Combinatorics

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Chapter 1

Elementary enumeration principles

Sequences

Theorem 1.1 There are n^k different sequences of length k that can be formed from elements of a set X consisting of n elements (elements are allowed to occur several times in a sequence).

Proof: For every element of the sequence, we have exactly n choices. Therefore, there are

$$\underbrace{n \cdot n \cdot \ldots \cdot n}_{k \text{ times}} = n^k$$

different possibilities. □

Example 1.1 Given an "alphabet" of n letters, there are exactly n^k k-letter words. For instance, there are 8 three-digit words (not necessarily meaningful) that can be formed from the letters S and O:

SSS, SSO, SOS, OSS, SOO, OSO, OOS, OOO.

The number of 100-letter words over the alphabet A,C,G,T is 4^{100}, which is a 61-digit number; DNA strings as they occur in cells of living organisms are much longer, of course. . .

Permutations

Theorem 1.2 The number of possibilities to arrange n (distinguishable) objects in a row (so-called *permutations*) is

$$n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

Proof: There are obviously n choices for the first position, then $n-1$ remaining choices for the second position (as opposed to the previous theorem), $n-2$ for the third position, etc. Therefore, one obtains the stated formula. □

Example 1.2 There are 6 possibilities to arrange the letters A,E,T in a row:

AET, ATE, EAT, ETA, TAE, TEA.

REMARK: By definition, $n!$ satisfies the equation $n! = n \cdot (n-1)!$, which remains true if one defines $0! = 1$ (informally, there is exactly one possibility to arrange 0 objects, and that is to do nothing at all).

Example 1.3 In how many ways can eight rooks be placed on an 8×8-chessboard in such a way that no horizontal or vertical row contains two rooks?

In order to solve this problem, let us assign coordinates (a-h and 1-8 respectively) to the squares of the chessboard. A possible configuration would then be a3, b5, c1, d8, e6, f2, g4, h7 (for instance). Generally, there must be exactly one rook on each vertical row (a-h), and analogously one rook on each horizontal row (1-8). Each permutation of the numbers 1 to 8 corresponds to exactly one feasible configuration (in the above case, 3-5-1-8-6-2-4-7), and so there are exactly $8! = 40320$ possibilities.

Sequences without repetitions

Theorem 1.3 The number of sequences of length k whose elements are taken from a set X comprising n elements is

$$n^{\underline{k}} = n \cdot (n-1) \cdot (n-2) \cdot \ldots (n-k+1) = \frac{n!}{(n-k)!}.$$

Proof: The proof is essentially the same as for Theorem 1.2: for the first element, there are n possible choices, then $n-1$ for the second element, etc. For the last element, there are $n-k+1$ choices left. □

REMARK: The case of permutations is clearly a special case of Theorem 1.3, corresponding to $k = n$. $n^{\underline{k}}$ is called a *falling factorial* (read: "n to the k falling").

Choosing a subset

Theorem 1.4 The number of possibilities to choose a subset of k elements from a set of n elements (the order being irrelevant) is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof: Let x be the number of possibilities that we are looking for. Once the k elements have been chosen (for which there are x possible ways), one has $k!$ possibilities (by Theorem 1.2) to arrange them in a sequence. Therefore, $x \cdot k!$ is exactly the number of possible sequences of k distinct elements, for which we have the formula

$$x \cdot k! = \frac{n!}{(n-k)!}$$

by Theorem 1.3, so that x is obtained immediately. □

Remark: The difference between Theorem 1.3 and Theorem 1.4 lies in the fact that the order plays a role in the former, which is does not in the latter. Each subset corresponds to exactly $k!$ sequences: for instance, the subset $\{A, E, T\}$ of the set $\{A, B, \ldots, Z\}$ corresponds to the sequences

AET, ATE, EAT, ETA, TAE, TEA.

The formula for the binomial coefficient only makes sense if $0 \leq k \leq n$. This is also quite intuitive as no subset can comprise more elements than the original set. It is often useful to define $\binom{n}{k} = 0$ if either $k < 0$ or $k > n$. Later we will also give a more general definition of the binomial coefficients.

Example 1.4 The number of six-element subsets of $\{1, 2, \ldots, 49\}$ (Lotto) is

$$\binom{49}{6} = \frac{49!}{6! \cdot 43!} = 13983816.$$

Remark: An obvious property of $\binom{n}{k}$ is the identity

$$\binom{n}{k} = \binom{n}{n-k},$$

which follows immediately from the formula. However, it also has a combinatorial meaning: choosing k elements is equivalent to not choosing $n - k$ elements. The generalisation of this principle leads us to the so-called *multinomial coefficient.*

Dividing a set into groups

Theorem 1.5 The number of possibilities to divide a set X into groups $X_1, X_2, \ldots, X_r$ whose sizes are prescribed to be $k_1, k_2, \ldots, k_r$ respectively (where $k_1 + k_2 + \ldots + k_r = n$) is given by

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_r!}.$$

Proof: By induction on r; for $r = 1$, the statement is trivial. For $r \geq 2$, one has $\binom{n}{k_1}$ choices for the elements of X_1, and by the induction hypothesis,

$$\frac{(n-k_1)!}{k_2!k_3! \ldots k_r!}$$

possible ways to divide the remaining $n - k_1$ elements. Therefore, we have exactly

$$\binom{n}{k_1} \cdot \frac{(n-k_1)!}{k_2!k_3! \ldots k_r!} = \frac{n!}{k_1!(n-k_1)!} \cdot \frac{(n-k_1)!}{k_2!k_3! \ldots k_r!} = \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_r!}$$

possibilities, as claimed. □

Choosing a multiset

Theorem 1.6 The number of ways to choose k elements from a set of n elements, repetitions allowed, is $\binom{n+k-1}{k}$.

Proof: Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set. A choice is characterised by the number of times that each of the elements is selected. If l_i denotes the multiplicity of x_i in our collection, then the problem is equivalent to determining the number of solutions of

$$l_1 + l_2 + \ldots + l_n = k,$$

where $l_1, l_2, \ldots, l_n$ have to be non-negative integers. Equivalently, we can write $m_i = l_i + 1$ and ask for the number of positive integer solutions to the equation

$$m_1 + m_2 + \ldots + m_n = k + n. \tag{1.1}$$

Let us imagine $k + n$ dots in a row. Each solution to the equation (1.1) corresponds to a way of separating the dots by inserting $n - 1$ bars at certain places (Figure 1.1). Since there are $n + k - 1$ positions for the bars, one has

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

possible ways to place the bars. □

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Figure 1.1: Dots and bars.

Remark: A finite sequence $m_1, m_2, \ldots, m_k$ of positive integers summing to n (that is, $n = m_1 + m_2 + \ldots + m_k$) is called a *composition* of n; the above argument ("dots and bars") shows that every positive integer n has exactly $\binom{n-1}{k-1}$ compositions into k summands.

Chapter 2

A combinatorial view of binomial coefficients

The identity $\binom{n}{k} = \binom{n}{n-k}$ is one of many interesting properties of binomial coefficients, many of which can also be interpreted combinatorially. In this chapter, some of these properties are considered.

2.1 The recursion

Theorem 2.1 The binomial coefficients satisfy the recursive formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad (0 \leq k \leq n).$$

Proof: The formula can be verified easily by algebraic manipulations:

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{k \cdot (n-1)!}{k!(n-k)!} + \frac{(n-k) \cdot (n-1)!}{k!(n-k)!} \\ &= \frac{n \cdot (n-1)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.\end{aligned}$$

However, a (perhaps simpler) way is to argue as follows: let x be an element of an n-element set X. If one wants to choose a subset of k elements, one can first decide whether to include x or not. In the former case, there are $\binom{n-1}{k-1}$ possibilities to choose the remaining $k-1$ elements. In the latter, we have $\binom{n-1}{k}$ possible choices, which already completes the proof. □

Remark: Note that the equation remains correct if one defines $\binom{n}{k}$ to be 0 if either $k < 0$ or $n > k$. A classical way to illustrate the recursion for the binomial coefficients is *Pascal's triangle*: the n-th line contains the binomial coefficients $\binom{n}{k}$ $(0 \leq k \leq n)$.

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

It is apparent that each number is the sum of the two numbers above it. If one defines the binomial coefficient $\binom{\alpha}{k}$ for arbitrary real (or even complex) numbers α by

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!},$$

Theorem 2.1 remains correct, even without the direct combinatorial interpretation. The following lemma relates the case of negative α to the more familiar case that α is positive:

Lemma 2.2 For any real number α and any non-negative integer k, one has

$$\binom{-\alpha}{k} = (-1)^k \binom{\alpha+k-1}{k}.$$

Proof:

$$\begin{aligned}
\binom{-\alpha}{k} &= \frac{-\alpha(-\alpha-1)(-\alpha-2)\ldots(-\alpha-k+1)}{k!} \\
&= (-1)^k \cdot \frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1)}{k!} = (-1)^k \binom{\alpha+k-1}{k}.
\end{aligned}$$

2.2 The binomial theorem revisited

The binomial theorem is certainly the most important theorem that involves the binomial coefficients; it can be stated as follows:

Theorem 2.3 (Binomial Theorem) For any integer $n \geq 0$, one has

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof: The classical proof proceeds by induction. However, it can also be proven combinatorially: suppose that we expand the product

$$(x+y)^n = (x+y)(x+y)\ldots(x+y) = x \cdot x \ldots x + y \cdot x \ldots x + \ldots.$$

Each summand is obtained by choosing either x or y for each of the factors. Therefore, we end up with summands of the form $x^k y^{n-k}$, and each of these summands occurs exactly $\binom{n}{k}$ times (which is the number of ways to select k out of n factors from which an x is taken). This readily proves the theorem. $\square$

Remark: It should be noted that one usually defines $0^0 = 1$ in this context, so that the binomial theorem remains correct even if $y = 0$ (and/or $x = 0$).

The binomial theorem has a generalisation to the case that n is not necessarily a positive integer, which is known as the *binomial series*:

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k. \tag{2.1}$$

Note that this is simply the Taylor series of $(1+x)^\alpha$ at $x = 0$. If $\alpha = n \geq 0$ is an integer, then this formula reduces to the binomial theorem, since all terms in the infinite sum that correspond to $k > n$ are 0 by definition.

It is possible to deduce several interesting identities directly from the binomial theorem, such as the following:

Theorem 2.4 One has the following relations:

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$$

for any integer $n \geq 0$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n} = 0$$

for any integer $n > 0$,

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} = 2^{n-1}$$

for any integer $n > 0$.

Proof: The first equation is obtained by setting $x = y = 1$ in the binomial theorem, the second equation for $x = -1$ and $y = 1$. The third equation is obtained by adding (subtracting) the first two equations and noting that

$$1 + (-1)^k = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \quad \text{resp.} \quad 1 - (-1)^k = \begin{cases} 0 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}$$

holds. $\square$

Remark: Let us interpret these equations combinatorially: the sum in the first equation is exactly the number of subsets of an n-element set (since $\binom{n}{k}$ counts subsets of size k). Therefore, one immediately ends up with the following result:

Corollary 2.5 An n-element set has precisely 2^n distinct subsets.

Example 2.1 The set {A,E,T} has 8 subsets:

$$\emptyset = \{\},\ \{A\},\ \{E\},\ \{T\},\ \{A,E\},\ \{A,T\},\ \{E,T\},\ \{A,E,T\}.$$

Of course, this corollary can be obtained without depending on the binomial theorem: each element can either be selected as a member of the subset or not, so that one has exactly $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ possible options. In this way, a subset of an n-element set corresponds directly to a 0-1-sequence of length n.

On the other hand, it follows from Theorem 2.4 that any finite non-empty set has exactly as many subsets of even cardinality as of odd cardinality. This is somewhat obvious if n is even (note the symmetry in Pascal's triangle!), but maybe a little more surprising if n is odd. However, there is again a combinatorial proof: If we distinguish an element x of the set, then one has a *bijection* between subsets of even cardinality and subsets of odd cardinality: two subsets are associated to each other if their only difference is that one of them contains x while the other one does not. In this way, we obtain 2^{n-1} pairs, each of which contains exactly one subset of even cardinality and one subset of odd cardinality.

Instead of considering even or odd cardinalities, let us see what happens if we restrict the cardinality of subsets to be divisible by 3:

Example 2.2 We prove the following equation for all positive integers n:

$$\sum_{k=0}^{n} \binom{3n}{3k} = \frac{1}{3}\left(2^{3n} + 2 \cdot (-1)^n\right).$$

In order to achieve this, we would like to use the same trick as before. Instead of plugging in $y = \pm 1$, we consider the *roots of unity*

$$\zeta_{1,2} = \frac{-1 \pm i\sqrt{3}}{2} = e^{\pm 2\pi i/3}.$$

One has $\zeta_1^3 = \zeta_2^3 = 1$ as well as

$$\zeta_1 + \zeta_2 + 1 = 0, \quad \zeta_1^2 = \zeta_2 \quad \text{and } \zeta_2^2 = \zeta_1.$$

If we now take $x = 1$ and $y = 1$, $y = \zeta_1$ and $y = \zeta_2$ respectively in the binomial theorem and add the resulting equations, we obtain

$$\sum_{l=0}^{3n} \binom{3n}{l} \left(1 + \zeta_1^l + \zeta_2^l\right) = 2^{3n} + (1+\zeta_1)^{3n} + (1+\zeta_2)^{3n}.$$

Now if $l = 3k$ is divisible by 3, then

$$1 + \zeta_1^l + \zeta_2^l = 1 + 1^k + 1^k = 3,$$

if $l = 3k + 1$, then

$$1 + \zeta_1^l + \zeta_2^l = 1 + 1^k \zeta_1 + 1^k \zeta_2 = 1 + \zeta_1 + \zeta_2 = 0,$$

and if $l = 3k + 2$, then

$$1 + \zeta_1^l + \zeta_2^l = 1 + 1^k \zeta_2 + 1^k \zeta_1 = 1 + \zeta_2 + \zeta_1 = 0.$$

Therefore, we end up with

$$\sum_{k=0}^{n} 3 \binom{3n}{3k} = 2^{3n} + (1 + \zeta_1)^{3n} + (1 + \zeta_2)^{3n}.$$

$1 + \zeta_{1,2} = \frac{1 \pm \sqrt{3} i}{2}$ are sixth roots of unity: it is easy to check that $(1 + \zeta_{1,2})^3 = -1$ (and thus $(1 + \zeta_{1,2})^6 = 1$). So it finally follows that

$$\sum_{k=0}^{n} \binom{3n}{3k} = \frac{1}{3} \left(2^{3n} + 2 \cdot (-1)^n \right).$$

This shows that the proportion of subsets whose cardinality is a multiple of 3 is approximately $\frac{1}{3}$ (it tends to $\frac{1}{3}$ as $n \to \infty$). Let us now have a look at another important identity that involves the binomial coefficients.

2.3 The Vandermonde identity

Theorem 2.6 (Vandermonde identity) For integers $N, M, n \geq 0$, one has

$$\sum_{k=0}^{n} \binom{N}{k} \binom{M}{n-k} = \binom{N+M}{n}.$$

Proof: Sums of the form

$$\sum_{k=0}^{n} a_k b_{n-k}$$

can be related to products of polynomials: indeed, if two polynomials $A(x) = \sum_{k=0}^{N} a_k x^k$ and $B(x) = \sum_{l=0}^{M} b_l x^l$ are given, then the product of the two is

$$A(x) \cdot B(x) = \sum_{k=0}^{N} \sum_{l=0}^{M} a_k b_l x^{k+l}.$$

The coefficient of x^n is now obtained as the sum of those summands for which $k + l = n$, or in other words $l = n - k$. Therefore this coefficient is equal to

$$\sum_{k=0}^{n} a_k b_{n-k},$$

where a_k is taken to be 0 if the degree N of the polynomial A is less than k (and analogously $b_l = 0$, if $l > M$). Similar ideas will be used extensively in Chapter 5.

In this specific case, we apply the binomial theorem to the polynomial $(1+x)^{N+M}$ that can also be written as the product of $(1+x)^N$ and $(1+x)^M$:

$$\begin{aligned}(1+x)^{N+M} &= (1+x)^N \cdot (1+x)^M = \left(\sum_{k=0}^{N}\binom{N}{k}x^k\right)\left(\sum_{l=0}^{M}\binom{M}{l}x^l\right)\\ &= \sum_{k=0}^{N}\sum_{l=0}^{M}\binom{N}{k}\binom{M}{l}x^{k+l} = \sum_{n=0}^{N+M}\sum_{k=0}^{n}\binom{N}{k}\binom{M}{n-k}x^n.\end{aligned}$$

Comparing coefficients with

$$(1+x)^{N+M} = \sum_{n=0}^{N+M}\binom{N+M}{n}x^n,$$

we obtain the desired identity.

Once again it is possible to provide a proof by counting arguments as well: a set of $N+M$ elements is divided into two groups consisting of N and M elements respectively. Choosing n elements from the set is equivalent to choosing k elements from the first group and the remaining $n-k$ elements from the second group, where k can be any integer between 0 and n. The number of possible choices for fixed k is clearly $\binom{N}{k}\binom{M}{n-k}$. Summing over all k, we obtain the total number of choices, which is $\binom{N+M}{n}$. □

The following special case of the Vandermonde identity is quite remarkable:

Corollary 2.7

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

In a similar vein, we can prove the following:

Theorem 2.8 For a non-negative integer n, one has

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases}(-1)^{n/2}\binom{n}{n/2} & n \text{ even},\\ 0 & n \text{ odd}.\end{cases}$$

Proof: We use the same type of argument as before, comparing coefficients in the equation

$$\left(\sum_{k=0}^{n}\binom{n}{n-k}x^{n-k}\right)\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^k\right) = (1+x)^n(1-x)^n = (1-x^2)^n = \sum_{k=0}^{n}\binom{n}{k}(-x^2)^k.$$

The coefficient of x^n on the right hand side is $(-1)^{n/2}\binom{n}{n/2}$ if n is even, and otherwise 0. The coefficient on the left hand side obtained by expanding the product is

$$\sum_{k=0}^{n} \binom{n}{n-k} \cdot (-1)^k \binom{n}{k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 .$$

□

Chapter 3

The principle of inclusion and exclusion

3.1 A simple example

A frequently occurring problem is to determine the size of the union or intersection of a number of sets, as in the following example:

Example 3.1 All second-year science students may choose either mathematics, or physics, or both. The mathematics course is attended by 50 students, the physics course by 30 students. 15 students attend both courses. How many second-year science students are there?

Let M be the set of students taking mathematics and P the set of all students who take physics. By our conditions, the set of all students is the union $S = M \cup P$. If we add the sizes of the two sets, all students who attend both courses are counted twice. Therefore, we have to subtract the size of $M \cap P$, which yields the formula

$$|S| = |M \cup P| = |M| + |P| - |M \cap P|,$$

where $|X|$ is the number of elements of a set X. Plugging in, we find that there are $50 + 30 - 15 = 65$ students.

This principle can be generalised to unions (or intersections) of an arbitrary number of sets. Before we discuss the general formula, let us extend Example 3.1:

Example 3.2 Third-year science students also have the opportunity to attend chemistry, but every student has to take at least one of the three courses. Altogether, there are 40 students in the mathematics class, 25 who attend physics, and 20 who attend chemistry. Furthermore, we know that 10 students do both mathematics and physics, 8 both mathematics and chemistry, and 7 physics and chemistry. There are two particularly keen students who attend all three courses. How many third-year science students are there?

Let M, P, C denote the respective sets of students attending mathematics, physics and chemistry. Once again, we are looking for the size of the union $M \cup P \cup C$. To this end, we first add $|M|$, $|P|$ and $|C|$. Students taking both mathematics and physics are double-counted, so we have to subtract $|M \cap P|$. The same applies to $|M \cap C|$ and $|P \cap C|$. However, those two students that attend all the three courses are added three times and subtracted three times as well. Hence we have to add $|M \cap P \cap C|$ to make up for this. Formally, we have

$$|M \cup P \cup C| = |M| + |P| + |C| - |M \cap P| - |M \cap C| - |P \cap C| + |M \cap P \cap C|.$$

This means that there are $40 + 25 + 20 - 10 - 8 - 7 + 2 = 62$ third-year students.

3.2 The general formula

The simple example presented in the previous section can be generalized to an arbitrary number of sets as follows:

Theorem 3.1 (Inclusion-exclusion) Let $X_1, X_2, \ldots, X_k$ be arbitrary finite sets. For a set $I \subseteq \{1, 2, \ldots, k\}$, we denote the intersection of all sets X_i with $i \in I$ by $\bigcap_{i \in I} X_i$. Then the following formula holds:

$$\left| \bigcup_{i=1}^{k} X_i \right| = \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ I \neq \emptyset}} (-1)^{|I|+1} \left| \bigcap_{i \in I} X_i \right|, \tag{3.1}$$

where the sum is taken over all non-empty subsets of $\{1, 2, \ldots, k\}$.

