The same argument applies, of course, to the case that the second or third party gets a majority, and so we end up with

$$\binom{402}{2} - 3 \cdot \binom{201}{2} = 20301$$

possible distributions.

(b) Arguing in the exact same way as in (a), we obtain a number of

$$\binom{402}{2} - 3 \cdot \binom{135}{2} = 53466$$

possible distributions.

30. (a) Altogether, there are $\binom{52}{13} = 635013559600$ different hands. There are three possibilities for a total of three points:

- One king: 4 choices for the king, $\binom{36}{12}$ choices for the remaining cards.
- One queen and one jack: 4^2 choices for the queen and the jack, $\binom{36}{11}$ choices for the remaining cards.
- Three jacks: $4 = \binom{4}{3}$ choices for the jacks, $\binom{36}{10}$ choices for the remaining cards.

Therefore, there are

$$4 \cdot \binom{36}{12} + 4^2 \cdot \binom{36}{11} + 4 \cdot \binom{36}{10} = 15636342960$$

decks with a score of three points.

(b) If we want to determine the number of all decks with a score of at least three points, we have to subtract all decks with a score ≤ 2. These are

- One queen: 4 choices for the queen, $\binom{36}{12}$ choices for the remaining cards.
- Two jacks: $6 = \binom{4}{2}$ choices for the jacks, $\binom{36}{11}$ choices for the remaining cards.
- One jack: 4 choices for the jack, $\binom{36}{12}$ choices for the remaining cards.

- No high cards: $\binom{36}{13}$ choices.

Hence there are

$$\binom{52}{13} - 2 \cdot 4 \cdot \binom{36}{12} - 6 \cdot \binom{36}{11} - \binom{36}{13} = 619084516624$$

decks with a score ≥ 3.

31. If the middle number is i, then there are i possible choices for the smallest number and $n-i$ choices for the largest number. Summing over all possible values of i, we must obtain the total number of subsets of size 3, which is $\binom{n+1}{3}$. This proves the stated identity. Generally, if we choose $r+\ell+1$ numbers from the set $\{0, 1, \ldots, n\}$ such that the $(r+1)$-th is i, there are $\binom{i}{r}$ choices for the first r numbers and $\binom{n-i}{\ell}$ choices for the last ℓ numbers. Summing over all i, we get the identity

$$\sum_{i=0}^{n} \binom{i}{r}\binom{n-i}{\ell} = \binom{n+1}{r+\ell+1}.$$

32. There are $\binom{n}{p}$ ways to choose the fixed points. Once these fixed points have been chosen, the remaining $n-p$ numbers must form a derangement. Hence we find that the number of such permutations is

$$\binom{n}{p} \cdot (n-p)! \sum_{k=0}^{n-p} (-1)^k \frac{1}{k!} = \frac{n!}{p!} \cdot \sum_{k=0}^{n-p} (-1)^k \frac{1}{k!}.$$

The probability that a randomly chosen permutation has p fixed points is therefore

$$\frac{1}{p!} \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} = \frac{1}{e \cdot p!}$$

as $n \to \infty$.

33. We consider the polynomial identity

$$\left(\frac{(1+x)^{2n}}{2} + \frac{(1-x)^{2n}}{2} \right)^2 = \frac{(1+x)^{4n} + (1-x)^{4n}}{4} + \frac{(1-x^2)^{2n}}{2}.$$

On the left hand side, we have

$$\begin{aligned} \frac{(1+x)^{2n}}{2} + \frac{(1-x)^{2n}}{2} &= \frac{1}{2} \left(\sum_{k=0}^{2n} \binom{2n}{k} x^k + \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} x^k \right) \\ &= \sum_{\substack{k=0 \\ k \text{even}}}^{2n} \binom{2n}{k} x^k = \sum_{k=0}^{n} \binom{2n}{2k} x^{2k}, \end{aligned}$$

which means that the coefficient of x^{2n} in

$$\left(\frac{(1+x)^{2n}}{2} + \frac{(1-x)^{2n}}{2} \right)^2$$

is

$$\sum_{k=0}^{n} \binom{2n}{2k} \binom{2n}{2n-2k} = \sum_{k=0}^{n} \binom{2n}{2k}^2.$$

On the other hand,

$$\frac{(1+x)^{4n}+(1-x)^{4n}}{4}+\frac{(1-x^2)^{2n}}{2}=\frac{1}{2}\left(\frac{(1+x)^{4n}+(1-x)^{4n}}{2}+(1-x^2)^{2n}\right)$$
$$=\frac{1}{2}\left(\sum_{\substack{k=0\\k\text{even}}}^{4n}\binom{4n}{k}+\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k x^{2k}\right),$$

and the coefficient of x^{2n} in this expression is

$$\frac{1}{2}\left(\binom{4n}{2n}+(-1)^n\binom{2n}{n}\right).$$

34. (a) The characteristic equation is $q^2-3q+2=0$, with the two solutions $q_1=1$ and $q_2=2$. Since 3 is not a solution of the characteristic equation, the solution of the recursion must have the form

$$a_n = A + B\cdot 2^n + C\cdot 3^n.$$

First we determine the nonhomogeneous solution:

$$C\cdot 3^n = 3\cdot C\cdot 3^{n-1} - 2\cdot C\cdot 3^{n-2} + 3^n.$$

Dividing by 3^n, we find

$$C = C - \frac{2C}{9} + 1$$

or $C=\frac{9}{2}$. Now we can use the initial values to determine A and B:

$$\begin{array}{ccccccc} A & + & B & + & \frac{9}{2} & = & -3, \\ A & + & 2B & + & \frac{27}{2} & = & 6, \end{array}$$

so that $A=-\frac{15}{2}$ and $B=0$ and finally $a_n=\frac{9\cdot 3^n-15}{2}$.

(b) The characteristic equation is $q^2-q-2=0$, with the two solutions $q_1=-1$ and $q_2=2$. Since 2 is a solution of the characteristic equation, the solution of the recursion must have the form

$$a_n = A\cdot(-1)^n + (B+Cn+Dn^2)\cdot 2^n.$$

First we determine the nonhomogeneous solution:

$$(Cn+Dn^2)\cdot 2^n = (C(n-1)+D(n-1)^2)\cdot 2^{n-1}+2\cdot(C(n-2)+D(n-2)^2)\cdot 2^{n-2}+9n2^n.$$

Dividing by 2^n, we find

$$Cn+Dn^2 = \frac{Cn}{2}-\frac{C}{2}+\frac{Dn^2}{2}-Dn+\frac{D}{2}+\frac{Cn}{2}-C+\frac{Dn^2}{2}-2Dn+2D+9n,$$

which simplifies to

$$0=(-3D+9)n+\frac{5D-3C}{2}.$$

Therefore, we must have $D=3$ and $C=5$. Now we can use the initial values to determine A and B:

$$\begin{array}{ccccccc} A & + & B & & & = & 0, \\ -A & + & 2B & + & 16 & = & 4, \end{array}$$

so that $A=4$ and $B=-4$ and finally

$$a_n = \left(3n^2+5n-4\right)2^n + 4(-1)^n.$$

35. We use induction on n: for $n = 0$, we obtain

$$\sum_{k=0}^{0} F_k = F_0 = 0 = F_2 - 1,$$

which is obviously true. Now suppose that

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1$$

holds. Then we also have

$$\sum_{k=0}^{n+1} F_k = F_{n+1} + \sum_{k=0}^{n} F_k = F_{n+1} + F_{n+2} - 1 = F_{n+3} - 1$$

by definition of F_n, which is what we had to prove to complete the induction.

An alternative approach makes use of generating functions: the generating function for the Fibonacci numbers is $\frac{x}{1-x-x^2}$, hence the generating function for the sum $\sum_{k=0}^{n} F_k$ is

$$\frac{1}{1-x} \cdot \frac{x}{1-x-x^2}.$$

On the other hand, the generating function for $F_{n+2} - 1$ is

$$x^{-2} \cdot \left(\frac{x}{1-x-x^2} - x \right) - \frac{1}{1-x} = \frac{x}{(1-x)(1-x-x^2)},$$

and the two clearly agree, so we must have $\sum_{k=0}^{n} F_k = F_{n+2} - 1$.

36. (a) $\displaystyle\sum_{n=0}^{\infty} 3^n a_n x^n = \sum_{n=0}^{\infty} a_n (3x)^n = A(3x)$,

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} k a_k \right) x^n = \frac{x}{1-x} A'(x)$ (Combine Theorem 5.4, 6. and 8.).