REMARK: The cases $k = 2$ and $k = 3$ correspond to our two examples; for instance, the formula

$$|X_1 \cup X_2 \cup X_3| = |X_1| + |X_2| + |X_3| - |X_1 \cap X_2| - |X_1 \cap X_3| - |X_2 \cap X_3| + |X_1 \cap X_2 \cap X_3|$$

is obtained for $k = 3$.

Proof: By induction on k. For $k = 1$, the formula reduces to the trivial identity $|X_1| = |X_1|$. The induction step from k to $k + 1$ makes use of the special case $k = 2$ that was discussed in our first example:

$$\begin{aligned} \left| \bigcup_{i=1}^{k+1} M_i \right| &= \left| \left(\bigcup_{i=1}^{k} M_i \right) \cup M_{k+1} \right| \\ &= \left| \bigcup_{i=1}^{k} M_i \right| + |M_{k+1}| - \left| \left(\bigcup_{i=1}^{k} M_i \right) \cap M_{k+1} \right| \\ &= \left| \bigcup_{i=1}^{k} M_i \right| + |M_{k+1}| - \left| \bigcup_{i=1}^{k} (M_i \cap M_{k+1}) \right| \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1}\left|\bigcap_{i\in I} M_i\right| + |M_{k+1}| - \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1}\left|\bigcap_{i\in I}(M_i\cap M_{k+1})\right| \\
&= \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1}\left|\bigcap_{i\in I} M_i\right| + |M_{k+1}| - \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1}\left|\bigcap_{i\in I\cup\{k+1\}} M_i\right| \\
&= \sum_{\substack{I\subseteq\{1,\ldots,k+1\}\\ k+1\notin I, I\neq\emptyset}} (-1)^{|I|+1}\left|\bigcap_{i\in I} M_i\right| + \sum_{\substack{I\subseteq\{1,\ldots,k+1\}\\ k+1\in I}} (-1)^{|I|+1}\left|\bigcap_{i\in I} M_i\right| \\
&= \sum_{\substack{I\subseteq\{1,\ldots,k+1\}\\ I\neq\emptyset}} (-1)^{|I|+1}\left|\bigcap_{i\in I} M_i\right| .
\end{aligned}
$$

This completes the induction. □

REMARK: The same formula holds true if union and intersection are interchanged (the proof being completely analogous):

$$\left|\bigcap_{i=1}^{k} X_i\right| = \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1}\left|\bigcup_{i\in I} X_i\right| . \tag{3.2}$$

REMARK: A common interpretation of the inclusion-exclusion principle involves probabilities: suppose that $X_1, X_2, \ldots, X_k$ are subsets of a set X of outcomes. Then $P(X_i) = \frac{|X_i|}{|X|}$ is the probability that one of the events in X_i occurs. Dividing formulas (3.1) and (3.2) by $|X|$, we obtain

$$P\left(\bigcup_{i=1}^{k} X_i\right) = \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1} P\left(\bigcap_{i\in I} X_i\right)$$

and

$$P\left(\bigcap_{i=1}^{k} X_i\right) = \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1} P\left(\bigcup_{i\in I} X_i\right) .$$

Note that $P\left(\bigcup_{i=1}^{k} X_i\right)$ is the probability that at least one of the events associated to $X_1, X_2, \ldots$ occurs, while $P\left(\bigcap_{i=1}^{k} X_i\right)$ is the probability that all of them occur.

REMARK: If we take $X_1 = X_2 = \ldots = X_n = \{x\}$ in (3.1) or (3.2), then all intersections and unions have size 1, so that we obtain another proof of the identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n} = 0,$$

see Theorem 2.4.

Quite frequently, the sets X_i are subsets of some base set, and one is interested in the number of elements that are contained in none of the X_i or not in all of the X_i. In the former case, one has to determine the cardinality of $X \setminus (X_1 \cup X_2 \ldots \cup X_k)$, which is equal to

$$|X| - \sum_{I \subseteq \{1,\ldots,k\}, I \neq \emptyset} (-1)^{|I|+1} \left| \bigcap_{i \in I} X_i \right|. \tag{3.3}$$

The second problem amounts to determining the cardinality of $X \setminus (X_1 \cap X_2 \ldots \cap X_k)$, which is given by

$$|X| - \sum_{I \subseteq \{1,\ldots,k\}, I \neq \emptyset} (-1)^{|I|+1} \left| \bigcup_{i \in I} X_i \right|. \tag{3.4}$$

Naturally, the sizes of the sets $\bigcap_{i \in I} X_i$ are not always explicitly given as in our first two examples. However, they are often easier to obtain than those that one is actually interested in. In the following section, some applications are discussed.

3.3 Applications

Example 3.3 How many n-digit numbers are there that do not contain the digits $0, 1, 2$, but have to contain the three digits $3, 4, 5$?

By the stated conditions, the digits have to be taken from the set $\{3, 4, \ldots, 9\}$; let $M_n(D)$ be the number of all n-digit numbers whose digits are taken from the set D. Then the set whose size we want to determine consists of all numbers in $M_n(\{3, 4, \ldots, 9\})$ that are not contained in any of $M_n(\{4, 5, \ldots, 9\})$, $M_n(\{3, 5, \ldots, 9\})$ or $M_n(\{3, 4, 6, \ldots, 9\})$ (none of the digits $3, 4, 5$ may be missing). Note also that

$$M_n(D_1) \cap M_n(D_2) = M_n(D_1 \cap D_2).$$

By Theorem 1.1, we have $|M_n(D)| = |D|^n$ (if $0 \notin D$; otherwise, one would have to exclude leading zeros). Therefore, the inclusion-exclusion principle yields

$$\begin{aligned}
&|M_n(\{3,4,\ldots,9\}) \setminus (M_n(\{4,5,\ldots,9\}) \cup M_n(\{3,5,\ldots,9\}) \cup M_n(\{3,4,6,\ldots,9\}))| \\
&\quad = |M_n(\{3,4,\ldots,9\})| - |M_n(\{4,5,\ldots,9\})| - |M_n(\{3,5,\ldots,9\})| - |M_n(\{3,4,6,\ldots,9\})| \\
&\qquad + |M_n(\{5,6,\ldots,9\})| + |M_n(\{4,6,\ldots,9\})| + |M_n(\{3,6,\ldots,9\})| - |M_n(\{6,7,\ldots,9\})| \\
&\quad = 7^n - 3 \cdot 6^n + 3 \cdot 5^n - 4^n.
\end{aligned}$$

Euler's totient function

Example 3.4 A classical application of the inclusion-exclusion principle stems from number theory: *Euler's totient function* $\phi(n)$ is defined as the number of integers x such that

$0 \le x < n$ and x and n do not have a common divisor (other than 1). If $p_1, p_2, \dots, p_k$ are the prime factors of n, then this occurs precisely if x is not divisible by any of $p_1, p_2, \dots, p_k$. Note now that the number of integers $0 \le x < n$ that are divisible by a specific prime factor p_i is exactly n/p_i. Likewise, if I is any subset of $\{1, 2, \dots, k\}$, then the number of integers $0 \le x < n$ that are divisible by all p_i with $i \in I$ (and thus by their product $\prod_{i \in I} p_i$) is exactly $n / \prod_{i \in I} p_i$. Therefore, formula (3.3) yields

$$\begin{aligned}\phi(n) &= n - \sum_{I \subseteq \{1,\dots,k\}, I \neq \emptyset} (-1)^{|I|+1} \frac{n}{\prod_{i \in I} p_i} \\ &= n \left(1 + \sum_{I \subseteq \{1,\dots,k\}, I \neq \emptyset} (-1)^{|I|} \prod_{i \in I} \frac{1}{p_i} \right) \\ &= n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i} \right)\end{aligned}$$

The last step follows from the fact that the term $\prod_{i \in I} \frac{1}{p_i}$ occurs in the expansion of

$$\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right)$$

with a coefficient of $(-1)^{|I|}$. Generally, the product

$$\prod_{i=1}^{k} (1 - x_i) = (1 - x_1)(1 - x_2) \dots (1 - x_k)$$

expands to

$$\sum_{I \subseteq \{1,\dots,k\}} (-1)^{|I|} \prod_{i \in I} x_i,$$

where the product over the empty set is taken to be 1, while the product

$$\prod_{i=1}^{k} (1 + x_i) = (1 + x_1)(1 + x_2) \dots (1 + x_k)$$

is simply

$$\sum_{I \subseteq \{1,\dots,k\}} \prod_{i \in I} x_i.$$

Derangements

Example 3.5 A *derangement* is a permutation of $\{1, 2, \dots, n\}$ without fixed points; that is, the number i does not occur in position i ($1 \le i \le n$). For instance, 34251 is a derangement, but 32451 is not (since 2 occurs in second position). The number of derangements can be determined by the inclusion-exclusion principle: for a subset I of $\{1, 2, \ldots, n\}$, let $p_n(I)$ be the number of permutations for which all elements of I (and possible others) are fixed points, i.e., they occur in their respective positions. Then, we are looking for the number of permutations that belong to none of $p_n(\{1\}), p_n(\{2\}), \ldots$. The size of $p_n(I)$ is clearly $(n-|I|)!$ (the $n-|I|$ remaining elements can be arranged in any order), so that (3.3) yields the following formula for the number of derangements:

$$n! - \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} p_n(I).$$

There are exactly $\binom{n}{k}$ subsets I of size $|I| = k$, and so this reduces to

$$n! + \sum_{k=1}^{n} (-1)^k \binom{n}{k} (n-k)! = n! + \sum_{k=1}^{n} (-1)^k \frac{n!}{k!} = n! \cdot \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

Therefore, the probability that a randomly selected permutation of $\{1, 2, \ldots, n\}$ is a derangement is

$$\sum_{k=0}^{n} (-1)^k \frac{1}{k!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \ldots \pm \frac{1}{n!}.$$

As $n \to \infty$, this value approaches the infinite sum

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} = \frac{1}{e} = 0.367879.$$

Chapter 4

Counting by recursion

Many counting problems can be solved by setting up recursions and solving them. We have already encountered the recursive relation that is satisfied by the binomial coefficients. In the following, we discuss three examples that exhibit the basic idea. The following chapter provides a general method to solve recursions as they occur in the study of enumeration problems.

Example 4.1 We draw n lines in the plane in such a way that there are no parallel lines and no intersections of three or more lines. These lines divide the plane into several regions; how many such regions are there?

Let us first discuss some trivial cases: if there is no line ($n = 0$), then there is precisely one region; for $n = 1$, there are two regions; for $n = 2$ and $n = 3$ we obtain four and seven regions, respectively. One might conjecture now that the number of regions increases by $n+1$ if we add a line to n existing lines. Indeed, this is the case, as the following argument shows.

If we add an additional line to n lines in the plane, then we obtain exactly n new points of intersection that divide the new line into $n+1$ segments. Each of these segments divides one of the old regions into two new regions, while all other regions remain the same. Therefore, if a_n denotes the total number of regions, we have the recursion

$$a_{n+1} = a_n + (n + 1).$$

An explicit form can be deduced immediately:

$$\begin{aligned} a_n &= a_{n-1} + n = a_{n-2} + n + (n-1) = a_{n-2} + n + (n-1) + (n-2) = \dots \\ &= a_0 + n + (n-1) + (n-2) + \dots + 1 = a_0 + \frac{n(n+1)}{2} = \frac{n^2 + n + 2}{2} \end{aligned}$$

by the well-known formula for the sum $1 + 2 + \dots + n$, which solves the problem.

Example 4.2 Tom gets an allowance of R100 every month, which he spends entirely on ice cream (R5), chocolate (R10) or cookies (R10). Every day, he buys exactly one of these until he runs out of money. In how many possible ways can Tom spend his money? His older brother Phil gets an allowance of R150, which he also spends on ice cream, chocolate and cookies only. How many possible ways does he have?

We study a more general problem: if the allowance is $5n$ (n any positive integer; note that the prices are all divisible by R5), how many ways are there? Let this number be denoted by a_n; then it is easy to see that $a_1 = 1$ and $a_2 = 3$. Next we deduce a recursion for a_n: the first thing that Tom buys can be either ice cream (so that he is left with $5(n-1)$ and has a_{n-1} possibilities for the rest) or chocolate or cookies (in each of these cases, the remaining amount is $5(n-2)$, so that he is left with a_{n-2} possibilities). This shows that

$$a_n = a_{n-1} + 2a_{n-2}$$

holds. Note that this even remains true if we set $a_0 = 1$ (without money, there is only one option, which is to buy nothing at all). One obtains the following sequence:

n	1	2	3	4	5	6	7	8	9	10
a_n	1	3	5	11	21	43	85	171	341	683

An explicit formula is given by

$$a_n = \tfrac{1}{3}\left(2^{n+1} + (-1)^n\right),$$

which can be proven by means of induction. A method to determine such an explicit formula from a recursion will be discussed in the following chapter. Plugging in $n = 20$ and $n = 30$ respectively, we find that Tom has 699051 options, while Phil has 715827883 different ways to spend his money.

Our final example leads to the famous *Fibonacci numbers*:

Example 4.3 All the houses on one side of a certain street are to be painted either yellow or red. In how many ways can this be done if there are n houses and there may not be two red houses next to each other?

Let a_n be the number that we want to determine. We distinguish two cases:

- If the first house is painted yellow, then the remaining houses can be painted in any of the feasible a_{n-1} ways, the first house can be neglected.
- If the first house is painted red, then the second house must be painted yellow. Using the same argument as before, we see that there are a_{n-2} possibilities for the remaining houses.

Hence we have the recursion

$$a_n = a_{n-1} + a_{n-2}$$

with initial values $a_1 = 2$ and $a_2 = 3$. This yields the sequence $2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$ of so-called *Fibonacci numbers*. They are usually defined by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$; it is easy to see that $a_n = f_{n+2}$ in our example. There is also an explicit formula for the Fibonacci numbers, see the following chapter.

Chapter 5

Generating functions

5.1 Solving recursions

Generating functions provide a method to solve recursions, but they are actually much more versatile, as we will see later. We associate a power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

to any sequence $a_0, a_1, \ldots$ of real (or complex) numbers; in the context of combinatorics, the coefficients a_n are typically integers. The n-th coefficient of such a power series $A(x)$ is also written as $a_n = [x^n]A(x)$. For now, we do not care too much about convergence and mostly regard generating functions as formal objects. A typical example of a generating function is the geometric series

$$\frac{1}{1-qx} = \sum_{n=0}^{\infty} q^n x^n,$$

which is the generating function of a geometric sequence $1, q, q^2, q^3, \ldots$. In particular, the generating function of $1, 1, 1, \ldots$ is $\frac{1}{1-x}$. As a first example of the use of generating functions, let us discuss the recursion

$$a_n = a_{n-1} + 2a_{n-2}$$

with initial vales $a_0 = a_1 = 1$ that we encountered in the previous chapter. If $A(x)$ denotes the associated generating function, then we obtain

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} (a_{n-1} + 2a_{n-2}) x^n \\ &= 1 + x + \sum_{n=2}^{\infty} a_{n-1} x^n + 2 \sum_{n=2}^{\infty} a_{n-2} x^n \end{aligned}$$

$$= 1 + x + \sum_{n=1}^{\infty} a_n x^{n+1} + 2 \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$= 1 + x + x(A(x) - a_0) + 2x^2 A(x).$$

Solving for $A(x)$, we find

$$A(x) = \frac{1}{1 - x - 2x^2},$$

which can also be written as

$$A(x) = \frac{2/3}{1 - 2x} + \frac{1/3}{1 + x},$$

making use of partial fractions. We expand the two summands into geometric series to obtain

$$A(x) = \frac{2}{3} \sum_{n=0}^{\infty} 2^n x^n + \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} \frac{2^{n+1} + (-1)^n}{3} \cdot x^n,$$

so that we can simply read off the coefficients: $a_n = [x^n]A(x) = \frac{2^{n+1}+(-1)^n}{3}$, as claimed. In the very same style, one finds *Binet's formula* for the Fibonacci numbers, whose generating function is $\frac{x}{1-x-x^2}$:

$$f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right) \tag{5.1}$$

for any $n \geq 0$. The method can actually be generalised to the entire class of *linear recursions*, whose general form is

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_r a_{n-r}.$$

To see how such a recursion can be solved for arbitrary coefficients $c_1, c_2, \ldots, c_r$, we need the following lemma:

Lemma 5.1 The power series associated to the function $\frac{1}{(1-qx)^k}$ is given by

$$\frac{1}{(1 - qx)^k} = \sum_{n=0}^{\infty} \binom{n + k - 1}{k - 1} q^n x^n.$$

Proof: Combine the binomial series (2.1) with Lemma 2.2 to obtain

$$\frac{1}{(1 - qx)^k} = \sum_{n=0}^{\infty} \binom{-k}{n} (-qx)^n = \sum_{n=0}^{\infty} (-1)^n \binom{n + k - 1}{n} (-qx)^n = \sum_{n=0}^{\infty} \binom{n + k - 1}{k - 1} q^n x^n.$$

Note that for fixed k, $\binom{n+k-1}{k}$ is a polynomial (in n) of degree $k - 1$:

$$\binom{n + k - 1}{k - 1} = \frac{(n + k - 1)(n + k - 2) \ldots (n + 1)}{(k - 1)!},$$

for instance

$$\binom{n+1}{1} = n+1, \qquad \binom{n+2}{2} = \frac{(n+2)(n+1)}{2} = \frac{n^2}{2} + \frac{3n}{2} + 1, \qquad \ldots$$

Now we are ready to prove the following theorem:

Theorem 5.2 The general solution of a linear recursion of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_r a_{n-r}$$

is given by

$$a_n = \sum_{i=1}^{s} P_i(n) q_i^n,$$

where $q_1, q_2, \ldots, q_s$ are the (possibly complex) solutions of the *characteristic equation* $q^r = c_1 q^{r-1} + c_2 q^{r-2} + \ldots + c_r$ and $P_1(n)$, $P_2(n)$, $\ldots$, $P_s(n)$ are polynomials. The degree of P_i is strictly less than the multiplicity of q_i as a solution of the characteristic equation (that is, the number of times the factor $q - q_i$ occurs in the factorisation of the polynomial $q^r - c_1 q^{r-1} - c_2 q^{r-2} - \ldots - c_r$).

Proof: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $a_0, a_1, \ldots$. Then we have

$$\begin{aligned}
A(x) &= \sum_{n=0}^{r-1} a_n x^n + \sum_{n=r}^{\infty} a_n x^n \\
&= \sum_{n=0}^{r-1} a_n x^n + \sum_{n=r}^{\infty} (c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_r a_{n-r}) x^n \\
&= \sum_{n=0}^{r-1} a_n x^n + \sum_{j=1}^{r} c_j \sum_{n=r}^{\infty} a_{n-j} x^n \\
&= \sum_{n=0}^{r-1} a_n x^n + \sum_{j=1}^{r} c_j \sum_{n=r-j}^{\infty} a_n x^{n+j} \\
&= \sum_{n=0}^{r-1} a_n x^n + \sum_{j=1}^{r} c_j \left(\sum_{n=0}^{\infty} a_n x^{n+j} - \sum_{n=0}^{r-j-1} a_n x^{n+j} \right) \\
&= \sum_{n=0}^{r-1} a_n x^n + \sum_{j=1}^{r} c_j x^j A(x) - \sum_{j=1}^{r} c_j \sum_{n=0}^{r-j-1} a_n x^{n+j}.
\end{aligned}$$

Now write

$$N(x) = \sum_{n=0}^{r-1} a_n x^n - \sum_{j=1}^{r} c_j \sum_{n=0}^{r-j-1} a_n x^{n+j}$$

and

$$D(x) = 1 - \sum_{j=1}^{r} c_j x^j.$$

Both $N(x)$ and $D(x)$ are polynomials; solving the equation for $A(x)$ gives $A(x) = \frac{N(x)}{D(x)}$, so that $A(x)$ is invariably a rational function. Now factor the denominator:

$$D(x) = (1 - q_1 x)^{\ell_1}(1 - q_2 x)^{\ell_2} \dots (1 - q_s x)^{\ell_s}$$

Note that a factor $(1 - q_i x)^{\ell_i}$ in the factorisation occurs if and only if $D(\frac{1}{q_i}) = 0$, which is equivalent to

$$0 = 1 - \sum_{j=1}^{r} c_j q_i^{-j} = q^{-r}\left(q^r - \sum_{j=1}^{r} c_j q_i^{r-j}\right),$$

i.e., q_i is a solution of the characteristic equation. The multiplicity of q_i as a solution is precisely the exponent ℓ_i. Now expand the generating function $A(x)$ into partial fractions:

$$\sum_{n=0}^{\infty} a_n x^n = A(x) = \sum_{i=1}^{s} \sum_{j=1}^{\ell_i} \frac{K_{ij}}{(1 - q_i x)^j}.$$

We can apply Lemma 5.1 to each of the summands to obtain

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{i=1}^{s} \sum_{j=1}^{\ell_i} K_{ij} \sum_{n=0}^{\infty} \binom{n+j-1}{j-1} q_i^n x^n,$$

and comparing coefficients yields

$$a_n = \sum_{i=1}^{s} \left(\sum_{j=1}^{\ell_i} K_{ij} \binom{n+j-1}{j-1} \right) q_i^n = \sum_{i=1}^{s} P_i(n) q_i^n$$

for certain polynomials P_i whose degree is less than ℓ_i (the highest degree that occurs is that of $\binom{n+\ell_i-1}{\ell_i-1}$, which is $\ell_i - 1$). This proves the theorem. □

Knowing the general form of a solution, one can solve a recursion by means of the method of *undetermined coefficients* without translating it to the world of generating functions; this is exhibited in the following example:

Example 5.1 Suppose we want to determine an explicit formula for the sequence that is defined by

$$a_n = 5a_{n-1} - 3a_{n-2} - 9a_{n-3}$$

with initial values $a_0 = -1$, $a_1 = 1$ and $a_2 = 5$.

The characteristic equation is $q^3 - 5q^2 + 3q + 9 = (q-3)^2(q+1) = 0$, so that $q_1 = 3$ and $q_2 = -1$, the multiplicities being 2 and 1 respectively. Therefore, Theorem 5.2 shows that the solution must have the form

$$a_n = (An + B)3^n + C(-1)^n.$$

Plugging in $n = 0, 1, 2$ yields the system of equations

$$\begin{array}{rcrcrcr} & & B & + & C & = & -1, \\ 3A & + & 3B & - & C & = & 1, \\ 18A & + & 9B & + & C & = & 5, \end{array}$$

which leads to the solution $A = \frac{1}{2}$, $B = -\frac{3}{8}$, $C = -\frac{5}{8}$, so that finally

$$a_n = \frac{4A - 3}{8} \cdot 3^n - \frac{5(-1)^n}{8}.$$

Theorem 5.2 actually only treats the *homogeneous* case; the method can be extended to *non-homogeneous* recursions of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_r a_{n-r} + b_n,$$

the only difference being the non-homogeneous term b_n. If $B(x) = \sum_{n=0}^{\infty} b_n x^n$ is the generating function of b_n, then one obtains

$$A(x) = \frac{N(x) + B(x)}{D(x)}$$

as in the proof of Theorem 5.2. If the non-homogenous term is of the form $Q(n)q^n$ for a polynomial Q, then one can deduce an explicit formula for a_n once again (the proof being similar to that of Theorem 5.2):

Theorem 5.3 Suppose that $b_n = Q(n)q^n$, where Q is a polynomial of degree d. If q is not a solution of the characteristic equation, then the solution of the linear recursion

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_r a_{n-r} + b_n$$

is of the form

$$a_n = \sum_{i=1}^{s} P_i(n) q_i^n + P^*(n) q^n,$$

where $q_1, q_2, \ldots, q_s$ are the (possibly complex) solutions of the characteristic equation $q^r = c_1 q^{r-1} + c_2 q^{r-2} + \ldots + c_r$ and $P_1(n)$, $P_2(n)$, $\ldots$, $P_s(n)$, $P^*(n)$ are polynomials. The degree of P_i is strictly less than the multiplicity of q_i as a solution of the characteristic equation, and the degree of $P^*(n)$ is d.

If, on the other hand, q is a solution of the characteristic equation (without loss of generality, $q = q_1$), then the solution has the form

$$a_n = \sum_{i=1}^{s} P_i(n) q_i^n,$$

$q_1, q_2, \ldots, q_s$ and $P_1(n)$, $P_2(n)$, $\ldots$, $P_s(n)$ as before, except for the degree of P_1, which is equal to the multiplicity of q_1 plus the degree of Q.

A sum of non-homogeneous terms of this form gives rise to a sum in the solution; let us discuss a modified version of Example 5.1 to exhibit the method:

Example 5.2 Suppose we want to determine an explicit formula for the sequence that is defined by

$$a_n = 5a_{n-1} - 3a_{n-2} - 9a_{n-3} - 4 \cdot 3^n + 4n + 4$$

with initial values $a_0 = -1$, $a_1 = 1$ and $a_2 = 5$.

The summand $4n + 4$ can be interpreted as $(4n + 4) \cdot 1^n$ and gives rise to a linear term $An+B$ in the solution; since 3 is a double solution of the characteristic equation and $-4 \cdot 3^n$ occurs in the non-homogeneous term, we must have a summand $(Cn^2 + Dn + E) \cdot 3^n$ in the solution; finally, -1 is a solution of the characteristic equation, so that we end up with

$$a_n = An + B + (Cn^2 + Dn + E) \cdot 3^n + F(-1)^n.$$

Plugging into the recursion yields

$$\begin{aligned} &An + B + (Cn^2 + Dn + E) \cdot 3^n + F(-1)^n \\ &\qquad = 5\left(A(n-1) + B + (C(n-1)^2 + D(n-1) + E) \cdot 3^{n-1} + F(-1)^{n-1}\right) \\ &\qquad\quad - 3\left(A(n-2) + B + (C(n-2)^2 + D(n-2) + E) \cdot 3^{n-2} + F(-1)^{n-2}\right) \\ &\qquad\quad - 9\left(A(n-3) + B + (C(n-3)^2 + D(n-3) + E) \cdot 3^{n-3} + F(-1)^{n-3}\right) \\ &\qquad\quad - 4 \cdot 3^n + 4n + 4 \end{aligned}$$

and after collecting terms

$$\left(\frac{8}{3}C + 4\right) \cdot 3^n + (8A - 4)n + (-28A + 8B - 4) = 0.$$

Therefore, we have $A = \frac{1}{2}$, $B = \frac{9}{4}$ and $C = -\frac{3}{2}$. Finally, the initial values can be used to determine D, E, F, and the solution is found to be

$$a_n = \left(\frac{n}{2} + \frac{9}{4}\right) + \left(-\frac{3n^2}{2} + 5n - \frac{31}{8}\right) \cdot 3^n + \frac{5}{8}(-1)^n.$$

REMARK: The parts that belong to a solution of the homogeneous equation can be left out in the first step (in the example above, this means that one simply plugs in $An+B+Cn^2 3^n$ into the recursion formula, since the other parts have to cancel anyway). This principle can be stated as follows: the general solution of a non-homogeneous recursion is the sum of a particular solution and the solution to the homogeneous equation (compare this to linear differential equations!).

5.2 General rules for generating functions

Before we turn to more advanced applications of generating functions, let us describe the effect that certain operations on sequences have on the generating function level.

Theorem 5.4 If $\{a_n\}$ and $\{b_n\}$ $(n \geq 0)$ are sequences and $A(x)$ and $B(x)$ their associated generating functions, then

1. the sequence $c_n = a_n + b_n$ has generating function $C(x) = A(x) + B(x)$,
2. the sequence $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ has generating function $C(x) = A(x) \cdot B(x)$,
3. the sequence $c_n = \alpha a_n$ has generating function $C(x) = \alpha A(x)$ (α any constant),
4. the sequence
$$c_n = \begin{cases} a_{n-m} & n \geq m, \\ 0 & \text{otherwise,} \end{cases}$$
has generating function $C(x) = x^m A(x)$,
5. the sequence $c_n = a_{n+m}$ has generating function $C(x) = \dfrac{A(x) - \sum_{n=0}^{m-1} a_n x^n}{x^m}$,
6. the sequence $c_n = na_n$ has generating function $C(x) = xA'(x)$,
7. the sequence
$$c_n = \begin{cases} \frac{a_n}{n} & n > 0, \\ 0 & \text{otherwise,} \end{cases}$$
has generating function $C(x) = \displaystyle\int_0^x \frac{A(t) - a_0}{t}\, dt$,
8. the sequence $c_n = \sum_{k=0}^{n} a_k$ has generating function $C(x) = \dfrac{A(x)}{1-x}$.

Proof: Each of the statements can be obtained by simple arithmetic:

1.
$$C(x) = \sum_{n=0}^{\infty}(a_n + b_n)x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = A(x) + B(x).$$

2.
$$\begin{aligned} C(x) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} x^n = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} a_k b_{n-k} x^n = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} a_k x^k b_{n-k} x^{n-k} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_k x^k b_l x^l = \left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{l=0}^{\infty} b_l x^l\right) = A(x)\cdot B(x). \end{aligned}$$

3.
$$C(x) = \sum_{n=0}^{\infty} \alpha a_n x^n = \alpha \sum_{n=0}^{\infty} a_n x^n = \alpha A(x).$$

4.
$$C(x) = \sum_{n=m}^{\infty} a_{n-m} x^n = \sum_{n=0}^{\infty} a_n x^{n+m} = x^m \sum_{n=0}^{\infty} a_n x^n = x^m A(x).$$

5.
$$C(x) = \sum_{n=0}^{\infty} a_{n+m} x^n = \sum_{n=m}^{\infty} a_n x^{n-m} = x^{-m} \sum_{n=m}^{\infty} a_n x^n = \frac{A(x) - \sum_{n=0}^{m-1} a_n x^n}{x^m}.$$

6.
$$C(x) = \sum_{n=0}^{\infty} n a_n x^n = x \sum_{n=0}^{\infty} n a_n x^{n-1} = x \sum_{n=0}^{\infty} a_n \frac{d}{dx} x^n = x \frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = x A'(x).$$

7.
$$C(x) = \sum_{n=1}^{\infty} \frac{a_n}{n} x^n = \sum_{n=1}^{\infty} a_n \int_0^x t^{n-1}\,dt = \int_0^x \sum_{n=1}^{\infty} a_n t^{n-1}\,dt = \int_0^x \frac{A(x) - a_0}{t}\,dt.$$

8.
$$\begin{aligned} C(x) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k x^n = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} a_k x^n = \sum_{k=0}^{\infty} a_k \sum_{n=0}^{\infty} x^{n+k} \\ &= \sum_{k=0}^{\infty} a_k x^k \sum_{n=0}^{\infty} x^n = \sum_{k=0}^{\infty} a_k x^k \cdot \frac{1}{1-x} = \frac{A(x)}{1-x}. \end{aligned}$$

Remark: Generating functions can be regarded as formal objects without considering convergence. The sum, difference, product or quotient of two power series is a power series again (for the quotient, one has to assume that the denominator has a non-zero constant coefficient). For instance, one can formally multiply

$$(a_0+a_1x+a_2x^2+\ldots)(b_0+b_1x+b_2x^2+\ldots) = a_0b_0+(a_1b_0+a_0b_1)x+(a_0b_2+a_1b_1+a_2b_0)x^2+\ldots$$

or divide

$$\frac{a_0+a_1x+a_2x^2+\ldots}{b_0+b_1x+b_2x^2+\ldots} = c_0+c_1x+c_2x^2+\ldots,$$

where the coefficients $c_0, c_1, c_2, \ldots$ can be found by comparing coefficients in the identity

$$(b_0+b_1x+b_2x^2+\ldots)(c_0+c_1x+c_2x^2+\ldots) = a_0+a_1x+a_2x^2+\ldots,$$

so that $c_0 = \frac{a_0}{b_0}$, $c_1 = \frac{a_1b_0-a_0b_1}{b_0^2}$, etc.

An application of rule 2. for products of generating functions was given in Section 2.3 (Vandermonde identity); let us now discuss an application of the last rule for cumulative sums, which is an extension of Example 4.2:

Example 5.3 Tom decides that he might not spend his entire allowance of R100 and perhaps save some part of it instead. How many ways does he have now to spend his money?

In this setting, Tom can spend nothing, or R5, R10, …, R100. Generally, if his allowance is $5n$ (n any non-negative integer), then the amount that he spends is of the form $5k$, $0 \le k \le n$; if a_n denotes the number of ways to spend exactly $5n$ Rand, then the number of ways to spend at most $5n$ is exactly

$$c_n = \sum_{k=0}^{n} a_k.$$

Recall that the generating function of a_n is $A(x) = \frac{1}{1-x-2x^2}$. Therefore, Theorem 5.4 shows that the generating function of c_n is $C(x) = \frac{1}{(1-x)(1-x-x^2)} = \frac{1}{(1-x)(1+x)(1-2x)}$. Making use of partial fractions once again, we find

$$C(x) = \frac{4}{3(1-2x)} - \frac{1}{2(1-x)} + \frac{1}{6(1+x)}$$

and thus $c_n = [x^n]C(x) = \frac{4}{3}\cdot 2^n - \frac{1}{2} + \frac{1}{6}(-1)^n$. In particular, if $n = 20$ (which corresponds to an amount of R100), then this number is 1398101.

5.3 Nonlinear recursions

While linear recursions are usually solved more quickly by means of the method of undetermined coefficients, generating functions are actually far more versatile and can be applied to many non-linear recursions as well (which the method of undetermined coefficients can not). In this section, we treat two such examples.

Example 5.4 We want to count the number of stacks that can be formed from contiguous rows of coins in such a way that (from the second row on) every coin touches the two coins below it, where the number of coins in the first row is n (see Figure 5.1, which illustrates the case $n = 3$).

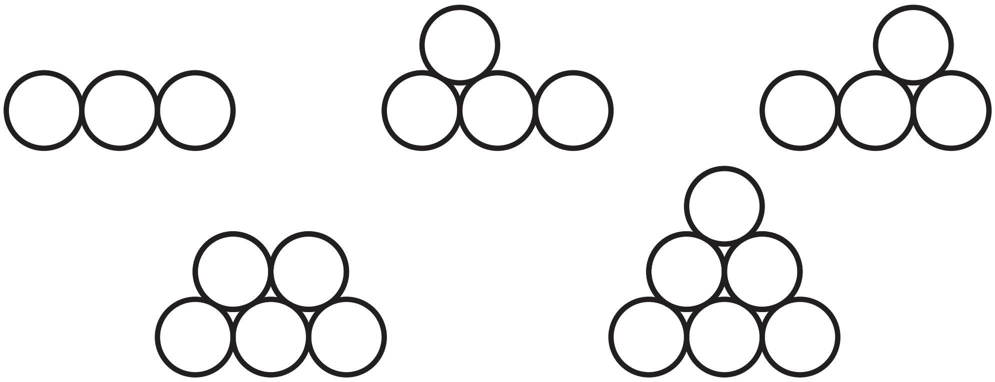

Figure 5.1: Different stacks in the case $n = 3$.

Let a_n denote the number of configurations with n coins in the bottom row. Each such configuration that is not just a single row of coins is obtained by placing a configuration whose bottom row consists of k coins in any of $n - k$ possible positions (there are $n - 1$ available positions for coins altogether, so the first of the k coins can be in either the first, or the second, ..., or the $(n-k)$-th of these positions), where k can be any number between 1 and $n - 1$. This results in the recursion

$$a_n = 1 + \sum_{k=1}^{n-1} (n-k) a_k$$

for $n > 1$ (with the initial value $a_1 = 1$). For convenience, we set $a_0 = 0$ and rewrite this recursion as

$$a_n = 1 + \sum_{k=0}^{n} (n-k) a_k.$$

Let $A(x)$ be the generating function of the sequence a_n; note that $\sum_{k=0}^{n} (n-k) a_k$ is the n-th coefficient in the product of $A(x)$ and

$$\sum_{n=0}^{\infty} n x^n = x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = x \frac{d}{dx} \frac{1}{1-x} = \frac{x}{(1-x)^2}.$$

Therefore, we obtain the equation

$$A(x) = \sum_{n=1}^{\infty} x^n + \frac{x}{(1-x)^2} \cdot A(x) = \frac{x}{1-x} + \frac{x}{(1-x)^2} \cdot A(x).$$

Solving for $A(x)$, we find

$$A(x) = \frac{x(1-x)}{1-3x+x^2}.$$

If one computes the first few elements of the sequence, one finds $a_2 = 2$, $a_3 = 5$, $a_4 = 13$, $a_6 = 34$, ... One notices that all of these numbers are Fibonacci numbers. Indeed, one can show that $a_n = f_{2n-1}$, where f_n denotes the n-th Fibonacci number as in Example 4.3: $f_0 = 0$, $f_1 = 1$, ... One possibility to prove the identity is to determine an explicit formula for a_n from the generating function (by means of partial fractions) and compare it to Binet's formula (5.1). However, we will go another way: if $F(x) = \sum_{n=0}^{\infty} f_n x^n = \frac{x}{1-x-x^2}$ denotes the generating function for the Fibonacci numbers, then we find the generating function for the odd-indexed Fibonacci numbers by the same method that was also applied in the proof of Theorem 2.4:

$$\sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} f_n x^n = \sum_{n=0}^{\infty} \frac{1^n - (-1)^n}{2} f_n x^n = \frac{1}{2}\left(\sum_{n=0}^{\infty} f_n x^n - \sum_{n=0}^{\infty} f_n (-x)^n\right) = \frac{F(x) - F(-x)}{2},$$

which shows that

$$\sum_{n=1}^{\infty} f_{2n-1} x^{2n-1} = \frac{1}{2}\left(\frac{x}{1-x-x^2} - \frac{-x}{1+x-x^2}\right) = \frac{x(1-x^2)}{1-3x^2+x^4},$$

thus

$$\sum_{n=1}^{\infty} f_{2n-1} x^{2n} = \frac{x^2(1-x^2)}{1-3x^2+x^4}$$

and finally

$$\sum_{n=1}^{\infty} f_{2n-1} x^{n} = \frac{x(1-x)}{1-3x+x^2}$$

upon replacing x^2 by x, which is exactly the generating function $A(x)$ that we found before.

The following example introduces the *Catalan numbers*, which will be treated in more detail in Section 7.1.

Example 5.5 Consider $2n$ points on a circle. How many ways are there to connect them by n lines such that there are no points of intersection (and each point is connected to exactly one other point)? Figure 5.2 shows all configurations in the case $n = 3$ (think of $2n$ people shaking hands).

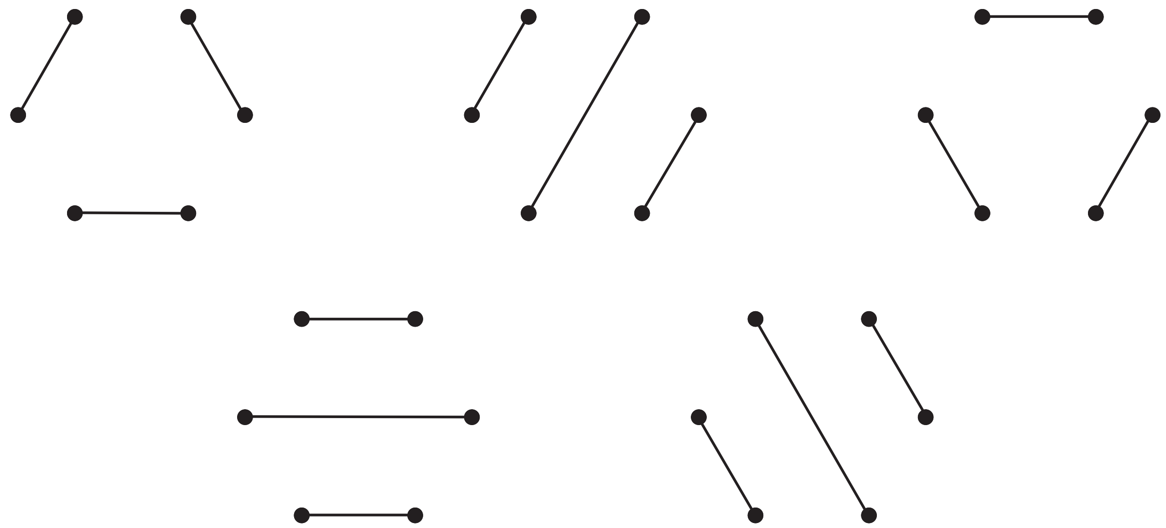

Figure 5.2: Connecting six points on a circle.

Let the points be numbered from 1 to $2n$. If point 1 is connected to point k, this leaves two groups of sizes $k-2$ and $n-k$ that cannot be connected any more; therefore, $k = 2l$ must be even. Now each of the two groups can be treated separately. If a_n denotes the number of possibilities, then we obtain the recursion

$$a_n = \sum_{l=1}^{n} a_{l-1}a_{n-l},$$

the initial value being $a_0 = 1$. Once again, we want to translate this to a functional equation for the generating function $A(x)$. Note that

$$\sum_{l=1}^{n} a_{l-1}a_{n-l} = \sum_{m=0}^{n-1} a_m a_{n-m-1}$$

is exactly the coefficient of x^{n-1} in $A(x)^2$, which is the coefficient of x^n in $xA(x)^2$. Therefore,

$$A(x) = xA(x)^2 + 1,$$

where the last summand takes the initial value into account. Solving the quadratic equation, we find

$$A(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

The negative sign has to be chosen to make sure that $A(0) = a_0 = 1$, as it should be. How can this be turned into a formula for the coefficients a_n? Recall the binomial series (2.1) that we can apply now (in the specific case $\alpha = \frac{1}{2}$):

$$A(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{1}{2x} - \frac{1}{2x}(1-4x)^{1/2} = \frac{1}{2x} - \frac{1}{2x}\sum_{n=0}^{\infty}\binom{1/2}{n}(-4x)^n$$

$$
\begin{aligned}
&= -\frac{1}{2x}\sum_{n=1}^{\infty}\binom{1/2}{n}(-4x)^n = -\frac{1}{2}\sum_{n=1}^{\infty}\binom{1/2}{n}(-4)^n x^{n-1} \\
&= -\frac{1}{2}\sum_{m=0}^{\infty}\binom{1/2}{m+1}(-4)^{m+1}x^m,
\end{aligned}
$$

which shows that $a_n = [x^n]A(x) = -\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1}$; however, this can be simplified as follows:

$$
\begin{aligned}
-\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1} &= -\frac{1}{2}\cdot\frac{(1/2)\cdot(-1/2)\cdot(-3/2)\cdots(1/2-n-1)}{(n+1)!}\cdot(-4)^{n+1} \\
&= -\frac{1}{2}\cdot\frac{2^{-n-1}(-1)^n\cdot 1\cdot 3\cdots(2n-1)}{(n+1)!}\cdot(-4)^{n+1} \\
&= 2^n\cdot\frac{1\cdot 3\cdots(2n-1)}{(n+1)!} = 2^n\cdot\frac{(2n)!}{(n+1)!\cdot 2\cdot 4\ldots(2n)} \\
&= 2^n\cdot\frac{(2n)!}{(n+1)!\cdot 2^n\cdot 1\cdot 2\cdots n} = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}
$$

The numbers $\frac{1}{n+1}\binom{2n}{n}$ are known as Catalan numbers; the first few elements of the sequence are $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, $a_4 = 14$.