37. Let the vertices be numbered from 1 to n; the edge between 1 and 2 has to form a triangle with one of the other $n - 2$ points. If k is this point, then the n-gon is divided into a $(k-1)$-gon and a $(n-k+2)$-gon which have to be triangulated separately. Therefore, if t_n denotes the number of triangulations, we have

$$t_n = \sum_{k=3}^{n} t_{k-1} t_{n-k+2}$$

for $n \geq 3$, where we define t_2 to be 1. This is exactly the recursion for the Catalan numbers, albeit shifted: write $a_n = t_{n+2}$ to get

$$a_n = \sum_{k=3}^{n+2} t_{k-1} t_{n-k+4} = \sum_{k=3}^{n+2} a_{k-3} t_{n-k+2} = \sum_{m=0}^{n-1} a_m t_{n-m-1},$$

which describes the Catalan numbers $a_0 = t_2 = 1$, $a_1 = t_3 = 1$, $a_2 = t_4 = 2$, $a_3 = t_5 = 5$, ... (see p.31 in your notes).

38. A valid password can either be a digit, followed by an arbitrary valid password, or a letter or special character, followed by a digit, followed by an arbitrary valid password. Therefore, one has the recursion

$$a_n = 10a_{n-1} + 60 \cdot 10a_{n-2} = 10a_{n-1} + 600a_{n-2},$$

where a_n is the number of passwords of length n. The initial values are $a_1 = 10$ (any one-character password consisting of a single digit is valid) and $a_2 = 700$ (the first character is arbitrary, the second one has to be a digit). The recursion remains true if we set $a_0 = 1$. Then we obtain

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} (10a_{n-1} + 600a_{n-2}) x^n \\ &= 1 + 10x + 10 \sum_{n=1}^{\infty} a_n x^{n+1} + 600 \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= 1 + 10x + 10x(A(x) - a_0) + 600x^2 A(x) = 1 + (10x + 600x^2)A(x). \end{aligned}$$

It follows that

$$A(x) = \frac{1}{1 - 10x - 600x^2} = \frac{3/5}{1 - 30x} + \frac{2/5}{1 + 20x},$$

so that the number of valid passwords of length eight is found to be $\frac{3}{5} \cdot 30^8 + \frac{2}{5} \cdot (-20)^8 = 403900000000$. If one is interested in all passwords of length ≤ 8, one has to consider the generating function for cumulative sums, which is

$$\frac{1}{1-x} A(x) = \frac{18/29}{1 - 30x} - \frac{1/609}{1 - x} + \frac{8/21}{1 + 20x}.$$

So the total number of passwords of length ≤ 8 is found to be $\sum_{k=0}^{8} a_k = \frac{18}{29} \cdot 30^8 - \frac{1}{609} + \frac{8}{21} \cdot 20^8 = 416986863711$ (this includes the empty word of length 0).

39. Let a_n denote the number of configurations with n squares in the bottom row. Each such configuration is either a single row or obtained by placing a smaller stack whose bottom row consists of $1 \leq k \leq n-2$ squares on top of the bottom row. There are $n - 1 - k$ possible positions for the smaller stack. Hence,

$$a_n = 1 + \sum_{k=1}^{n-2} (n - 1 - k) a_k = 1 + \sum_{k=0}^{n-1} (n - 1 - k) a_k$$

for $n \geq 1$ if we set $a_0 = 0$ for convenience. The sum on the right hand side of the equation is exactly the coefficient of x^{n-1} in the product of the two generating functions $\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$ and $\sum_{n=0}^{\infty} a_n x^n = A(x)$. Therefore,

$$A(x) = \sum_{n=1}^{\infty} x^n + x \cdot \frac{x}{(1-x)^2} \cdot A(x) = \frac{x}{1-x} + \frac{x^2}{(1-x)^2} A(x).$$

Solving this equation yields

$$A(x) = \frac{\frac{x}{1-x}}{1 - \frac{x^2}{(1-x)^2}} = \frac{x(1-x)}{1-2x}.$$

It remains to determine the coefficients from this explicit formula:

$$A(x) = \frac{x(1-x)}{1-2x} = -\frac{1}{4} + \frac{x}{2} + \frac{1}{4}\frac{1}{1-2x} = -\frac{1}{4} + \frac{x}{2} + \frac{1}{4}\sum_{n=0}^{\infty} 2^n x^n,$$

which shows that $a_n = \frac{1}{4} \cdot 2^n = 2^{n-2}$ for $n \geq 2$ ($a_0 = 0$ and $a_1 = 1$).

40. The generating function for odd integers $\mathcal{O}$ is given by

$$x + x^3 + x^5 + \ldots = \sum_{n=0}^{\infty} x^{2n+1} = x \sum_{n=0}^{\infty} x^{2n} = \frac{x}{1-x^2}.$$

Since compositions into odd summands can be specified as $\mathsf{Seq}(\mathcal{O})$, we obtain the generating function

$$\frac{1}{1 - \frac{x}{1-x^2}} = \frac{1-x^2}{1-x-x^2} = 1 + \frac{x}{1-x-x^2},$$

which shows that the number of partitions of n into odd summands is the Fibonacci number F_n ($n \geq 1$).