Chapter 6

The symbolic method

The detour via recursions is often not necessary if one wants to obtain the generating function for a certain family of combinatorial objects; the *symbolic method* that will be presented in this chapter works for many important combinatorial structures, such as words, permutations, compositions, trees, and many others, that also play a frequent role in computer science and other sciences.

6.1 Unlabelled structures

We consider combinatorial structures that are made up of certain *atoms*. The size of such an object is defined as the number of its atoms. For instance, the atoms of a word over a given alphabet are its letters, and the size is its length. For a tree (a common data structure, see Figure 6.1), a node is an atom, and the size is the number of nodes, and there are many other examples. If there are a_n elements of size n in a certain family $\mathcal{A}$ of combinatorial objects, then the associated generating function is $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

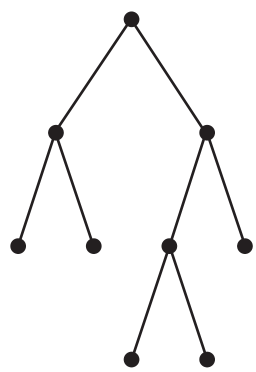

Figure 6.1: A simple binary tree.

Example 6.1 Let us start with a trivial example: *piles* of coins. The atoms are the coins, and the size of a pile is its height (the number of coins); if the coins are indistinguishable, then there is exactly one pile of every non-negative integer size (if we consider the empty pile of size 0 as well), so that the family $\mathcal{P}$ of piles can be described as

$$\mathcal{P} = \{\emptyset, \bullet, \bullet\bullet, \bullet\bullet\bullet, \ldots\},$$

where $\bullet$ denotes an atom (a single coin), and the associated generating function is $P(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

There are certain natural transformations that can be performed on families of combinatorial objects:

Unions

If $\mathcal{A}$ and $\mathcal{B}$ are disjoint families of combinatorial objects, then their union $\mathcal{A} \cup \mathcal{B}$ consists of all objects that are elements of either $\mathcal{A}$ or $\mathcal{B}$, and it is obvious that the associated generating function is $A(x) + B(x)$. The empty family (that does not contain anything at all) has associated generating function 0 and acts as the neutral element in this regard.

Pairs

If $\mathcal{A}$ and $\mathcal{B}$ are families of combinatorial objects (not necessarily disjoint), then we can form pairs (A, B) of objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The family of all such pairs is typically denoted by $\mathcal{A} \times \mathcal{B}$ (Cartesian product of sets). The size of a pair (A, B) is the sum of the sizes of A and B, so if a pair (A, B) is to have size n, then the sizes of A and B have to be k and $n - k$ respectively, where k is any integer between 0 and n. Therefore, if a_n (b_n) denotes the number of elements in $\mathcal{A}$ ($\mathcal{B}$, respectively) whose size is n, and c_n denotes the number of pairs of size n, we find

$$c_n = \sum_{k=0}^{n} a_k b_{n-k},$$

which is exactly the coefficient of x^n in the product of the associated generating functions. Therefore, the generating function for the family $\mathcal{A} \times \mathcal{B}$ is exactly $A(x) \cdot B(x)$. Of course, this principle can be generalised to triples, quadruples, etc. The family $\mathcal{E}$ that only contains one object ϵ (of size 0) has generating function 1 and acts as the neutral element.

Sequences

This builds on the ideas of the previous construction; if $\mathcal{A}$ is any family, then $\mathcal{A} \times \mathcal{A}$ is the family of pairs, $\mathcal{A} \times \mathcal{A} \times \mathcal{A}$ the family of triples, etc. altogether, we obtain the set of all finite *sequences* of elements of $\mathcal{A}$. If we include the empty sequence ϵ as an element of size 0, then we obtain the specification for sequences $\mathsf{Seq}(\mathcal{A})$ as follows:

$$\mathcal{B} = \mathsf{Seq}(\mathcal{A}) = \{\epsilon\} \cup \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) \cup \ldots$$

This translates to the world of generating functions as follows:

$$B(x) = 1 + A(x) + A(x)^2 + A(x)^3 + \ldots = \frac{1}{1 - A(x)}.$$

Here, $\mathcal{A}$ itself must be assumed not to contain elements of size 0 to avoid redundancies. One can also consider restricted sequences: for instance, $A(x)^k$ is the generating function for sequences of length k (for which we write Seq_k),

$$1 + A(x) + A(x)^2 + A(x)^3 + \ldots + A(x)^k = \frac{1 - A(x)^{k+1}}{1 - A(x)}$$

is the generating function for sequences of size at most k (denoted $\mathsf{Seq}_{\leq k}$), and

$$A(x)^k + A(x)^{k+1} + A(x)^{k+2} + \ldots = \frac{A(x)^k}{1 - A(x)}$$

is the generating function for sequences of size at least k (denoted $\mathsf{Seq}_{\geq k}$).

Example 6.2 Recall from Chapter 1 that a *composition* of n is a finite sequence of positive integers adding up to n, such as $3+1+4+1$, which is a composition of 9. The generating function for positive integers $\mathcal{I}$ is obviously

$$x + x^2 + x^3 + \ldots = \frac{x}{1 - x},$$

and so the generating function for compositions $\mathcal{C} = \mathsf{Seq}(\mathcal{I})$ is

$$C(x) = \frac{1}{1 - \frac{x}{1-x}} = \frac{1 - x}{1 - 2x}$$

Now we can make use of the geometric series once again:

$$\frac{1-x}{1-2x} = \frac{1}{2}\left(1 + \frac{1}{1-2x}\right) = \frac{1}{2}\left(1 + \sum_{n=0}^{\infty} 2^n x^n\right) = \frac{1}{2} + \sum_{n=0}^{\infty} 2^{n-1} x^n,$$

which shows that there are exactly 2^{n-1} compositions of n if $n \geq 1$, a result that can also obtained by the "dots and bars" argument. If we are interested in compositions of length k, then we find the generating function to be

$$\left(\frac{x}{1-x}\right)^k = \frac{x^k}{(1-x)^k} = x^k \sum_{n=0}^{\infty} \binom{-k}{n} x^n = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^{n+k} = \sum_{n=k}^{\infty} \binom{n-1}{n-k} x^n$$

by virtue of Lemma 2.2. Therefore, the number of compositions of length k of a positive integer n is given by $\binom{n-1}{n-k} = \binom{n-1}{k-1}$, compare Theorem 1.6 and the remark thereafter.

Compositions into restricted sets of integers can be considered as well: for instance, if only summands 1 and 2 are allowed, then the associated family is $\mathsf{Seq}(\{1, 2\})$, and the generating function is $\frac{1}{1-x-x^2}$, which yields yet another interpretation of the Fibonacci numbers.

Example 6.3 Consider *words* over the simple alphabet $\mathcal{A} = \{a, b\}$. Then we can regard a and b as our atoms, so that the generating function is simply $A(x) = 2x$. The family $\mathcal{W}$ of words over $\mathcal{A}$ is now specified by

$$\mathcal{W} = \mathsf{Seq}(\mathcal{A}) = \mathsf{Seq}(\{a, b\}),$$

so that the generating function is $W(x) = \frac{1}{1-2x}$ (in accordance with the fact that there are 2^n words of length n, see Theorem 1.1). For a general finite alphabet of size k, the associated generating function for words is $\frac{1}{1-kx}$.

Let us now consider a restricted family of words: only a-b-words that do not contain two adjacent letters b are allowed. Such a word starts with an arbitrary (possibly empty) sequence of a's, followed by a b, followed by a non-empty sequence of a's, followed by a b, followed by another non-empty sequence of a's, etc. After the last sequence of a's, we may attach either one more b or nothing at all. This leads to the specification

$$\mathsf{Seq}(\{a\}) \times \mathsf{Seq}(\{b\} \times \mathsf{Seq}_{\geq 1}(\{a\})) \times \{\epsilon, b\},$$

where ϵ denotes an empty word (of length 0). This can be translated to the generating function

$$\frac{1}{1-x} \cdot \frac{1}{1 - \frac{x^2}{1-x}} \cdot (1+x) = \frac{1+x}{1-x-x^2},$$

and we find that the number of such words is a Fibonacci number (note that this example is essentially equivalent to Example 4.3).

Example 6.4 How many different ways are there to put 50 tennis balls into 5 boxes, if the first box can only hold at most 9 balls and the second box at most 7 balls?

While the inclusion-exclusion principle could be used for this purpose, generating functions are somewhat faster. If $\bullet$ stands for a single ball, then our situation is equivalent to the specification

$$\mathsf{Seq}_{\leq 9}(\{\bullet\}) \times \mathsf{Seq}_{\leq 7}(\{\bullet\}) \times \mathsf{Seq}(\{\bullet\}) \times \mathsf{Seq}(\{\bullet\}) \times \mathsf{Seq}(\{\bullet\}),$$

for which the generating function is

$$\frac{1-x^{10}}{1-x} \cdot \frac{1-x^8}{1-x} \cdot \left(\frac{1}{1-x}\right)^3 = \frac{1-x^8-x^{10}+x^{18}}{(1-x)^5}$$

we are interested in the coefficient of x^{50}, which is

$$\begin{aligned}
[x^{50}]\frac{1-x^8-x^{10}+x^{18}}{(1-x)^5} &= [x^{50}]\frac{1}{(1-x)^5} - [x^{50}]\frac{x^8}{(1-x)^5} - [x^{50}]\frac{x^{10}}{(1-x)^5} + [x^{50}]\frac{x^{18}}{(1-x)^5} \\
&= [x^{50}]\frac{1}{(1-x)^5} - [x^{42}]\frac{1}{(1-x)^5} - [x^{40}]\frac{1}{(1-x)^5} + [x^{32}]\frac{1}{(1-x)^5}
\end{aligned}$$

in view of Theorem 5.4. Now, Lemma 2.2 can be applied to find that this number is

$$\begin{aligned}\binom{-5}{50} - \binom{-5}{42} - \binom{-5}{40} + \binom{-5}{32} &= \binom{54}{50} - \binom{46}{42} - \binom{44}{40} + \binom{36}{32} \\ &= \binom{54}{4} - \binom{46}{4} - \binom{44}{4} + \binom{36}{4} = 76220.\end{aligned}$$

The following constructions are somewhat more advanced, but they often prove useful as well:

Powersets

The *powerset* $\mathsf{PSet}(\mathcal{A})$ of $\mathcal{A}$ consists of all subsets of $\mathcal{A}$; if the size of a subset is the sum of the sizes of all its elements (as in the case of sequences), then we find that

$$\mathsf{PSet}(\mathcal{A}) = \bigotimes_{A \in \mathcal{A}} \{A, \epsilon\},$$

since every element A can either be present or not; if a_n is the number of elements in $\mathcal{A}$ of size n (assume that $a_0 = 0$), then we find that the generating function of $\mathcal{B} = \mathsf{PSet}(\mathcal{A})$ is

$$\begin{aligned}B(x) = \prod_{n=1}^{\infty}(1+z^n)^{a_n} &= \exp\left(\sum_{n=1}^{\infty} a_n \log(1+z^n)\right) \\ &= \exp\left(\sum_{n=1}^{\infty} a_n \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} z^{mn}\right) = \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{n=1}^{\infty} a_n z^{mn}\right) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} A(z^m)\right) = \exp\left(A(x) - \frac{A(x^2)}{2} + \frac{A(x^3)}{3} - \dots\right),\end{aligned}$$

making use of the Taylor series for the logarithm.

Multisets

Multisets $\mathsf{MSet}(A)$ are closely related to the powerset construction; we consider collections of elements of $\mathcal{A}$, where each element is allowed to occur more than once as well. This is equivalent to the formal expression

$$\mathsf{MSet}(\mathcal{A}) = \bigotimes_{A \in \mathcal{A}} \mathsf{Seq}(\{A\}),$$

which leads to the following generating function for $\mathcal{B} = \mathsf{MSet}(\mathcal{A})$:

$$B(x) = \prod_{n=1}^{\infty}(1-z^n)^{-a_n} = \exp\left(-\sum_{n=1}^{\infty} a_n \log(1-z^n)\right)$$

$$= \exp\left(\sum_{n=1}^{\infty} a_n \sum_{m=1}^{\infty} \frac{1}{m} z^{mn}\right) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} a_n z^{mn}\right)$$
$$= \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} A(z^m)\right) = \exp\left(A(x) + \frac{A(x^2)}{2} + \frac{A(x^3)}{3} + \dots\right)$$

Example 6.5 Formally, a *language* is a collection of distinct words (this notion of a language plays a role in computer science). If an alphabet $\mathcal{A}$ of k letters is given, how many languages are there that contain a total number of n letters? The according specification is

$$\mathcal{L} = \mathsf{PSet}(\mathsf{Seq}_{\geq 1}(\mathcal{A})),$$

giving rise to the generating function

$$L(x) = \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \cdot \frac{kx^m}{1 - kx^m}\right) = \prod_{n=1}^{\infty} (1 + x^n)^{k^n}.$$

While one cannot derive a closed formula from this generating function, it can be used to determine the first few coefficients (the sequence starts $l_1 = 2$, $l_2 = 5$, $l_3 = 16$, $l_4 = 42$, ... in the case $k = 2$), and further information about the coefficients (in particular, their growth) can be extracted by means of analytic methods (which are beyond the scope of this course).

6.2 Labelled structures

In many instances, one is interested in structures where the atoms are *labelled*: if a structure consists of n atoms, then these atoms receive distinct labels from 1 to n. The most typical example are *permutations*: a permutation of n can be regarded as a sequence of atoms bearing labels from 1 to n; the $6 = 3!$ permutations of $\{1, 2, 3\}$ (see Theorem 1.2) are thus { ①②③, ①③②, ②①③, ②③①, ③①②, ③②①}. In the context of labelled structures, it is useful to work with *exponential generating functions*: if a_n is a sequence, then its exponential generating function is defined by

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

The exponential generating function associated to permutations is thus

$$\sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

since there are $n!$ permutations of $\{1, 2, \dots, n\}$. Note that, on the other hand, the ordinary generating function $\sum_{n=0}^{\infty} n! x^n$ is not an elementary function, and it is even divergent for all $x \neq 0$. The name "exponential" is due to the fact that the exponential generating function of the sequence $1, 1, \ldots$ is e^x (the associated structure is sometimes called "urns": an urn is merely a collection of atoms labelled 1 to n (think of balls marked with these labels) without additional structure.

Example 6.6 In how many ways can one form *cycles* of the numbers $1, 2, \ldots, n$? Figure 6.2 shows all possibilities in the case $n = 4$; generally, if one starts at 1, then there are $(n-1)!$ possible orders for the remaining numbers. Therefore, the associated exponential generating function is

$$\sum_{n=0}^{\infty} \frac{(n-1)!}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n} = -\log(1-x).$$

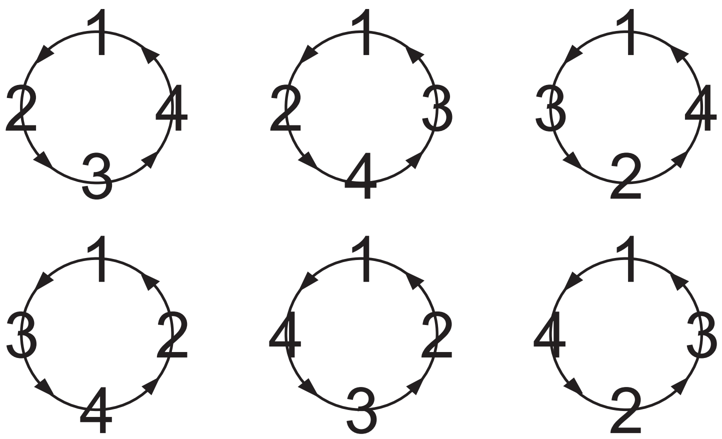

Figure 6.2: All cycles of length 4.

For instance, one can think of these cycles as necklaces made of four beads in different colours.

Again, there are several useful transformations on families of labelled objects:

Unions

As in the case of unlabelled structures, one can define the union of two disjoint families $\mathcal{A}$ and $\mathcal{B}$, and the associated generating function is exactly the sum of the generating functions of $\mathcal{A}$ and $\mathcal{B}$.

Pairs

The situation is slightly different to the unlabelled case, since an ordinary pair of labelled objects does not constitute a proper labelled object again (since labels are duplicated). If one combines two labelled object A and B of sizes k and $n-k$ respectively to form a new object of size n, then one first has to distribute the labels among the two objects, which can be done in $\binom{n}{k}$ ways (choose the k labels that are given to A). Therefore, if two families $\mathcal{A}$ and $\mathcal{B}$ with exponential generating functions $A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!}$ and $B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!}$ are given, then the number of pairs of size n that can be formed in this way is

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_k b_{n-k},$$

which can be rewritten as

$$\frac{c_n}{n!} = \sum_{k=0}^{n} \frac{a_k}{k!} \cdot \frac{b_{n-k}}{(n-k)!}.$$

Note that this is exactly the coefficient of x^n in the product $A(x) \cdot B(x)$, so that the generating function of the family $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ of pairs is exactly $A(x) \cdot B(x)$. This fact is one of the main reasons why exponential generating functions are used in this context.

Sequences

As in the case of unlabelled structure, one can extend the above reasoning to triples, quadruples, etc. Generally, *sequences* of length k of elements from a family $\mathcal{A}$ with exponential generating function $A(x)$ have exponential generating function $A(x)^k$ (we denote this family by $\mathsf{Seq}_k(\mathcal{A})$), and the exponential generating functions for sequences $\mathsf{Seq}_{\leq k}(\mathcal{A})$ of length at most k and sequences $\mathsf{Seq}_{\geq k}(\mathcal{A})$ of length at least k are

$$\frac{1 - A(x)^{k+1}}{1 - A(x)} \qquad \text{and} \qquad \frac{A(x)^k}{1 - A(x)}$$

respectively. Arbitrary sequences $\mathsf{Seq}(\mathcal{A})$ have exponential generating function $\frac{1}{1-A(x)}$.

Example 6.7 *Words* over an alphabet of size k can also be regarded as labelled objects, namely sequences of urns (the i-th urn records the positions in the word where the i-th letter occurs) of length k. Therefore, the exponential generating function is

$$(e^x)^k = e^{kx} = \sum_{n=0}^{\infty} \frac{k^n}{n!} x^n,$$

in agreement with the fact that there are k^n words of length n.

Sets

Sets of elements from a family $\mathcal{A}$ of combinatorial objects are somewhat easier to handle in the labelled case than in the unlabelled one. This is due to the fact that objects are now distinguishable by their labels: to every set of k objects (among which the labels 1 to n are distributed), there are exactly $k!$ sequences (since a sequence is nothing but an ordered set); therefore, the exponential generating function for $\mathsf{Set}_k(\mathcal{A})$ is simply $\frac{A(x)^k}{k!}$, and summing over all k yields the exponential generating function for $\mathcal{B} = \mathsf{Set}(\mathcal{A})$ (arbitrary sizes):

$$B(x) = \sum_{k=0}^{\infty} \frac{A(x)^k}{k!} = \exp(A(x)).$$

Various examples will be discussed in the following section.

Chapter 7

Special numbers

7.1 Catalan numbers

We have already encountered the Catalan numbers in Example 5.5. However, there are many other intuitive counting problems that lead to the sequence $1, 2, 5, 14, 42, \ldots$ of Catalan numbers. In this section, a few more examples are treated.