41. Note first: if $A(x)$ is the generating function of the sequence a_n, then the generating function of $\sum_{k=0}^{n} k a_k$ is given by

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} k a_k \right) x^n = \frac{x}{1-x} A'(x),$$

as can be seen by combining 6. and 8. of Theorem 5.4. Now the recursion translates to the differential equation

$$A(x) = \frac{x}{1-x} A'(x).$$

This differential equation is separable: we find

$$\int \frac{1}{A}\, dA = \int \frac{1-x}{x}\, dx$$

or

$$\ln A = \ln x - x + C$$

and thus

$$A(x) = x e^{C-x}.$$

The initial conditions $a_0 = 0$ and $a_1 = 1$ imply that $A(0) = a_0 = 0$ and $A'(0) = a_1 = 1$. The first condition is satisfied for any value of C, for the second we must have

$$e^{C-x} - x e^{C-x} \Big|_{x=0} = 1$$

and thus $C = 0$. So finally

$$A(x) = x e^{-x} = x \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{(n-1)!},$$

which shows that $a_n = \frac{(-1)^{n-1}}{(n-1)!}$ for $n \geq 1$.

42. Grandma's problem can be written symbolically as

$$\mathsf{Seq}_{\geq 25}(\{\bullet\}) \times \mathsf{Seq}(\{\bullet\}) \times \mathsf{Seq}(\{\bullet\}) \times \mathsf{Seq}_{\leq 20}(\{\bullet\}),$$

where $\bullet$ stands for a single rand. This translates to the generating function

$$\frac{x^{25}}{1-x} \cdot \left(\frac{1}{1-x}\right)^2 \cdot \frac{1-x^{21}}{1-x} = \frac{x^{25}-x^{46}}{(1-x)^4}.$$

Now we have to extract the coefficient of x^{100}:

$$\begin{aligned}[x^{100}]\frac{x^{25}-x^{46}}{(1-x)^4} &= [x^{75}](1-x)^{-4} - [x^{54}](1-x)^{-4} = (-1)^{75}\binom{-4}{75} - (-1)^{54}\binom{-4}{54} \\ &= \binom{78}{75} - \binom{57}{54} = \binom{78}{3} - \binom{57}{3} = 46816.\end{aligned}$$

43. Words with this property can be regarded as a collection of three urns (each contains the positions of one of the letters) that have to contain at least two elements each. The exponential generating function for one such urn is

$$\sum_{n=2}^{\infty} \frac{x^n}{n!} = e^x - x - 1,$$

so that we obtain the exponential generating function $(e^x - x - 1)^3$ for our problem.

Remark: If one expands this function, one obtains the explicit formula $3^n - 3(\frac{n}{2}+1)2^n + 3(n^2+n+1)$ for words of length $n \geq 6$ with the prescribed property.

44. A Motzkin path (other than the trivial path of length 0) can either start with a level step, followed by an arbitrary Motzkin path, or with an "up" step, followed by a path that stays above the line $y = 1$, followed by a "down" step (this is the first time that the x-axis is reached again), followed by another (arbitrary) Motzkin path. This shows that we have

$$\mathcal{M} = \epsilon \cup (\rightarrow \mathcal{M}) \cup (\nearrow \mathcal{M} \searrow \mathcal{M}).$$

This translates to the functional equation

$$M(x) = 1 + xM(x) + x^2 M(x)^2$$

for the generating function, which has the solution

$$M(x) = \frac{1 - x - \sqrt{(1-x)^2 - 4x^2}}{2x^2} = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

We have to take the negative sign in order to ensure $\lim_{x\to 0} M(x) = 1$.

45. Unary-binary trees can be recursively defined as

$$\mathcal{UB} = \{\bullet\} \cup (\{\bullet\} \times \mathcal{UB}) \cup (\{\bullet\} \times \mathcal{UB} \times \mathcal{UB}).$$

If $U(x)$ is the generating function, this translates to

$$U(x) = x(1 + U(x) + U(x)^2).$$

Solving the quadratic equation for $U(x)$ yields

$$U(x) = \frac{1 - x \pm \sqrt{1 - 2x - 3x^2}}{2x}.$$

In order to have $\lim_{x \to 0} U(x) = 0$, one must choose the negative sign, so that

$$U(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x}.$$

Compare this generating function to the generating function for Motzkin paths above; the coefficients in the expansion are known as *Motzkin numbers.*

46. The exponential generating function for cycles $\mathcal{C}$ is known to be $-\log(1-x)$; if we want to exclude cycles of length 1, we have to subtract x to get $-\log(1-x) - x$ as the exponential generating function for cycles $\mathcal{C}_{\geq 2}$ of length ≥ 2. Derangements $\mathcal{D}$ can be seen as sets of such cycles:

$$\mathcal{D} = \mathsf{Set}(\mathcal{C}_{\geq 2}).$$

On the generating function level, this means

$$D(x) = \exp(-\log(1-x) - x) = \frac{e^{-x}}{1-x},$$

as desired. Now it only remains to extract coefficients:

$$\begin{aligned} n![x^n]D(x) = n![x^n]e^{-x} \cdot \frac{1}{1-x} &= n![x^n] \left(\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \right) \left(\sum_{l=0}^{\infty} x^l \right) \\ &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \cdot 1 = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \end{aligned}$$

47. (a) We assume that the person is walking on the real axis, starting at 0. Furthermore, assume that the first step is a step to the right, the other case being symmetric. If the first return to position 0 occurs after $2n+2$ steps, then the first step has to be followed by $2n$ steps (n to the left, n to the right) during which the person remains on the positive part of the axis. The final step is from 1 to 0. We can interpret the $2n$ steps in the middle as a Dyck path of length $2n$, and we know that there are $\frac{1}{n+1}\binom{2n}{n}$ such paths. Therefore, we obtain a total of $2 \cdot \frac{1}{n+1}\binom{2n}{n}$ sequences of steps that take the person back to the origin for the first time after $2n+2$ steps.

    (b) The probability for any such sequence to occur is $p^{n+1}q^{n+1}$, since it consists of $n+1$ steps to the left and $n+1$ steps to the right. Summing over all possible sequences, we obtain the total probability

    $$\sum_{n=0}^{\infty} \frac{2}{n+1} \binom{2n}{n} p^{n+1} q^{n+1}.$$

    (c) The generating function of the Catalan numbers is $\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n = \frac{1-\sqrt{1-4x}}{2x}$.

Therefore, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{2}{n+1}\binom{2n}{n}p^{n+1}q^{n+1} &= 2pq\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}(pq)^n \\ &= 2pq\cdot\frac{1-\sqrt{1-4pq}}{2pq} = 1-\sqrt{1-4pq} \\ &= 1-\sqrt{1-4p(1-p)} = 1-\sqrt{1-4p+4p^2} \\ &= 1-\sqrt{(2p-1)^2} = 1-|2p-1| = 1-|p-(1-p)| \\ &= 1-|p-q|.\end{aligned}$$

48. (a) If a permutation of $\{1,2,\ldots,n\}$ has $n-2$ cycles, then there are two possibilities: one cycle of length 3 or two cycles of length 2 (all other cycles must be 1-cycles). In the former case, we have $\binom{n}{3}$ choices for the three elements of the 3-cycle and two possible orientations. In the latter case, we have $\binom{n}{4}$ choices for the four elements that form the 2-cycles and three possibilities to form two cycles. Therefore,

$$\begin{bmatrix} n \\ n-2 \end{bmatrix} = 2\binom{n}{3} + 3\binom{n}{4}.$$

(b) Using the same argument as in (a), we find

$$\begin{Bmatrix} n \\ n-2 \end{Bmatrix} = \binom{n}{3} + 3\binom{n}{4}.$$

The only difference lies in the fact that a 3-cycle can be oriented in two ways, while a set of three elements has no orientation at all.

49. This problem can be interpreted as the number of surjections from the set $\{1,2,\ldots,8\}$ to the set {red, blue, green, yellow}. The number of these surjections is

$$4!\cdot\begin{Bmatrix} 8 \\ 4 \end{Bmatrix} = 40824.$$

50. Note that the bars that separate the runs are inserted precisely at the descents. Hence a permutation with four runs is the same as a permutation with three descents, and so there are $\left\langle {9 \atop 3} \right\rangle = 88234$ such permutations.

51. There are precisely

$$\begin{bmatrix} 7 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 735 + 175 + 21 + 1 = 932$$

such permutations.

52. (a) The inversion table is $6,1,3,2,0,1,0$.
(b) The permutation associated with this inversion table is 6743125.