Binary trees

A *binary tree* is a tree with the property that all its nodes have either two children (internal node) or no children (external node). Figure 6.1 shows a binary tree with four internal and five external nodes. Such trees provide an important data structure in computer science. It is easy to show, by means of induction, that there is always exactly one external node more than internal nodes (so that the total number of nodes is necessarily odd). Making use of the symbolic method, we can almost effortlessly determine a generating function for the number of binary trees: if $\mathcal{B}$ is the family of binary trees, then

$$\mathcal{B} = \{\bullet\} \cup (\{\bullet\} \times \mathcal{B} \times \mathcal{B}) .$$

(A binary tree is either a single node or a root with two binary trees attached.) This translates to the functional equation

$$B(x) = x + xB(x)^2,$$

with the solution

$$B(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x}.$$

Note the similarity with Example 5.5; using the same method as in that example, we find a formula for the number of binary trees with a prescribed number of nodes:

Theorem 7.1 The number of binary trees with $2n + 1$ nodes ($n \geq 0$) is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Alternatively, one can consider *pruned binary trees*, which are obtained by leaving out all external nodes. In such a case, the remaining nodes can either have two children, or only a left child, or only a right child, or no children at all, see Figure 7.1.

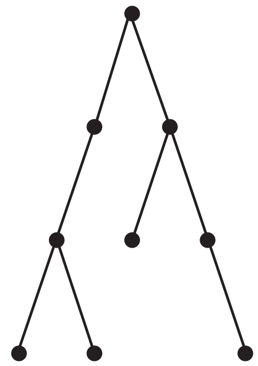

Figure 7.1: A pruned binary tree.

Pruned binary trees can be defined symbolically by

$$\mathcal{PB} = \{\bullet\} \cup (\{\bullet\} \times \mathcal{PB}) \cup (\mathcal{PB} \times \{\bullet\}) \cup (\mathcal{PB} \times \{\bullet\} \times \mathcal{PB})$$

to the effect that the generating function satisfies

$$PB(x) = x + 2xPB(x) + xPB(x)^2 = x(1 + PB(x))^2,$$

and thus

$$PB(x) = \frac{1 - 2x - \sqrt{1-4x}}{2x},$$

in accordance with the fact that the number of pruned binary trees with $n > 0$ nodes is exactly the number of binary trees with $2n + 1$ nodes.

Plane trees

In contrast to binary trees, the nodes of a *plane tree* can have any number of children, see Figure 7.2 for an example.
Their construction can be written symbolically as

$$\mathcal{P} = \{\bullet\} \times \mathsf{Seq}(\mathcal{P})$$

(a plane tree consists of a root to which a sequence, possibly empty, of plane trees is attached). Therefore, we have the following equation for the generating function:

$$P(x) = x \cdot \frac{1}{1 - P(x)},$$

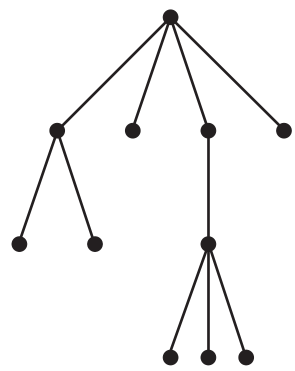

Figure 7.2: A plane tree.

which leads to a quadratic equation whose solution is

$$P(x) = \frac{1 - \sqrt{1 - 4x}}{2}.$$

Again, we have to choose the minus sign in order to have $P(0) = 0$. Therefore, the following theorem holds:

Theorem 7.2 The number of plane trees with n nodes ($n \geq 1$) is the Catalan number $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$.

There is a simple combinatorial explanation to the fact that the number of plane trees with $n > 1$ nodes is exactly the number of pruned binary trees with $n - 1$ nodes. This so-called *rotation correspondence* is exhibited in Figure 7.3.

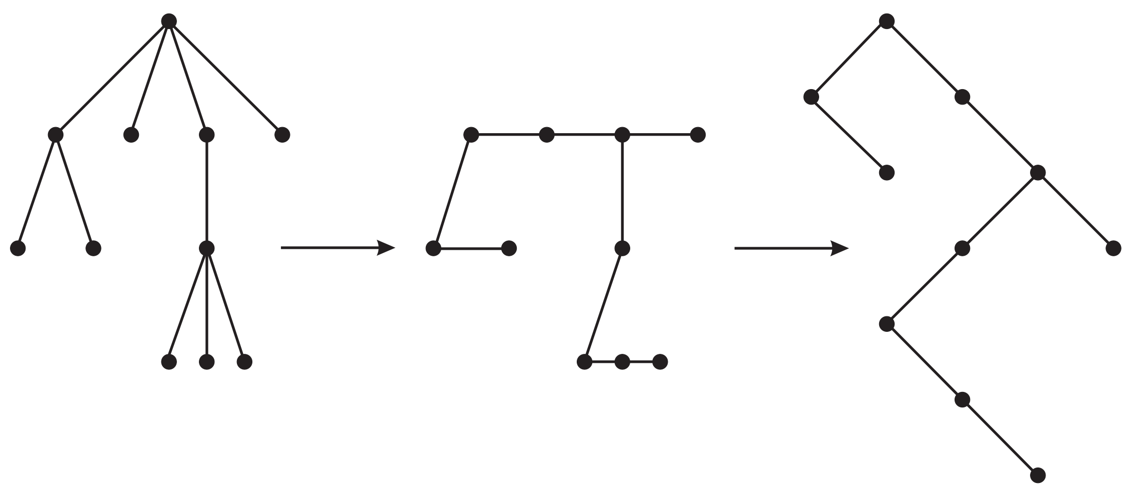

Figure 7.3: The rotation correspondence.

In order to obtain a pruned binary tree from a plane tree, remove the root and all connections to children that are not leftmost children. Instead, connect siblings. Now perform a 45 degrees rotation, so that siblings become right children. This process can be reversed, which means that there is a direct $1-1$ correspondence between pruned binary trees and plane trees.

Dyck paths

A *Dyck path* is a path consisting of "up" and "down" steps that start and end on the x-axis (say, at $(0,0)$ and $(2n,0)$ respectively) and have the property that they never go below the x-axis (see Figure 7.4).

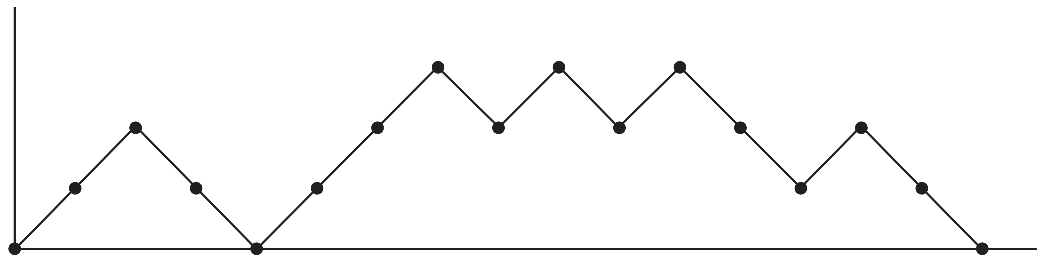

Figure 7.4: An example of a Dyck path.

What is the number of such paths? One way to determine this number is to decompose Dyck paths: each such path (that is not the trivial path of length 0) consists of an initial "up" step, followed by a path that stays above the line $y=1$, followed by a "down" step (this is the first time that the x-axis is reached again), followed by another (arbitrary) Dyck path. This shows that we have

$$\mathcal{D} = \epsilon \cup (\{\nearrow\} \times \mathcal{D} \times \{\searrow\} \times \mathcal{D}),$$

where ϵ denotes the path of length 0. This translates to the functional equation

$$D(x) = 1 + x^2 D(x)^2$$

for the generating function; compare this to the functional equation obtained for binary trees. One obtains the following theorem:

Theorem 7.3 The number of Dyck paths of length $2n$ ($n \geq 0$) is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

This theorem can also be obtained by means of a simple correspondence between plane trees and Dyck paths; if one moves along the edges of a tree, the path traced exactly describes a Dyck path (see Figure 7.5); this process (that is also easy to reverse) is known as the *glove bijection*.

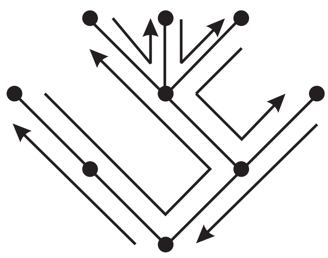

Figure 7.5: The glove bijection: the plane tree that is shown corresponds to the Dyck path in Figure 7.4.

There is a clever proof (called *André's reflection principle*) that yields the formula for the Catalan numbers directly from the definition of Dyck paths; first note that we need exactly $n \nearrow$ steps and $n \searrow$ steps, for which there are a priori $\binom{2n}{n}$ possible arrangements (ways to choose the n positions for the $\nearrow$ steps); we have to exclude those that contain a part below the x-axis. For any such path, consider the first time that the level -1 is reached. If we reflect the entire path up to this point about the line $y = -1$, we obtain a path between $(0, -2)$ and $(2n, 0)$ that consists of $n + 1 \nearrow$ steps and $n - 1 \searrow$ steps (Figure 7.6).

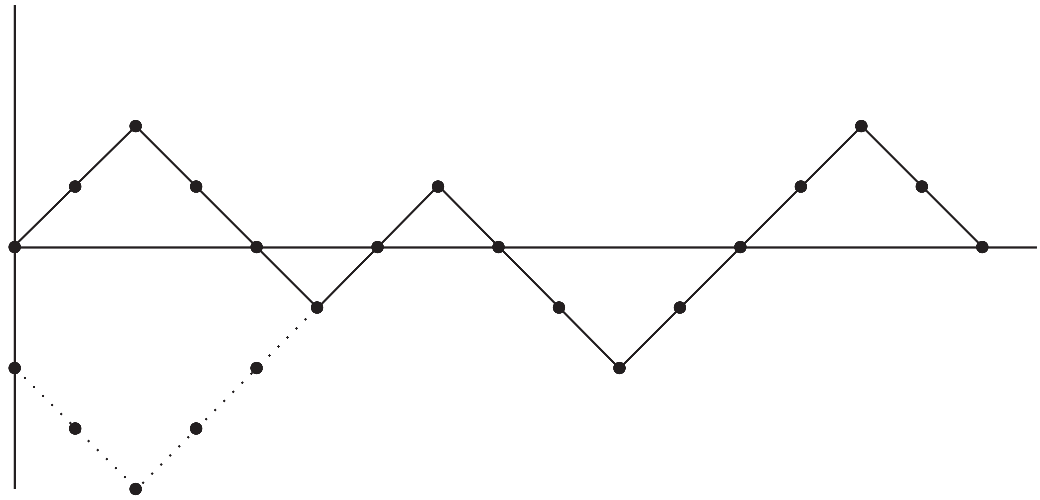

Figure 7.6: The reflection principle.

This process can be reversed: given any such path between $(0, -2)$ and $(2n, 0)$, consider the first time that the level -1 is reached (it must be, by continuity), and reflect the path up to this point. Therefore, the number of paths to be excluded is also exactly the number of paths between $(0, -2)$ and $(2n, 0)$ consisting of $n + 1 \nearrow$ steps and $n - 1 \searrow$ steps. The number of such paths is $\binom{2n}{n+1}$ (by the same argument as before), so that we obtain the formula for the Catalan numbers once again:

$$\begin{aligned} C_n &= \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} \\ &= \frac{(n+1)(2n)!}{(n+1)!n!} - \frac{n(2n)!}{(n+1)!n!} = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

This idea can even be generalised: the so-called *Ballot problem* can be stated as follows: suppose that two candidates (Alice and Bob) at an election receive n and m votes, respectively, where $n \geq m$. Altogether, the counting (that is done vote by vote) can thus proceed in $\binom{n+m}{n}$ ways, and each of them corresponds to a path that consists of n $\nearrow$ steps and m $\searrow$ steps that connects $(0,0)$ and $(n+m, n-m)$. In this setting, the y-coordinate describes Alice's current lead. How many of these paths stay above the x-axis (so that Bob never leads throughout the counting procedure)? The argument that was used for Dyck paths is applicable again, to the effect that one obtains the *Ballot number*

$$\begin{aligned} B_{n,m} &= \binom{n+m}{n} - \binom{n+m}{n+1} = \frac{(n+m)!}{n!m!} - \frac{(n+m)!}{(n+1)!(m-1)!} \\ &= \frac{(n+1)(n+m)!}{(n+1)!m!} - \frac{m(n+m)!}{(n+1)!m!} = \frac{(n-m+1)(n+m)!}{(n+1)!m!} = \frac{n-m+1}{n+1}\binom{n+m}{n}. \end{aligned}$$

The Catalan number C_n arises as the special case $m = n$. The fraction $\frac{n-m+1}{n+1}$ can be interpreted as the probability that Alice leads all the way through.

7.2 Stirling cycle numbers

Permutations can be regarded as functions from $\{1, 2, \ldots, n\}$ to itself. For instance, the permutation 3 9 4 7 5 2 8 1 6 can be seen as a function with $\sigma(1) = 3$, $\sigma(2) = 9$, $\sigma(3) = 4$, ... Representing the permutation by arrows pointing from k to $\sigma(k)$, one ends up with a natural decomposition into *cycles* (see Figure 7.7) that is particularly useful if the set of permutations of $\{1, 2, \ldots, n\}$ is regarded as a group (the so-called *symmetric group*) in the sense of abstract algebra. Permutations are often written as a collection of cycles; for instance, the partition shown in Figure 7.7 is written as (13478)(296)(5).

Note that derangements (as discussed in Example 3.5) are precisely those permutations that do not have cycles of length 1. If $\mathcal{C}$ is the family of cycles and $\mathcal{P}$ the family of permutations, then this decomposition means that

$$\mathcal{P} = \mathsf{Set}(\mathcal{C}).$$

Recall that the exponential generating functions for cycles (Example 6.6) and permutations are $C(x) = -\log(1-x)$ and $P(x) = \frac{1}{1-x}$, respectively, in agreement with the fact that one must have $P(x) = \exp(C(x))$ by the above correspondence. Let us now ask the following question: how many permutations of $1, 2, \ldots, n$ have exactly k cycles? This number is

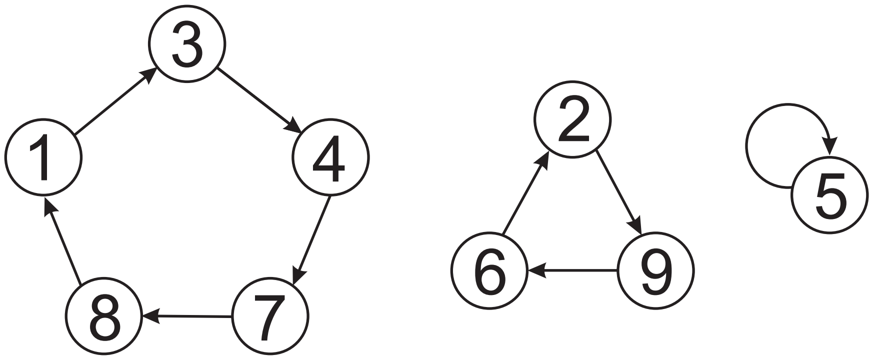

Figure 7.7: The cycle representation.

known as the *Stirling cycle number* or *(unsigned) Stirling number of the first kind* and is denoted by $\left[{n \atop k}\right]$. Consider the case $n = 3$:

$$123, \quad 132, \quad 213, \quad 231, \quad 312, \quad 321.$$

The number of cycles is 3, 2, 2, 1, 1 and 2 respectively. Therefore, $\left[{3 \atop 1}\right] = 2$, $\left[{3 \atop 2}\right] = 3$ and $\left[{3 \atop 3}\right] = 1$. Let us collect a few properties:

Theorem 7.4 The Stirling cycle numbers $\left[{n \atop k}\right]$ satisfy the following properties:

1. $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$, $\begin{bmatrix} n \\ n \end{bmatrix} = 1$ and $\begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}$ for any $n \geq 1$,

2. $\displaystyle\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!$,

3. $\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$.

Proof:

1. The first formula follows from the discussion in Example 6.6. If there are n cycles, then each of the numbers $1, 2, \ldots, n$ must be a fixed point (that is, it is mapped to itself), so that there is only one possible permutation; this implies $\left[{n \atop n}\right] = 1$. Finally, $n-1$ cycles are only possible if there are $n-2$ fixed points and one cycle of length 2. Therefore, $\left[{n \atop n-1}\right] = \binom{n}{2}$, which is exactly the number of choices for the two elements that form the 2-cycle.

2. The sum of all the $\left[{n \atop k}\right]$ must obviously be the total number of permutations, which is $n!$.

3. A permutation of $1, 2, \ldots, n$ can be obtained from a permutation of $1, 2, \ldots, n-1$ in two different ways:

- By inserting n into one of the cycles. A cycle of length r has r possible places of insertion. Since the sum of the cycle lengths must be the total number of elements (which is $n-1$), there are $n-1$ possible positions where n can be inserted; the number of cycles remains the same, so this gives rise to the summand $(n-1)\left[{n-1 \atop k}\right]$.
- By adding n as a single cycle of length 1; this increases the number of cycles by 1, which explains the summand $\left[{n-1 \atop k-1}\right]$.

Summing the two contributions, one obtains the formula. □

The recursion for the Stirling cycle numbers can be used to produce a table in analogy to Pascal's triangle:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 2 & & 3 & & 1 & & & & \\
 & & & 6 & & 11 & & 6 & & 1 & & & \\
 & & 24 & & 50 & & 35 & & 10 & & 1 & & \\
 & 120 & & 274 & & 225 & & 85 & & 15 & & 1 & \\
720 & & 1764 & & 1624 & & 735 & & 175 & & 21 & & 1
\end{array}$$

The Stirling cycle numbers have an interesting generating function that can also be derived from the recursion or by means of the symbolic method: the cycle decomposition implies that permutations with exactly k cycles can be specified by

$$\mathcal{P}_k = \mathsf{Set}_k(\mathcal{C}).$$

Therefore, the associated exponential generating function is

$$\sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!} = \frac{1}{k!}(-\log(1-x))^k. \tag{7.1}$$

Now consider the *bivariate generating function*

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{u^k x^n}{n!} = \sum_{k=0}^{\infty} \frac{1}{k!} u^k (-\log(1-x))^k = e^{-u\log(1-x)} = (1-x)^{-u},$$

where we set

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{cases} 1 & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

for convenience, so that (7.1) remains correct for $k = 0$. This shows that in the expansion of the bivariate function $(1-x)^{-u}$, the coefficient of $u^k x^n$ is the Stirling cycle number $\left[{n \atop k}\right]$. If we extract the coefficient of x^n (making use of the binomial series), we obtain

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} u^k = n![x^n](1-x)^{-u} = n! \cdot (-1)^n \binom{-u}{n} = n! \cdot \binom{n+u-1}{n}$$

$$= n! \cdot \frac{(n+u-1)(n+u-2)\dots u}{n!} = u(u+1)\dots(u+n-1) = u^{\overline{n}},$$

which is called the *rising factorial* of u (read: "u to the n rising"). Check, for instance, that

$$u(u+1)(u+2)(u+3) = 6u + 11u^2 + 6u^3 + u^4,$$

in agreement with the table above.

7.3 Stirling partition numbers and Bell numbers

The *Stirling partition numbers* stem from a different yet related question: in how many ways can the numbers $1, 2, \dots, n$ (or any other set of size n) be partitioned into k groups? This number is called the Stirling partition number (or *Stirling number of the second kind*) $\left\{{n \atop k}\right\}$. For instance, there are 7 ways to split the numbers $1, 2, 3, 4$ into 2 groups (this is known as a *set partition*):

$$\begin{array}{llll} \{1\},\{2,3,4\}; & \{2\},\{1,3,4\}; & \{3\},\{1,2,4\}; & \{4\},\{1,2,3\}; \\ \{1,2\},\{3,4\}; & \{1,3\},\{2,4\}; & \{1,4\},\{2,3\}. & \end{array}$$

Hence, $\left\{{4 \atop 2}\right\} = 7$. The following theorem collects some of the important properties of the Stirling partition numbers:

Theorem 7.5 The Stirling partition numbers $\left\{{n \atop k}\right\}$ satisfy the following properties:

1. $\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$ and $\left\{{n \atop n-1}\right\} = \binom{n}{2}$ for any $n \geq 1$,
2. $\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$.

Proof: The proofs are very similar to those for Theorem 7.4.

1. The first formula is obvious, since we don't have any choice if either $k = 1$ (all elements belong to one group) or $k = n$ (each group is a singleton). Furthermore, $\left\{{n \atop n-1}\right\} = \binom{n}{2}$, since there are $\binom{n}{2}$ ways to choose the two elements that form the group of size 2 (all others being singletons, compare the proof of Theorem 7.4).

2. A set partition of $1, 2, \dots, n$ can be obtained from a set partition of $1, 2, \dots, n-1$ in two different ways:

   - By adding n to one of the existing groups; this does not change the number of groups, so that we obtain the summand $k\left\{{n-1 \atop k}\right\}$ (there are k groups to add the element n to, hence the factor k).
   - By adding n as a single group of length 1; this increases the number of groups by 1, which explains the summand $\left\{{n-1 \atop k-1}\right\}$.