53. We know that the rising factorials can be written in terms of the Stirling cycle numbers:

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}x^k = x^{\overline{n}} = x(x+1)(x+2)\ldots(x+n-1).$$

Replacing x by $-x$, we find

$$\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}x^k = (-1)^n\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-x)^k = (-1)^n\cdot(-x)(-x+1)(-x+2)\dots(-x+n-1)$$
$$= x(x-1)(x-2)\dots(x-n+1) = x^{\underline{n}}.$$

54. If the number 1 is in a cycle of length $n+1-j$, then we have $\binom{n}{n-j}$ choices for the other elements of this cycle and $(n-j)!$ possible arrangements for these elements within the cycle. The remaining j elements can form the remaining k cycles in $\left[{j \atop k}\right]$ ways. Therefore, we obtain

$$\begin{bmatrix}n+1\\k+1\end{bmatrix} = \sum_{j=k}^{n}\binom{n}{n-j}(n-j)!\begin{bmatrix}j\\k\end{bmatrix} = \sum_{j=k}^{n}\frac{n!}{j!}\begin{bmatrix}j\\k\end{bmatrix}.$$

55. If the number 1 is in a set of size $n+1-j$, then we have $\binom{n}{n-j}=\binom{n}{j}$ choices for the other elements of this set. The remaining j elements can be grouped into the remaining k sets of the set partition in $\left\{{j \atop k}\right\}$ ways. Therefore, we obtain

$$\begin{Bmatrix}n+1\\k+1\end{Bmatrix} = \sum_{j=k}^{n}\binom{n}{j}\begin{Bmatrix}j\\k\end{Bmatrix}.$$

56. If the range is known to have cardinality k, then there are $\binom{m}{k}$ possibilities for the elements of the range. A function, restricted to its range, is always surjective. Therefore, there are $k!\left\{{n \atop k}\right\}$ functions whose range is a given set of k elements (the number of surjections to that set). It follows that the total number of functions from $\{1,2,\dots,n\}$ to $\{1,2,\dots,m\}$, which is m^n, is also equal to

$$\sum_{k=1}^{m}\binom{m}{k}k!\begin{Bmatrix}n\\k\end{Bmatrix} = \sum_{k=1}^{m}\begin{Bmatrix}n\\k\end{Bmatrix}m^{\underline{k}}.$$

57. The number of permutations with an even number of cycles is given by

$$\sum_{\substack{k=1\\k\text{ even}}}^{n}\begin{bmatrix}n\\k\end{bmatrix} = \sum_{k=1}^{n}\frac{1+(-1)^k}{2}\begin{bmatrix}n\\k\end{bmatrix},$$

using the familiar fact that

$$1+(-1)^k = \begin{cases}2 & \text{if } k \text{ is even,}\\ 0 & \text{otherwise.}\end{cases}$$

Now recall that

$$\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}u^k = u^{\overline{n}} = u(u+1)(u+2)\dots(u+n-1).$$

Therefore,

$$\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k = (-1)\cdot 0\cdot 1\cdots(n-2) = 0$$

and thus

$$\sum_{k=1}^{n}\frac{1+(-1)^k}{2}\begin{bmatrix}n\\k\end{bmatrix} = \frac{1}{2}\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix} = \frac{1}{2}n!,$$

which is exactly half of the total number of permutations, proving the statement.

58. (a) The inverse of a permutation is obtained by reversing the directions of all the cycles, hence the number of cycles stays the same.

(b) Only 1- and 2-cycles have the property that they remain the same if the direction is reversed. Hence only permutations consisting entirely of 1- and 2-cycles can be identical to their inverses.

(c) If all cycles in a permutation of n elements are 1-or 2-cycles, then there have to be at least $n/2$ cycles (since the total number of elements in the cycles has to be n). Thus, by (b), if $k < n/2$, any permutation of n elements with k cycles cannot be equal to its own inverse. This means that we can group all such permutations into pairs (each permutation is paired with its inverse). This can only be possible if their number, which is the Stirling cycle number $\left[{n \atop k}\right]$, is even.

59. If we number the levels $1, 2, 3, \dots$ and assign a label to each node that is exactly the number of its level, then we see that Joyce trees are nothing else but binary increasing trees (see p.58 in your notes). Therefore, the number of Joyce trees with $2n+1$ nodes is exactly the tangent number t_{2n+1}.

60. The Lagrange inversion formula, applied to the functional equation

$$A(x) = \frac{x}{(1 - A(x))^2} = x(1 - A(x))^{-2},$$

yields

$$a_n = [x^n]A(x) = \frac{1}{n}[t^{n-1}](1-t)^{-2n} = \frac{1}{n}(-1)^{n-1}\binom{-2n}{n-1} = \frac{1}{n}\binom{3n-2}{n-1}.$$

61. Since $\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1}$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^{n/2-1}(4^n - 2^n)B_n}{n!} x^n &= -\sum_{n=0}^{\infty} \frac{B_n}{n!}(-1)^{n/2}4^n x^n + \sum_{n=0}^{\infty} \frac{B_n}{n!}(-1)^{n/2}2^n x^n \\
&= -\sum_{n=0}^{\infty} \frac{B_n}{n!}(4ix)^n + \sum_{n=0}^{\infty} \frac{B_n}{n!}(2ix)^n \\
&= -\frac{4ix}{e^{4ix} - 1} + \frac{2ix}{e^{2ix} - 1} = -\frac{4ix}{e^{4ix} - 1} + \frac{2ix(e^{2ix} + 1)}{e^{4ix} - 1} \\
&= \frac{2ix}{e^{4ix} - 1}\left(-2 + e^{2ix} + 1\right) = \frac{2ix(e^{2ix} - 1)}{e^{4ix} - 1} = \frac{2ix}{e^{2ix} + 1} \\
&= ix + ix\left(\frac{2}{e^{2ix} + 1} - 1\right) = ix + ix \cdot \frac{1 - e^{2ix}}{e^{2ix} + 1} \\
&= ix - ix \cdot \frac{e^{2ix} - 1}{e^{2ix} + 1} = ix - ix \cdot \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} \\
&= ix + x \cdot \frac{(e^{ix} - e^{-ix})/(2i)}{(e^{ix} + e^{-ix})/2} = ix + x \cdot \frac{\sin x}{\cos x} \\
&= ix + x \tan x.
\end{aligned}$$

Now we compare the coefficients of x^n: since $x \tan x = \sum_{k=0}^{\infty} \frac{t_k}{k!} x^{k+1} = \sum_{k=0}^{\infty} \frac{kt_k}{(k+1)!} x^{k+1}$, the coefficient of x^n in the expansion of $ix + x\tan x$ is $\frac{nt_{n-1}}{n!}$ for $n > 1$, which shows that

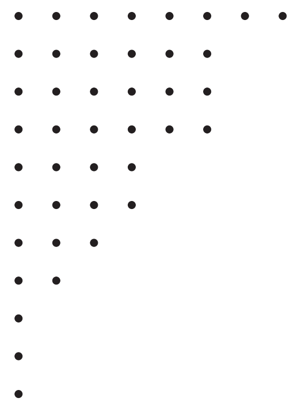
$$(-1)^{n/2-1}(4^n - 2^n)B_n = nt_{n-1}$$

or

$$B_n = (-1)^{n/2-1} \frac{n}{4^n - 2^n} t_{n-1}$$

for all even $n > 1$.

62. The conjugate partition is $11 + 8 + 7 + 6 + 4 + 4 + 1 + 1$, see the figure.



63. The generating function is given by

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2},$$

which has the partial fraction decomposition

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2} = \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1-x)^2}.$$

Now we can expand into a series:

$$\begin{aligned}
\frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1-x)^2} &= \frac{1}{4}\sum_{n=0}^{\infty} x^n + \frac{1}{4}\sum_{n=0}^{\infty}(-x)^n + \frac{1}{2}\sum_{n=0}^{\infty}\binom{-2}{n}x^n \\
&= \frac{1}{4}\sum_{n=0}^{\infty} x^n + \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n + \frac{1}{2}\sum_{n=0}^{\infty}\binom{n+1}{n}x^n \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{4} + \frac{1}{4}(-1)^n + \frac{n+1}{2}\right)x^n \\
&= \sum_{n=0}^{\infty}\frac{2n+3+(-1)^n}{4} \cdot x^n.
\end{aligned}$$

It follows that the number of possibilities is

$$\frac{2n+3+(-1)^n}{4} = \begin{cases} \frac{n}{2}+1 & n \text{ even,} \\ \frac{n+1}{2} & n \text{ odd.} \end{cases}$$

64. (a) The generating function is

$$(1 + x^2 + x^4 + \cdots)(1-x^3)^{-1}(1-x^4)^{-1}\cdots$$

(b) The generating function is

$$(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}\cdots$$

Since

$$1+x^2+x^4+\ldots=\frac{1}{1-x^2},$$

the two are equal.