Summing the two contributions, one obtains the formula. □

Again, one can conveniently compute the numbers in a triangular scheme:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 1 & & & & \\
 & & & 1 & & 7 & & 6 & & 1 & & & \\
 & & 1 & & 15 & & 25 & & 10 & & 1 & & \\
 & 1 & & 31 & & 90 & & 65 & & 15 & & 1 & \\
1 & & 63 & & 301 & & 350 & & 140 & & 21 & & 1
\end{array}
$$

The horizontal row sums $1, 2, 5, 15, 52, 203, 877, \ldots$ are known as the *Bell numbers*. The Bell number S_n gives the total number of set partitions of a set of n elements. Its generating function can be determined quite easily by the symbolic method: a set partition is a set of non-empty sets; hence, if $\odot$ stands for a single labelled atom, one has

$$\mathcal{S} = \mathsf{Set}(\mathsf{Set}_{\geq 1}(\odot))$$

for the family $\mathcal{S}$ of set partitions. This yields the exponential generating function

$$S(x) = \sum_{n=0}^{\infty} \frac{S_n x^n}{n!} = e^{e^x - 1}$$

Here, S_0 is defined to be 1 for convenience. The generating function can be used to determine a recursion for the Bell numbers: differentiate with respect to x to obtain

$$S'(x) = \sum_{n=1}^{\infty} \frac{S_n x^{n-1}}{(n-1)!} = e^x \cdot e^{e^x - 1}.$$

Some manipulations yield

$$\begin{aligned} S'(x) = \sum_{n=0}^{\infty} \frac{S_{n+1} x^n}{n!} &= \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \left(\sum_{m=0}^{\infty} \frac{S_m x^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{S_{n-k}}{k!(n-k)!} \right) x^n. \end{aligned}$$

Comparing coefficients, we find

$$S_{n+1} = \sum_{k=0}^{n} \binom{n}{k} S_{n-k} = \sum_{k=0}^{n} \binom{n}{k} S_k.$$

This can be interpreted as follows: in order to produce a set partition of $1, 2, \ldots, (n+1)$, choose the k elements ($0 \leq k \leq n$) that form a group with the element 1. The remaining $n - k$ form a set partition of their own.

Alternatively, Bell numbers can also be computed from *Dobinski's formula*:

Theorem 7.6 The Bell number S_n is given by

$$S_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}.$$

Proof: Making use of the exponential generating function once again, we find

$$\begin{aligned} S_n &= n![x^n]e^{e^x-1} = \frac{n!}{e}[x^n]e^{e^x} \\ &= \frac{n!}{e}[x^n]\sum_{k=0}^{\infty}\frac{e^{kx}}{k!} = \frac{n!}{e}\sum_{k=0}^{\infty}[x^n]\frac{e^{kx}}{k!} \\ &= \frac{n!}{e}\sum_{k=0}^{\infty}\frac{k^n}{n!k!} = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}. \end{aligned}$$

This proves the formula. □

Let us return to the Stirling partition numbers again. If we restrict ourselves to the class $\mathcal{S}_k$ of set partitions composed of exactly k subsets, we have

$$\mathcal{S}_k = \mathsf{Set}_k(\mathsf{Set}_{\geq 1}(\odot)),$$

so that

$$\sum_{n=0}^{\infty}\left\{ {n \atop k} \right\}\frac{x^n}{n!} = \frac{1}{k!}(e^x-1)^k. \tag{7.2}$$

This can be used to derive an explicit formula for the Stirling partition numbers:

Theorem 7.7 The Stirling partition number $\left\{ {n \atop k} \right\}$ is given by

$$\left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}m^n.$$

Proof: Expand the right hand side of (7.2) by means of the binomial theorem:

$$\begin{aligned} \left\{ {n \atop k} \right\} &= n![x^n]\frac{1}{k!}(e^x-1)^k = \frac{n!}{k!}[x^n]\sum_{m=0}^{k}\binom{k}{m}e^{mx}(-1)^{k-m} \\ &= \frac{n!}{k!}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}[x^n]e^{mx} = \frac{n!}{k!}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\frac{m^n}{n!} \\ &= \frac{1}{k!}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}m^n, \end{aligned}$$

which proves the claim. □

The Stirling partition numbers also have a simple bivariate generating function (compare the analogous considerations for Stirling cycle numbers):

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left\{{n \atop k}\right\}\frac{u^k x^n}{n!} = \sum_{k=0}^{\infty}\frac{1}{k!}u^k(e^x-1)^k = e^{u(e^x-1)},$$

where we define

$$\left\{{n \atop 0}\right\} = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$$

once again for convenience, so that (7.2) remains correct for $k = 0$. Just like the Stirling cycle numbers, the Stirling partition numbers also occur in an interesting relation between powers and factorials:

Theorem 7.8 The Stirling partition numbers satisfy the identity

$$\sum_{k=0}^{n}\left\{{n \atop k}\right\}x^{\underline{k}} = x^n$$

for $n \geq 0$ and arbitrary x.

Proof: By induction on n. The statement is trivial for $n = 0$ or $n = 1$. Now suppose that

$$\sum_{k=0}^{n-1}\left\{{n-1 \atop k}\right\}x^{\underline{k}} = x^{n-1}.$$

Then we have

$$\begin{aligned}
\sum_{k=0}^{n}\left\{{n \atop k}\right\}x^{\underline{k}} &= \sum_{k=0}^{n}\left(k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}\right)x^{\underline{k}} \\
&= \sum_{k=0}^{n-1}k\left\{{n-1 \atop k}\right\}x^{\underline{k}} + \sum_{k=1}^{n}\left\{{n-1 \atop k-1}\right\}x^{\underline{k}} \\
&= \sum_{k=0}^{n-1}k\left\{{n-1 \atop k}\right\}x^{\underline{k}} + \sum_{k=0}^{n-1}\left\{{n-1 \atop k}\right\}x^{\underline{k+1}} \\
&= \sum_{k=0}^{n-1}\left\{{n-1 \atop k}\right\}\left(kx^{\underline{k}} + x^{\underline{k+1}}\right) \\
&= \sum_{k=0}^{n-1}\left\{{n-1 \atop k}\right\}\left(kx^{\underline{k}} + (x-k)x^{\underline{k}}\right) \\
&= x\sum_{k=0}^{n-1}\left\{{n-1 \atop k}\right\}x^{\underline{k}}
\end{aligned}$$

$$= x \cdot x^{n-1} = x^n,$$

which completes the induction. □

Let us finally mention that Stirling partition numbers also arise in the counting of *surjections*: a surjection is a function from a set X to a set Y that is onto, i.e., every value $y \in Y$ is taken on at least once. One has the following theorem:

Theorem 7.9 The number of surjections from a set X of size n to a set Y of size k is $k!\left\{ {n \atop k} \right\}$.

Proof: Assume, without loss of generality, that $Y = \{1, 2, \ldots, k\}$, and consider a surjection f. The sets $X_1, X_2, \ldots, X_k$ that are defined by

$$X_i = \{x \in X \,:\, f(x) = i\}$$

induce a set partition on X. To each set partition, one has $k!$ corresponding surjections (since the k subsets can be arranged in any order). This correspondence can be reversed (an ordered set partition corresponds to a surjection in a unique way), so that the number of surjections from X to Y has to be the same as the number of ordered set partitions comprising k subsets. □

Remark: This theorem complements Theorem 1.3, which counts *injections*, i.e., one-to-one functions (every value is taken on at most once).

7.4 Eulerian numbers

If we regard a permutation of $1, 2, \ldots, n$ as a sequence $x_1, x_2, \ldots, x_n$, then we can ask how many times this sequence goes up (a so-called "ascent", $x_j > x_{j-1}$) and how many times it goes down (a "descent", $x_j < x_{j-1}$). The number of permutations of $1, 2, \ldots, n$ with exactly k ascents is the *Eulerian number* that is denoted by $\left\langle {n \atop k} \right\rangle$. Consider the following list of permutations of $1, 2, 3$:

$$123, \quad 132, \quad 213, \quad 231, \quad 312, \quad 321.$$

The number of ascents is, respectively, $2, 1, 1, 1, 1$, and 0. Therefore, we have $\left\langle {3 \atop 0} \right\rangle = \left\langle {3 \atop 2} \right\rangle = 1$ and $\left\langle {3 \atop 1} \right\rangle = 4$. The following theorem summarizes some of the elementary properties of Eulerian numbers:

Theorem 7.10 The Eulerian numbers $\left\langle {n \atop k} \right\rangle$ satisfy the following properties:

1. $\left\langle {n \atop 0} \right\rangle = \left\langle {n \atop n-1} \right\rangle = 1,$
2. $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle$ for any $0 \le k \le n-1,$

3. $\displaystyle\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle = n!,$

4. $\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle.$

Proof:

1. If there are are no ascents or only ascents (and no descents), then there is only one possibility: the permutation has to be $12\ldots n$ ($n\ldots 21$, respectively).

2. Reversing a permutation with k ascents, we obtain a permutation with k descents (and thus $n-1-k$ ascents), which proves the formula.

3. The sum of all the $\langle {n \atop k} \rangle$ must be the total number of permutations of $1, 2, \ldots, n$, which is $n!$.

4. This is the trickiest part of the theorem: a permutation of $1, 2, \ldots, n$ is obtained by inserting the number n at some position in a permutation of $1, 2, \ldots, n-1$. If it is not inserted in first position, it gives rise to exactly one ascent and one descent. Therefore, a permutation with k ascents can be obtained in two ways:

   - Given a permutation of $1, 2, \ldots, n-1$ with k ascents, insert n at the beginning or at one of the k ascents, or
   - given a permutation of $1, 2, \ldots, n-1$ with $k-1$ ascents, insert n at the end or at one of the $n-k-1$ descents.

   This yields the stated formula. □

The recursive formula in 4. can be used to determine $\langle {n \atop k} \rangle$ for small values of n and k, as shown in the following configuration (called *Euler's triangle* in analogy to Pascal's triangle):

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 1 & & & & \\
 & & & 1 & & 11 & & 11 & & 1 & & & \\
 & & 1 & & 26 & & 66 & & 26 & & 1 & & \\
 & 1 & & 57 & & 302 & & 302 & & 57 & & 1 & \\
1 & & 120 & & 1191 & & 2416 & & 1191 & & 120 & & 1
\end{array}$$

One important property of the Eulerian numbers is the fact that they occur in the generating function of n-th powers:

Theorem 7.11 The generating function of n-th powers is given by

$$\sum_{m=0}^{\infty} m^n x^m = (1-x)^{-n-1} \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^{k+1}.$$

Proof: Consider the polynomial

$$P_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^{k+1}.$$

By means of the recursion for the Eulerian numbers, we find

$$\begin{aligned}
P_n(x) &= \sum_{k=0}^{n-1} \left((k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle \right) x^{k+1} \\
&= \sum_{k=0}^{n-2}(k+1)\left\langle {n-1 \atop k} \right\rangle x^{k+1} + \sum_{k=1}^{n-1}(n-k)\left\langle {n-1 \atop k-1} \right\rangle x^{k+1} \\
&= \sum_{k=0}^{n-2}(k+1)\left\langle {n-1 \atop k} \right\rangle x^{k+1} + \sum_{k=0}^{n-2}(n-k-1)\left\langle {n-1 \atop k} \right\rangle x^{k+2} \\
&= (x-x^2)\sum_{k=0}^{n-2} \left\langle {n-1 \atop k} \right\rangle (k+1)x^k + nx\sum_{k=0}^{n-2} \left\langle {n-1 \atop k} \right\rangle x^{k+1} \\
&= (x-x^2)\frac{d}{dx}\left(\sum_{k=0}^{n-2} \left\langle {n-1 \atop k} \right\rangle x^{k+1} \right) + nxP_{n-1}(x) \\
&= x(1-x)P'_{n-1}(x) + nxP_{n-1}(x).
\end{aligned}$$

Now divide by $(1-x)^{n+1}$:

$$\frac{P_n(x)}{(1-x)^{n+1}} = \frac{xP'_{n-1}(x)}{(1-x)^n} + \frac{nxP_{n-1}(x)}{(1-x)^{n+1}} = x\frac{d}{dx}\frac{P_{n-1}(x)}{(1-x)^n} \tag{7.3}$$

by virtue of the product rule. The rest is an easy induction: If we set $P_0(x) = x$ for convenience, (7.3) remains true for $n = 1$, and we have

$$\frac{P_0(x)}{1-x} = \frac{x}{1-x} = \sum_{m=1}^{\infty} x^n = \sum_{m=1}^{\infty} m^0 x^m$$

since this is just a geometric series. Assuming that Theorem 7.11 holds for $n-1$, we now deduce that

$$\frac{P_n(x)}{(1-x)^{n+1}} = x\frac{d}{dx}\frac{P_{n-1}(x)}{(1-x)^{n+1}} = x\frac{d}{dx}\sum_{m=1}^{\infty} m^n x^m = x\sum_{m=1}^{\infty} m^n \cdot mx^{m-1} = \sum_{m=1}^{\infty} m^{n+1}x^m,$$

which completes the induction. □

Corollary 7.12 The Eulerian numbers can be determined by means of the explicit formula

$$\left\langle {n \atop k} \right\rangle = \sum_{l=0}^{k} (-1)^l \binom{n+1}{l} (k+1-l)^n.$$

Proof: By Theorem 7.11, $\left\langle {n \atop k} \right\rangle$ is the coefficient of x^{k+1} in $(1-x)^{n+1}\sum_{m=1}^{\infty} m^n x^m$, which is precisely

$$\sum_{l=0}^{k} (-1)^l \binom{n+1}{l} (k+1-l)^n,$$

making use of the coefficient formula for the product of two power series (in this case, $(1-x)^{n+1} = \sum_{l=0}^{n+1} \binom{n+1}{l} (-x)^l$ and $\sum_{m=1}^{\infty} m^n x^m$), see Theorem 5.4. □

7.5 Zigzag numbers

Let us now consider a special class of permutations that are known as *alternating permutations* or *zigzag permutations*: such a permutation has the property that ascents and descents are alternating, as in

$$9\ 5\ 7\ 1\ 8\ 4\ 6\ 2\ 3,$$

which is an alternating permutation of $\{1, 2, \ldots, 9\}$. How many such permutations are there? Without loss of generality, we may assume that an alternating permutation starts with a descent (clearly, the number of alternating permutations starting with an ascent is exactly the same as the number of alternating permutations starting with a descent). Now we can decompose these permutations: to the left of the number 1, there must be an alternating permutation of odd length (in fact, it can be *any* alternating permutation, since the element 1 automatically guarantees a descent followed by an ascent), while there can be any alternating permutation (starting with a descent) to the right of the element 1, and the length of this permutation is even or odd according to the length of the entire permutation:

$$9\ 5\ 7\ 1\ 8\ 4\ 6\ 2\ 3 \quad \mapsto \quad (9\ 5\ 7)\ \textcircled{1}\ (8\ 4\ 6\ 2\ 3)$$

Let us write s_n for the number of alternating permutations of even length n and t_n for the number of alternating functions of odd length ($s_n = 0$ if n is odd, and $t_n = 0$ if n is even). Then the decomposition argument presented above shows that

$$s_n = \sum_{k=0}^{n-1} \binom{n-1}{k} t_k s_{n-1-k},$$

where the summation index k is the number of elements to the left of the number 1 (so that $\binom{n-1}{k}$ is exactly the number of ways to distribute the numbers $2, 3, \ldots, n$ accordingly). Here, one has to define $s_0 = 1$. Analogously,

$$t_n = \sum_{k=0}^{n-1} \binom{n-1}{k} t_k t_{n-1-k}$$

for $n > 1$ and $t_1 = 1$. If $S(x)$ and $T(x)$ are the exponential generating functions of s_n and t_n, then this translates to

$$s_n = (n-1)![x^{n-1}]S(x)T(x)$$

and

$$t_n = (n-1)![x^{n-1}]T(x)^2$$

for $n > 1$. This is in agreement with the fact that we are forming pairs of alternating permutations whose combined size is $n-1$; compare the discussion of products of exponential generating functions in Section 6.2. Now note that

$$S'(x) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{s_n}{n!}x^n = \sum_{n=1}^{\infty}\frac{s_n}{(n-1)!}x^{n-1},$$

so that the coefficient of x^{n-1} in S' is exactly the coefficient of $S(x)T(x)$, implying $S'(x) = S(x)T(x)$ and analogously $T'(x) = 1 + T(x)^2$ (in this case, we have to take account of the fact that the equation for t_n only holds for $n > 1$). The differential equation

$$T' = 1 + T^2$$

is separable: one finds

$$\int \frac{dT}{1+T^2} = \int 1\,dx$$

or

$$\arctan T = x + C$$

and finally $T(x) = \tan x$ (the integration constant C has to be zero since $T(0) = t_0 = 0$). Now $S(x) = \sec x$ follows in very much the same way. The coefficients in the Taylor expansion of $\tan x$ at $x = 0$,

$$\tan x = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \frac{272x^7}{7!} + \frac{7936x^9}{9!} + \ldots,$$

which count alternating permutations of odd length, are known as the *tangent numbers* or *zag numbers*; for instance, there are 7936 alternating permutations of length 9. Accordingly, the coefficients in the expansion

$$\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \frac{1385x^8}{8!} + \ldots$$

are the *secant numbers* or *zig numbers*. These numbers also arise in other combinatorial problems: for instance, the study of *binary increasing trees* leads to these numbers; a binary increasing tree is a binary tree (all nodes have either two or no children) whose nodes are labelled from 1 to n in such a way that labels are increasing as one moves down from the root, see Figure 7.8.

Trees of this form occur in computer science in the analysis of algorithms (such as "Quicksort"). The decomposition (left subtree)-root-(right subtree) is immediate and provides a correspondence between binary increasing trees and alternating permutations; the tree shown in Figure 7.8 corresponds to the alternating permutation 9 5 7 1 8 4 6 2 3.

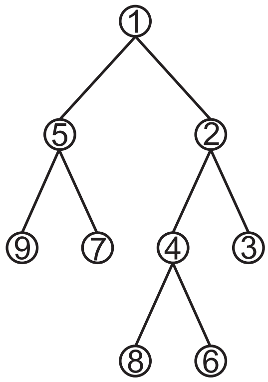

Figure 7.8: An increasing binary tree.

Let us finally describe an efficient way to compute the zigzag numbers: If $s_{n,r}$ and $t_{n,r}$ denote alternating permutations of even and odd length respectively whose last element is r. For odd n (with an ascent at the end), the permutation that remains if the last element r is dropped can have any number between 1 and $r-1$ at the end. This yields the formula

$$t_{n,r} = \sum_{k=1}^{r-1} s_{n-1,k}.$$

The same argument for even n shows that

$$s_{n,r} = \sum_{k=r}^{n-1} t_{n-1,k}.$$

Subtracting the formulas for $t_{n,r}$ and $t_{n,r-1}$ yields

$$t_{n,r} = t_{n,r-1} + s_{n-1,r-1}$$

with initial value $t_{n,1} = 0$ unless $n = 1$. Likewise,

$$s_{n,r} = s_{n,r+1} + t_{n-1,r}$$

with initial value $s_{n,n} = 0$. Now the numbers $s_{n,r}$ and $t_{n,r}$ (and thus also s_n and t_n) can be determined from the following triangular scheme:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & \leftarrow & 0 & & & & & & \\
 & & & & & 0 & \rightarrow & 1 & \rightarrow & 1 & & & & & \\
 & & & & 2 & \leftarrow & 2 & \leftarrow & 1 & \leftarrow & 0 & & & & \\
 & & & 0 & \rightarrow & 2 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 5 & & & \\
 & & 16 & \leftarrow & 16 & \leftarrow & 14 & \leftarrow & 10 & \leftarrow & 5 & \leftarrow & 0 & & \\
 & 0 & \rightarrow & 16 & \rightarrow & 32 & \rightarrow & 46 & \rightarrow & 56 & \rightarrow & 61 & \rightarrow & 61 & \\
272 & \leftarrow & 272 & \leftarrow & 256 & \leftarrow & 224 & \leftarrow & 178 & \leftarrow & 122 & \leftarrow & 61 & \leftarrow & 0
\end{array}$$

7.6 Bernoulli numbers

The *Bernoulli numbers* are typically defined by their exponential generating function

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1}.$$

Their importance lies in the fact that they occur in various formulas in analysis, one of which is treated here. First of all, let us note that a recursion for the Bernoulli numbers can be found by multiplying by $e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ and comparing coefficients:

$$\left(\sum_{n=1}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \right) = x$$

and thus

$$\sum_{k=1}^{n} \binom{n}{k} B_{n-k} = \begin{cases} 1 & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(Recall the rule for multiplication of exponential generating functions, see Section 6.2). Since $\binom{n}{k} = \binom{n}{n-k}$, we can rewrite the formula above (interchange k and $n-k$) as

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = \sum_{k=0}^{n-1} \binom{n}{n-k} B_k = \begin{cases} 1 & n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.4}$$

This can be used to determine the Bernoulli numbers recursively: $n = 1$ in the above formula yields $B_0 = 1$, then $n = 2$ yields $B_0 + 2B_1 = 0$ so that $B_1 = -\frac{1}{2}$, etc. The first few values are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \ldots$$

All odd-indexed Bernoulli numbers, except for B_1, are actually 0. This is due to the fact that

$$\frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1} = \frac{x}{2} \cdot \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} = \frac{x}{2} \coth \frac{x}{2}$$

is an even function. This identity also shows that the Bernoulli numbers occur in the power series of the cotangent.