65. If every term is only allowed to occur at most twice, then the resulting generating function is

$$\prod_{j=1}^{\infty}(1+x^j+x^{2j})=(1+x+x^2)(1+x^2+x^4)\cdots$$

Now we use the fact that $1+x^j+x^{2j}=\frac{1-x^{3j}}{1-x^j}$ to rewrite this generating function:

$$\begin{aligned}\prod_{j=1}^{\infty}(1+x^j+x^{2j}) &= \prod_{j=1}^{\infty}\frac{1-x^{3j}}{1-x^j}\\ &= \frac{(1-x^3)(1-x^6)(1-x^9)\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots}.\end{aligned}$$

All factors $1-x^j$ for which j is a multiple of 3 cancel, and we are left with

$$\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^4}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdots,$$

which is exactly the generating function for partitions into parts that are not divisible by 3.

66. We write the right hand side of the equation with a common denominator:

$$\begin{aligned}&\begin{bmatrix}n\\k\end{bmatrix}_q+q^{n+1-k}\begin{bmatrix}n\\k-1\end{bmatrix}_q\\
&=\frac{(1-q)(1-q^2)\ldots(1-q^n)}{(1-q)(1-q^2)\ldots(1-q^k)\cdot(1-q)(1-q^2)\ldots(1-q^{n-k})}\\
&\quad+q^{n+1-k}\frac{(1-q)(1-q^2)\ldots(1-q^n)}{(1-q)(1-q^2)\ldots(1-q^{k-1})\cdot(1-q)(1-q^2)\ldots(1-q^{n-k+1})}\\
&=\frac{(1-q)(1-q^2)\ldots(1-q^n)}{(1-q)(1-q^2)\ldots(1-q^k)\cdot(1-q)(1-q^2)\ldots(1-q^{n-k+1})}\\
&\quad\cdot\Big(1-q^{n-k+1}+q^{n+1-k}(1-q^k)\Big)\\
&=\frac{(1-q)(1-q^2)\ldots(1-q^n)}{(1-q)(1-q^2)\ldots(1-q^k)\cdot(1-q)(1-q^2)\ldots(1-q^{n-k+1})}\cdot(1-q^{n+1})\\
&=\frac{(1-q)(1-q^2)\ldots(1-q^n)(1-q^{n+1})}{(1-q)(1-q^2)\ldots(1-q^k)\cdot(1-q)(1-q^2)\ldots(1-q^{n-k+1})}\\
&=\begin{bmatrix}n+1\\k\end{bmatrix}_q,\end{aligned}$$

which proves the recursion.

67. We use induction on n: for $n = 1$, we have

$$\prod_{j=1}^{1}(1+q^j x) = 1 + qx = \sum_{k=0}^{1} \begin{bmatrix} 1 \\ k \end{bmatrix}_q x^k q^{k(k+1)/2}$$

since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q = 1$. Now assume that the identity holds for a certain n. We multiply by $1 + q^{n+1}x$ and make use of the previous problem to obtain

$$\begin{aligned}
\prod_{j=1}^{n+1}(1+q^j x) &= (1+q^{n+1}x) \cdot \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k q^{k(k+1)/2} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k q^{k(k+1)/2} + q^{n+1}x \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k q^{k(k+1)/2} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k q^{k(k+1)/2} + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{k+1} q^{n+1+k(k+1)/2} \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k q^{k(k+1)/2} + \sum_{k=1}^{n+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q x^k q^{n+1+k(k-1)/2} \\
&= \begin{bmatrix} n \\ 0 \end{bmatrix}_q + \sum_{k=1}^{n} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right) x^k q^{k(k+1)/2} + \begin{bmatrix} n \\ n \end{bmatrix}_q x^{n+1} q^{(n+1)(n+2)/2} \\
&= \begin{bmatrix} n+1 \\ 0 \end{bmatrix}_q + \sum_{k=1}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q x^k q^{k(k+1)/2} + \begin{bmatrix} n+1 \\ n+1 \end{bmatrix}_q x^{n+1} q^{(n+1)(n+2)/2} \\
&= \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q x^k q^{k(k+1)/2},
\end{aligned}$$

which completes the induction. Note that we may replace $\begin{bmatrix} n \\ 0 \end{bmatrix}_q$ and $\begin{bmatrix} n \\ n \end{bmatrix}_q$ by $\begin{bmatrix} n+1 \\ 0 \end{bmatrix}_q$ and $\begin{bmatrix} n+1 \\ n+1 \end{bmatrix}_q$, since they are all equal to 1.

68. As indicated in the figure, one can split any self-conjugate partition into “hooks” whose sizes must be odd by symmetry, and all distinct since the hooks have to fit inside each other. In this way, one can construct a partition into distinct odd parts from any self-conjugate partition, and vice versa, which proves the claim.