Consider now the sum $s_m(n) = \sum_{k=1}^{n} k^m$; it is well known that $s_1(n) = \frac{n(n+1)}{2}$, $s_2(n) = \frac{n(n+1)(2n+1)}{6}$, and $s_3(n) = \frac{n^2(n+1)^2}{4}$. Is there always an explicit formula for $s_m(n)$, regardless of the value of m? The following shows that this is indeed the case:

Theorem 7.13 The sum of the first $n-1$ m-th powers ($m \geq 1$), $s_m(n-1) = \sum_{k=1}^{n-1} k^m$, can be determined by means of the formula

$$s_m(n-1) = \frac{1}{m+1} \left(\sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k} \right).$$

Remark: The polynomial $B_m(y) = \sum_{k=0}^{m} \binom{m}{k} B_k y^{m-k}$ is known as the m-th *Bernoulli polynomial*. It satisfies $B_m(0) = B_m$ (by definition) as well as

$$B_m(1) = \begin{cases} B_m + 1 & m = 1, \\ B_m & \text{otherwise}, \end{cases}$$

by equation (7.4). Theorem 7.13 can now be written as $s_m(n-1) = \frac{B_{m+1}(n) - B_{m+1}}{m+1}$. Of course, $s_m(n) = s_m(n-1) + n^m$, so there is also a closed formula for $s_m(n)$.

Proof: Consider the exponential generating function of $s_m(n-1)$:

$$S(x) = \sum_{m=0}^{\infty} \frac{s_m(n-1)}{m!} x^m = \sum_{m=0}^{\infty} \sum_{k=1}^{n-1} \frac{k^m x^m}{m!}$$

Interchanging the order of summation, we find

$$S(x) = \sum_{k=1}^{n-1} \sum_{m=0}^{\infty} \frac{k^m x^m}{m!} = \sum_{k=1}^{n-1} e^{kx} = \frac{e^{nx} - e^x}{e^x - 1}$$

by means of the formula for a finite geometric series. Compare this to the exponential generating function for Bernoulli polynomials:

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{B_m(y)}{m!} x^m &= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{1}{m!} \binom{m}{k} B_k y^{m-k} x^m = \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \frac{1}{k!(m-k)!} B_k y^{m-k} x^m \\
&= \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} \sum_{m=k}^{\infty} \frac{1}{(m-k)!} y^{m-k} x^{m-k} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} \sum_{m=0}^{\infty} \frac{1}{m!} y^m x^m \\
&= \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} e^{xy} = \frac{x e^{xy}}{e^x - 1}.
\end{aligned}$$

Now we can extract the m-th coefficient from $S(x)$:

$$\begin{aligned}
s_m(n-1) = m![x^m]S(x) &= m![x^m] \frac{e^{nx} - e^x}{e^x - 1} = m![x^{m+1}] \frac{x(e^{nx} - e^x)}{e^x - 1} \\
&= m! \left([x^{m+1}] \frac{x e^{nx}}{e^x - 1} - [x^{m+1}] \frac{x e^x}{e^x - 1} \right) = m! \left(\frac{B_{m+1}(n)}{(m+1)!} - \frac{B_{m+1}(1)}{(m+1)!} \right) \\
&= \frac{B_{m+1}(n) - B_{m+1}}{m+1},
\end{aligned}$$

which proves the theorem. □

Example 7.1 For $m = 5$, we find the formula

$$\begin{aligned}
s_5(n) &= n^5 + s_5(n-1) = n^5 + \frac{1}{6} \left(\sum_{k=0}^{5} \binom{6}{k} B_k n^{6-k} \right) = n^5 + \frac{1}{6} \left(n^6 - 3n^5 + \frac{5}{2} n^4 - \frac{1}{2} n^2 \right) \\
&= \frac{2n^6 + 6n^5 + 5n^4 - n^2}{12} = \frac{n^2(n+1)^2(2n^2 + 2n - 1)}{12}.
\end{aligned}$$

Remark: Let us only mention one more application of the Bernoulli numbers: the series $\sum_{n=1}^{\infty} n^{-s}$ is known to converge for any $s > 1$; this function is known as the *zeta function* $\zeta(s)$. For even positive integers, there is an explicit formula for the zeta function, namely

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}B_{2n}}{2(2n)!}.$$

In particular, $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$, etc. Furthermore, they are related to the tangent numbers defined in Section 7.5 by the formula

$$B_n = (-1)^{n/2-1}\frac{n}{4^n - 2^n}t_{n-1}$$

for even $n > 0$.

Chapter 8

Trees and Lagrange inversion

Binary trees and plane trees, as encountered in Section 7.1, have the common property that their construction can be recursively defined as "root + some sequence of subtrees". This is a very general principle that holds for other interesting families of trees as well. Let us consider two more, namely *d-ary trees* and *Cayley trees* (labelled trees).

d-ary trees

A straightforward generalisation of binary trees is to consider d-ary trees, defined by the property that every vertex has either d children or no children at all (see Figure 8.1 for an example in the case $d = 4$). The same arguments that led to the functional equation for the generating function of binary trees now yield

$$T(x) = x + xT(x)^d = x(1 + T(x)^d),$$

if $T(x)$ is the generating function for d-ary trees. For general d, this equation does not have an explicit solution any more. However, it is still possible to determine the coefficients from such a functional equation, as we will see.

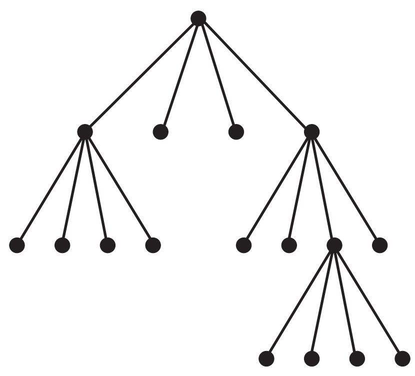

Figure 8.1: A 4-ary tree.

One can also consider *pruned* d-ary trees (children can be attached to nodes in any of d different positions, see Figure 8.2 for an example in the case $d = 3$). This gives rise to a functional equation of the form

$$T(x) = x(1 + T(x))^d$$

for the generating function $T(x)$.

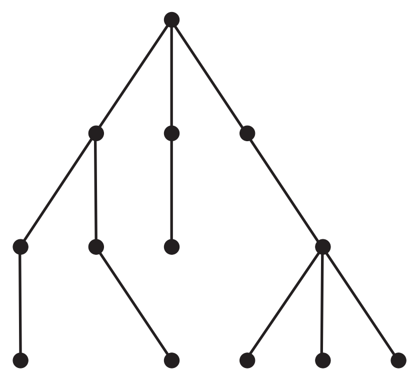

Figure 8.2: A pruned ternary tree.

Cayley trees

Cayley trees are rooted trees whose nodes are *labelled*, see Figure 8.3. The order of children does not matter in this context, one can think of the children as freely dangling from their parent node in space, in no particular order. In the language of Section 6.2, this can be described as

$$\mathcal{T} = \odot \times \mathsf{Set}(\mathcal{T}),$$

where $\mathcal{T}$ is the family of Cayley trees (such a tree consists of a root $\odot$ and a set of subtrees). Translating to the exponential generating function $T(x)$, we find

$$T(x) = xe^{T(x)}.$$

Note that the generating function $T(x)$ of any of the classes of trees (d-ary, pruned d-ary, plane, Cayley) that we considered satisfies a functional equation of the type

$$T(x) = x\,\Phi(T(x))$$

for some function Φ. The coefficients of such a generating function can be determined by means of a procedure that is known as *Lagrange inversion*.

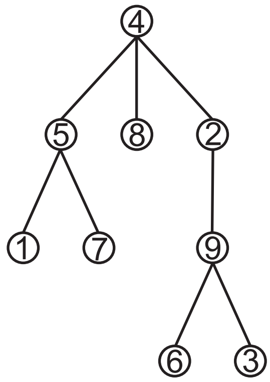

Figure 8.3: A Cayley tree.

Theorem 8.1 (Lagrange inversion formula) The coefficient of x^n in an implicitly defined function $T(x) = x\Phi(T(x))$ is given by

$$[x^n]T(x) = \frac{1}{n}[t^{n-1}]\Phi(t)^n$$

if $\Phi(t)$ has a power series expansion $\Phi(t) = \sum_{k=0}^{\infty} a_k t^k$ with $a_0 \neq 0$. More generally,

$$[x^n]T(x)^k = \frac{k}{n}[t^{n-k}]\Phi(t)^n.$$

Proof: In the following, we will work with power series that are allowed to contain negative powers:

$$b(t) = \sum_{n=-c}^{\infty} b_n t^n$$

for some $c \geq 0$. Note that the coefficient of t^{-1} in the derivative of such a power series is always zero, since

$$b'(t) = \sum_{n=-c}^{\infty} n b_n t^{n-1},$$

so that the coefficient of t^{-1} is exactly $0 \cdot b_0 = 0$. This is exploited in the following calculations: first of all, let the power series expansion of $T(x)^k$ be

$$T(x)^k = \sum_{m=k}^{\infty} s_m x^m.$$

Note that, since $T(x) = a_0 x + \ldots$ by definition, the first non-zero coefficient is that of x^k. Since $x = F(T(x)) = \frac{T(x)}{\Phi(T(x))}$, we have

$$t^k = \sum_{m=k}^{\infty} s_m F(t)^m,$$

substituting t for $T(x)$. Differentiate both sides to obtain

$$kt^{k-1} = \sum_{m=k}^{\infty} m s_m F(t)^{m-1} F'(t),$$

and divide by $F(t)^n$:

$$\frac{kt^{k-1}}{F(t)^n} = \sum_{m=k}^{\infty} m s_m F(t)^{m-n-1} F'(t).$$

Now we would like to take the coefficients of t^{-1} on both sides: since

$$F(t)^{m-n-1}F'(t) = \frac{d}{dt}\left(\frac{1}{m-n}F(t)^{m-n}\right)$$

for $m \neq n$, this coefficient must be zero by the above considerations for all summands except for the one that corresponds to $m = n$. Therefore, we have

$$[t^{-1}]\frac{kt^{k-1}}{F(t)^n} = [t^{-1}]\sum_{m=k}^{\infty} m s_m F(t)^{m-n-1}F'(t) = [t^{-1}]n s_n \frac{F'(t)}{F(t)}. \tag{8.1}$$

Now note that

$$\frac{F'(t)}{F(t)} = \frac{(t/\Phi(t))'}{t/\Phi(t)} = \frac{1/\Phi(t) - t\Phi'(t)/\Phi(t)^2}{t/\Phi(t)} = \frac{1}{t} - \frac{\Phi'(t)}{\Phi(t)}.$$

By our assumptions, $\frac{\Phi'(t)}{\Phi(t)}$ is just an ordinary power series without negative coefficients (since $\Phi(0) = a_0 \neq 0$); therefore, the coefficient of t^{-1} in $\frac{F'(t)}{F(t)}$ is exactly 1. Now (8.1) yields

$$[t^{-1}]\frac{kt^{k-1}}{F(t)^n} = n s_n$$

or

$$s_n = \frac{1}{n}[t^{-1}]\frac{kt^{k-1}}{F(t)^n} = \frac{1}{n}[t^{-1}]kt^{k-n-1}\Phi(t)^n = \frac{k}{n}[t^{n-k}]\Phi(t)^n,$$

which is what we wanted to prove. □

Remark: The second formula in Theorem 8.1 for powers of $T(x)$ is known as the *Lagrange-Bürmann formula*. It can be further extended to functions of the form $h(T(x))$, where $h(t)$ can be any function with a power series expansion $h(t) = \sum_{k=0}^{\infty} h_k t^k$:

$$\begin{aligned}
[x^n]h(T(x)) &= [x^n]\sum_{k=0}^{\infty} h_k T(x)^k = \sum_{k=0}^{\infty} \frac{kh_k}{n}[t^{n-k}]\Phi(t)^n \\
&= \sum_{k=0}^{\infty} \frac{kh_k}{n}[t^{n-1}]t^{k-1}\Phi(t)^n = \frac{1}{n}[t^{n-1}]\left(\sum_{k=0}^{\infty} kh_k t^{k-1}\right)\Phi(t)^n \\
&= \frac{1}{n}[t^{n-1}]h'(t)\Phi(t)^n.
\end{aligned}$$

Let us apply the Lagrange inversion formula to the various families of trees now. We have the following theorem:

Theorem 8.2 The number of d-ary trees with $dn+1$ nodes ($n \geq 0$) is the generalised Catalan number

$$\frac{1}{(d-1)n+1}\binom{dn}{n},$$

which is also the number of pruned d-ary trees with n vertices ($n \geq 1$). Furthermore, the number of Cayley trees with n nodes is n^{n-1} ($n \geq 1$).

Proof: The number of nodes in a d-ary tree must be of the form $dn+1$ (a proof by induction is not difficult), so that only this case needs to be considered. Since the generating function $T(x)$ satisfies $T(x) = x(1+T(x)^d)$, the Lagrange inversion formula yields

$$\begin{aligned}
[x^{dn+1}]T(x) &= \frac{1}{dn+1}[t^{dn}](1+t^d)^{dn+1} = \frac{1}{dn+1}[t^n](1+t)^{dn+1} = \frac{1}{dn+1}\binom{dn+1}{n} \\
&= \frac{1}{dn+1} \cdot \frac{(dn+1)!}{n!((d-1)n+1)!} = \frac{(dn)!}{n!((d-1)n+1)!} \\
&= \frac{1}{(d-1)n+1} \cdot \frac{(dn)!}{n!((d-1)n)!} = \frac{1}{(d-1)n+1}\binom{dn}{n}.
\end{aligned}$$

Likewise, the functional equation $T(x) = (1+T(x))^d$ for pruned d-ary trees yields

$$\begin{aligned}
[x^n]T(x) &= \frac{1}{n}[t^{n-1}](1+t)^{dn} = \frac{1}{n}\binom{dn}{n-1} = \frac{1}{n} \cdot \frac{(dn)!}{(n-1)!((d-1)n+1)!} \\
&= \frac{(dn)!}{n!((d-1)n+1)!} = \frac{1}{(d-1)n+1} \cdot \frac{(dn)!}{n!((d-1)n)!} = \frac{1}{(d-1)n+1}\binom{dn}{n},
\end{aligned}$$

as it should be. Finally, we can use the functional equation $T(x) = e^{T(x)}$ to obtain the number of Cayley trees with n nodes:

$$[x^n]T(x) = \frac{1}{n}[t^{n-1}](e^t)^n = \frac{1}{n}[t^{n-1}]e^{nt} = \frac{1}{n} \cdot \frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}.$$

Note that, since we are dealing with an exponential generating function here, a factor $\frac{1}{n!}$ occurs naturally. □

The last result concerning Cayley trees deserves more attention: consider *labelled trees* that do not have a distinguished root (but the nodes are still labelled 1 to n); in graph-theoretical terminology, a tree just is a connected acyclic graph (that is, there is no cycle of nodes). It is clear that any labelled tree with n nodes can be rooted in n different ways (at any of the nodes). Therefore, the number of labelled trees is $\frac{1}{n} \cdot n^{n-1} = n^{n-2}$, which is known as *Cayley's formula*. The first few terms in the resulting sequence are $1, 1, 3, 16, 125, \ldots$. There is a very elegant combinatorial proof for this formula by what is known as *Prüfer codes*.

To any labelled tree with n nodes, we associate a sequence $(a_1, a_2, \ldots, a_{n-2})$ of $n-2$ numbers between 1 and n as follows:

- Consider all leaves (i.e., nodes that have precisely one neighbour), and find the one whose label is largest.
- The label of its neighbour is the first element a_1 of our sequence.
- Remove the leaf and repeat the process with the remaining tree, until there are only two vertices left.

Figure 8.4 shows an example of this procedure. The resulting Prüfer code is $(3, 4, 1, 3)$. The process can also be reversed quite easily: determine the largest number between 1 and n that is not present in the code; the corresponding node must be a leaf and connected to the node labelled a_1; now remove a_1 from the code and repeat this step.

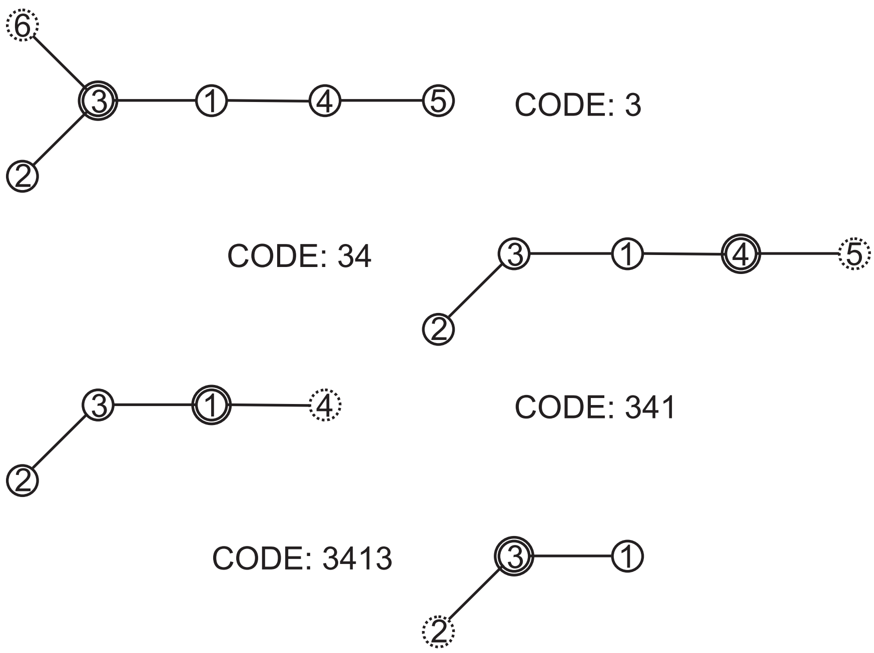

Figure 8.4: How to obtain the Prüfer code of a labelled tree.

Noting that there are exactly n^{n-2} possible codes, this shows that the number of labelled trees must also be n^{n-2}, which provides us with a different proof of Cayley's formula.

Chapter 9

Integer partitions

A *partition* of an integer n is a representation of n as a sum of positive integers, called the summands or parts of the partition. The order of the summands is irrelevant in this context (as opposed to compositions). Typically, one writes the summands in decreasing order; for instance, the partitions of 5 are

$$5, \quad 4+1, \quad 3+2, \quad 3+1+1, \quad 2+2+1, \quad 2+1+1+1, \quad 1+1+1+1+1.$$

Thus there are seven partitions of 5. The number of partitions of n is denoted by $p(n)$ ($p(5) = 7$, for instance).

9.1 Ferrers diagrams

A convenient way of visualising an integer partition is to draw what is known as a *Ferrers diagram*. Each summand is represented by a horizontal line of dots, as in Figure 9.1.

Ferrers diagrams are frequently used in the study of the properties of partitions. Let, for instance, the number of partitions of n whose largest summand is k be denoted by $p(n,k)$, and let the number of partitions of n whose largest summand is $\leq k$ be $P(n,k)$. Then, one has

$$p(n,k) = P(n-k,k).$$

This follows immediately from the following argument: removing the largest part k (which is equivalent to removing the first row in the Ferrers diagram) yields a partition of $n-k$ whose summands are all at most equal to k. Furthermore, it is clear from the definition that $p(n) = P(n,n)$ and that

$$P(n,k) = \sum_{l=1}^{k} p(n,l).$$

These formulas can be used to determine $p(n)$ recursively. However, a more efficient algorithm is given in the following section.

A particularly useful operation on Ferrers diagrams is *conjugation*: reading a partition by columns rather than by rows, one obtains the conjugate partition. For instance, the

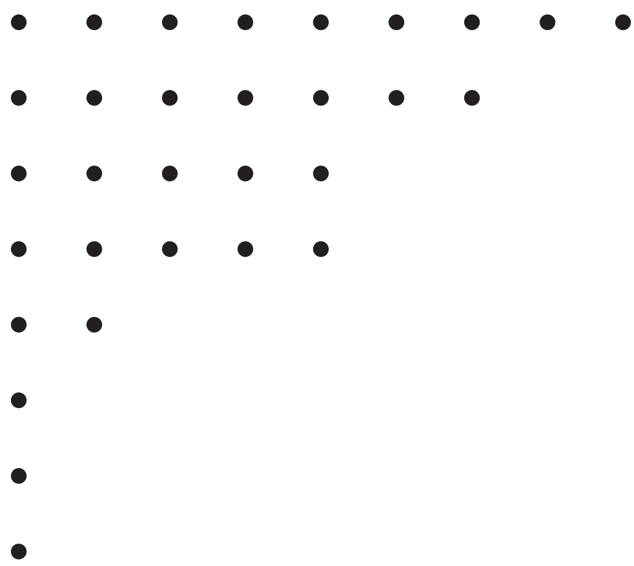

Figure 9.1: Ferrers diagram of the partition $9+7+5+5+2+1+1+1$.

conjugate of the partition shown in Figure 9.1 is $8+5+4+4+4+2+2+1+1$. Note that the length (number of summands) of a partition is exactly the maximum (largest summand) of its conjugate. The following theorem is an immediate consequence:

Theorem 9.1 The number of partitions of n whose largest summand is k is the same as the number of partitions of n into exactly k summands.

9.2 Generating functions

Generating functions play an important role in the study of partitions as well. In the language of Section 6.1, the summands in an integer partition form a multiset, so that we obtain the following representations for the generating function $P(x) = \sum_{n=1}^{\infty} p(n)x^n$:

$$P(x) = \prod_{k=1}^{\infty}(1-x^k)^{-1} = \exp\left(\sum_{k=1}^{\infty} \frac{x^k}{k(1-x^k)}\right).$$

The former representation as an infinite product is usually easier to work with. If one considers partitions with the property that all summands are distinct, then this corresponds exactly to the powerset construction, so that one obtains

$$Q(x) = \prod_{k=1}^{\infty}(1+x^k) = \exp\left(\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k(1-x^k)}\right)$$

for the generating function $Q(x) = \sum_{n=0}^{\infty} q(n)x^n$, where $q(n)$ is the number of partitions of n into distinct summands. More generally, one can consider partitions into elements from a prescribed set S. Then, the corresponding generating functions are

$$P_S(x) = \prod_{s \in S} (1 - x^s)^{-1}$$

and

$$Q_S(x) = \prod_{s \in S} (1 + x^s)$$

for arbitrary partitions and partitions into distinct summands, respectively. This can be used to treat so-called *money changing problems*:

Example 9.1 In how many ways can one give change of n Rand using R1, R2 or R5 coins? This problem is equivalent to asking for the number of partitions of n into summands $1, 2$ and 5. Hence the generating function is

$$C(x) = \frac{1}{(1-x)(1-x^2)(1-x^5)}.$$

This can be turned into an explicit formula (albeit complicated) by means of a partial fraction decomposition: one has

$$\begin{aligned} C(x) &= \frac{13}{40(1-x)} + \frac{1}{4(1-x)^2} + \frac{1}{10(1-x)^3} + \frac{1}{8(1+x)} + \frac{1+x+2x^2+x^3}{5(1+x+x^2+x^3+x^4)} \\ &= \frac{13}{40(1-x)} + \frac{1}{4(1-x)^2} + \frac{1}{10(1-x)^3} + \frac{1-x}{8(1-x^2)} + \frac{1+x^2-x^3-x^4}{5(1-x^5)} \\ &= \frac{13}{40}\sum_{n=0}^{\infty} x^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)x^n + \frac{1}{10}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n \\ &\quad + \frac{1-x}{8}\sum_{n=0}^{\infty} x^{2n} + \frac{1+x^2-x^3-x^4}{5}\sum_{n=0}^{\infty} x^{5n}, \end{aligned}$$

so that the number of possibilities is

$$\frac{n^2+8n}{20} + \begin{cases} 1 & n = 10k \text{ or } n = 10k+2, \\ \frac{11}{20} & n = 10k+1, \\ \frac{7}{20} & n = 10k+3 \text{ or } n = 10k+9, \\ \frac{3}{5} & n = 10k+4 \text{ or } n = 10k+8, \\ \frac{3}{4} & n = 10k+5 \text{ or } n = 10k+7, \\ \frac{4}{5} & n = 10k+6, \end{cases}$$

and it can also be shown that this is precisely the nearest integer to $\frac{(n+4)^2}{20}$.

Generating functions also prove useful when it comes to *partition identities*, of which the following theorem (due to Euler) is a famous example:

Theorem 9.2 The number of partitions of an integer n in which all parts are odd equals the number of partitions of n in which all parts are distinct.

Proof: The generating function for partitions into odd summands is

$$P_{\text{odd}}(x) = \prod_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (1-x^n)^{-1} = (1-x)^{-1}(1-x^3)^{-1}(1-x^5)^{-1}(1-x^7)^{-1}\dots$$

We introduce some artificial factors:

$$P_{\text{odd}}(x) = \frac{(1-x^2)(1-x^4)(1-x^6)\dots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\dots}$$

The factor $1-x^{2n}$ in the numerator can be written as $(1+x^n)(1-x^n)$, and the second factor also occurs in the denominator so that one can cancel. This yields

$$P_{\text{odd}}(x) = (1+x)(1+x^2)(1+x^3)\dots = \prod_{n=1}^{\infty}(1+x^n),$$

which is exactly the generating function for partitions into distinct summands. □

Let us finally discuss another result of Euler known as the *Pentagonal Number Theorem*:

Lemma 9.3 The infinite product $\prod_{n=1}^{\infty}(1-x^n)$ can be written as

$$\prod_{n=1}^{\infty}(1-x^n) = 1 + \sum_{k=1}^{\infty}(-1)^k\left(x^{k(3k-1)/2} + x^{k(3k+1)/2}\right).$$

Proof: The product $\prod_{n=1}^{\infty}(1-x^n)$ resembles the generating function for partitions into distinct summands, the only exception being the sign. If the product is expanded, then every summand corresponds to a partition into distinct elements, and the sign is $(-1)^l$, where l is the number of summands. Therefore, we have to consider the difference $q_{\text{even}}(n) - q_{\text{odd}}(n)$, where $q_{\text{even}}(n)$ denotes the number of partitions of n into distinct even summands and $q_{\text{odd}}(n)$ the number of partitions into distinct odd summands. This difference is exactly the constant in the expansion of the product, and we will show that it is almost always 0, except for two exceptional cases ($n = \frac{k(3k-1)}{2}$ and $n = \frac{k(3k+1)}{2}$), by constructing a bijection between the two.

To each partition λ, we assign two parameters called the *base* $b(\lambda)$ and the *slope* $s(\lambda)$. The base is simply the smallest summand, while the slope is the length of the NE-SW diagonal starting at the rightmost point of the top row (see Figure 9.2).

The bijection is now constructed as follows: if $b(\lambda) \leq s(\lambda)$, then we remove the base and attach it to the slope (as shown in Figure 9.3, top) to obtain another partition; if $b(\lambda) > s(\lambda)$, then we reverse the process by removing the slope and appending it at the end of the partition (Figure 9.3, bottom). This procedure reverses the parity of the length

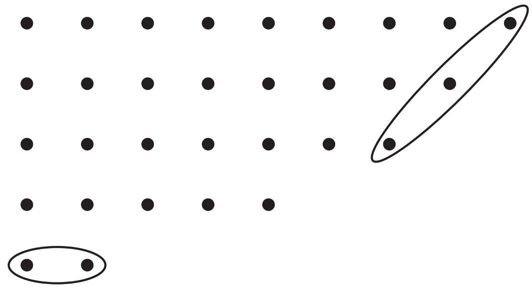

Figure 9.2: Base and slope.

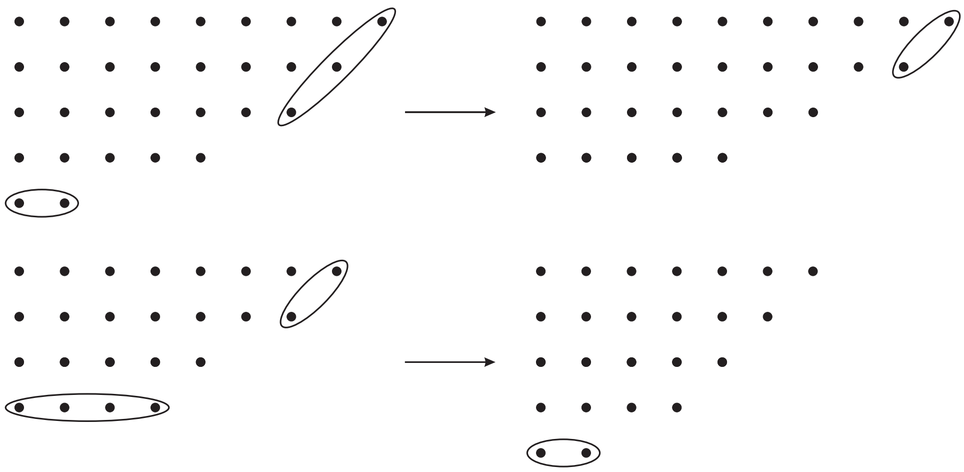

Figure 9.3: Transferring the base or the slope.

of a partition (note that the length changes by 1), and so it produces a $1-1$ correspondence between partitions into an even number of distinct summands and partitions into an odd number of distinct summands.

This procedure can only fail if the base and the slope overlap and if either $b(\lambda) = s(\lambda)$ (Figure 9.4, top) or $b(\lambda) = s(\lambda) - 1$ (Figure 9.4, bottom). In the first case, we do not get a proper partition at all, in the second case, we do not get a partition into distinct parts.

The first exception occurs if λ is the partition $(2k-1)+(2k-2)+\ldots+k$ (so that $k = b(\lambda) = s(\lambda)$); since

$$(2k-1)+(2k-2)+\ldots+k = k^2 + \sum_{m=0}^{k-1} m = k^2 + \frac{k(k-1)}{2} = \frac{k(3k-1)}{2},$$

this means that $q_{\text{even}}(n) - q_{\text{odd}}(n) = (-1)^k$ if $n = \frac{k(3k-1)}{2}$ (note that the length of the exceptional partition is k).

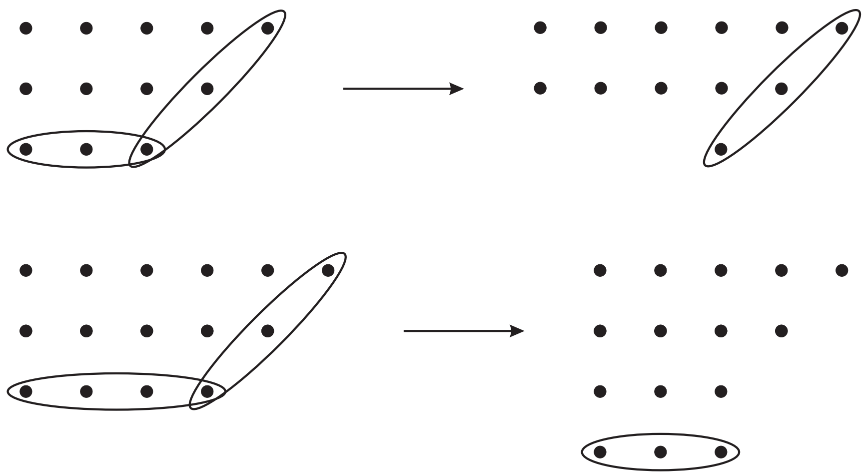

Figure 9.4: Exceptional cases.

Similarly, the second exception occurs if λ is the partition $(2k)+(2k-1)+\ldots+(k+1)$ (so that $k=b(\lambda)-1=s(\lambda)$); since

$$(2k)+(2k-1)+\ldots+(k+1)=k^2+\sum_{m=1}^{k} m = k^2+\frac{k(k+1)}{2}=\frac{k(3k+1)}{2},$$

this means that $q_{\text{even}}(n)-q_{\text{odd}}(n)=(-1)^k$ if $n=\frac{k(3k+1)}{2}$ (again, the length of the exceptional partition is k).

For all other $n>0$, we have $q_{\text{even}}(n)-q_{\text{odd}}(n)=0$, so that we obtain the theorem (clearly, the coefficient of x^0 has to be 1, which explains the first summand in the formula). $\square$

This result can now be used to prove a recursive formula for the number of partitions:

Theorem 9.4 For all $n>0$, the formula

$$p(n)=\sum_{k=1}^{\infty}(-1)^{k-1}\big(p(n-k(3k-1)/2)+p(n-k(3k+1)/2)\big)$$

holds, where we set $p(0)=1$ and $p(n)=0$ if $n<0$.

Proof: Since $\sum_{n=0}^{\infty}p(n)x^n=\prod_{k=1}^{\infty}(1-x^k)^{-1}$, Lemma 9.3 can be written as

$$\left(\sum_{n=0}^{\infty}p(n)x^n\right)\cdot\left(1+\sum_{k=1}^{\infty}(-1)^k\big(x^{k(3k-1)/2}+x^{k(3k+1)/2}\big)\right)=1.$$

Compare the coefficient of x^n on both sides: on the right hand side, it is clearly equal to 0 if $n>0$. On the left hand side, the coefficient is

$$p(n)+\sum_{k=1}^{\infty}(-1)^k\big(p(n-k(3k-1)/2)+p(n-k(3k+1)/2)\big),$$

which readily proves the theorem. □

This theorem allows one to determine the value of $p(n)$ for small n with comparatively little effort: $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, $p(6) = 11$, ...

REMARK: A celebrated theorem due to Hardy and Ramanujan states that the number of partitions of n is approximately

$$p(n) \approx \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n}.$$

9.3 q-binomial coefficients

We know from the previous section that the generating function for partitions into parts of size $\leq k$ (their number was denoted by $P(n, k)$) is given by

$$\sum_{n=0}^{\infty} P(n,k)x^n = \prod_{m=1}^{k} (1 - x^m)^{-1},$$

and by the arguments of Section 9.1, this is also the generating function for partitions of length $\leq k$. What if we restrict both at the same time (that is, require all parts to be $\leq k$ and the length to be $\leq l$)? The following theorem provides an answer:

Theorem 9.5 The generating function for the number $P(n, k, l)$ of partitions of n into at most l summands of size at most k is given by

$$\sum_{n=0}^{\infty} P(n,k,l)x^n = \frac{(x)_{k+l}}{(x)_k(x)_l},$$

where $(x)_k$ is defined by $(x)_k = \prod_{m=1}^{k}(1 - x^m)$.

REMARK: The sum in Theorem 9.5 is actually a finite sum, since $P(n, k, l) = 0$ if $n > kl$.

Proof: Note first that

$$P(n, k, l) - P(n, k, l - 1)$$

is the number of partitions of n into *exactly* l summands all of which are $\leq k$. If we remove the first column from the Ferrers diagram of such a partition, we are left with a partition of $n - l$ into at most l summands all of which are $\leq k - 1$ (and this can also be reversed). Hence,

$$P(n, k, l) - P(n, k, l - 1) = P(n - l, k - 1, l). \tag{9.1}$$

Now we can use induction to proceed. If $k = 0$, then the stated formula reads

$$\sum_{n=0}^{\infty} P(n, 0, l) = \frac{(x)_l}{(x)_l} = 1,$$

which holds if we define $P(0,0,l) = 1$ and $P(n,0,l) = 0$ otherwise (so that recursion (9.1) remains true in these cases). Likewise, the formula is also true if $l = 0$. All that remains is the induction step. Making use of (9.1), we find

$$\begin{aligned}\sum_{n=0}^{\infty} P(n,k,l)x^n &= \sum_{n=0}^{\infty} P(n,k,l-1)x^n + \sum_{n=l}^{\infty} P(n-l,k-1,l)x^n \\ &= \sum_{n=0}^{\infty} P(n,k,l-1)x^n + \sum_{n=0}^{\infty} P(n,k-1,l)x^{n+l} \\ &= \sum_{n=0}^{\infty} P(n,k,l-1)x^n + x^l \sum_{n=0}^{\infty} P(n-1,k-1,l)x^n \\ &= \frac{(x)_{k+l-1}}{(x)_k(x)_{l-1}} + x^l \cdot \frac{(x)_{k+l-1}}{(x)_{k-1}(x)_l},\end{aligned}$$

where the last step follows from the induction hypothesis. Making use of the fact that $(x)_k = (x)_{k-1}(1-x^k)$ by definition, we can simplify further to obtain

$$\begin{aligned}\frac{(x)_{k+l-1}}{(x)_k(x)_{l-1}} + x^l \cdot \frac{(x)_{k+l-1}}{(x)_{k-1}(x)_l} &= (1-x^l) \cdot \frac{(x)_{k+l-1}}{(x)_k(x)_l} + x^l(1-x^k) \cdot \frac{(x)_{k+l-1}}{(x)_k(x)_l} \\ &= (1-x^{k+l}) \cdot \frac{(x)_{k+l-1}}{(x)_k(x)_l} = \frac{(x)_{k+l}}{(x)_k(x)_l}.\end{aligned}$$

This completes the induction. □

The generating function for the number $p(n,k,l)$ of partitions of n whose length and largest part are *exactly* l and k respectively is closely related; the techniques of Section 9.1 can be applied once again:

Corollary 9.6 The generating function for the number $p(n,k,l)$ of partitions of n into l summands of which the largest is equal to k is given by

$$\sum_{n=0}^{\infty} p(n,k,l)x^n = x^{k+l-1} \frac{(x)_{k+l-2}}{(x)_{k-1}(x)_{l-1}}.$$

Proof: If we remove the first row and the first column of the Ferrers diagram of a partition of n whose length and largest part are l and k respectively, we obtain a partition of $n-k-l+1$ whose length is at most $l-1$ and whose largest part is at most $k-1$. Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} p(n,k,l)x^n &= \sum_{n=k+l-1}^{\infty} P(n-k-l+1,k-1,l-1)x^n \\ &= \sum_{n=0}^{\infty} P(n,k-1,l-1)x^{n+k+l-1} = x^{k+l-1} \frac{(x)_{k+l-2}}{(x)_{k-1}(x)_{l-1}}\end{aligned}$$

by Theorem 9.5. □

$P(n, k, l)$ counts precisely those partitions whose Ferrers diagram fits inside a $k \times l$-rectangle; let us assume that the lower left corner of this rectangle is $(0, 0)$, while the upper right corner is (k, l). The boundary of the Ferrers diagram can also be interpreted as a lattice path consisting of "up" and "right" steps that connects $(0, 0)$ and (k, l), see Figure 9.5. Therefore, $P(n, k, l)$ also counts paths of this type with the property that the area above the path (or, by symmetry, below the path) is n.

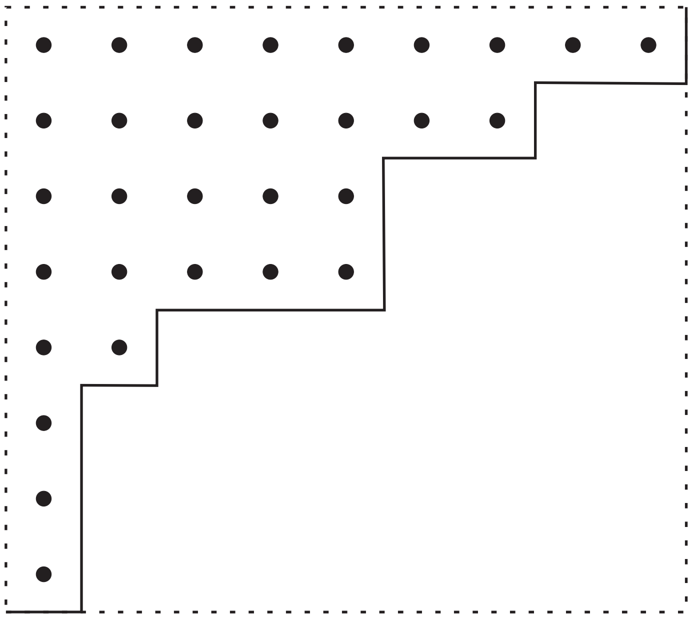

Figure 9.5: The lattice path corresponding to a partition.

The total number of all such lattice paths is $\binom{k+l}{k}$ (the number of ways to choose which of the $k + l$ steps are "right" steps). Therefore, one has

$$\sum_{n=0}^{\infty} P(n, k, l) = \binom{k+l}{k},$$

and so the limit of the right hand side of the formula in Theorem 9.5 must be

$$\lim_{x \to 1} \frac{(x)_{k+l}}{(x)_k (x)_l} = \binom{k+l}{k}.$$

Note that one has to take the limit, since both numerator and denominator are zero at $x = 1$. This is the reason why one calls

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}} = \frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)(1-q^2)\dots(1-q^k) \cdot (1-q)(1-q^2)\dots(1-q^{n-k})}$$

a *q-binomial coefficient* (it is customary to use the letter q in this context). One has

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k},$$

and the proof of Theorem 9.5 shows that

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

Analogously, one also finds

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

Note that as $q \to 1$, both formulas reduce to the ordinary recursion for the binomial coefficients.

Example 9.2 The q-binomial coefficient $\left[\begin{smallmatrix} 5 \\ 3 \end{smallmatrix}\right]_q$ is given by

$$\begin{aligned} \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q &= \frac{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}{(1-q)(1-q^2)(1-q^3)(1-q)(1-q^2)} = \frac{(1-q^4)(1-q^5)}{(1-q)(1-q^2)} \\ &= (1+q^2)(1+q+q^2+q^3+q^4) = 1+q+2q^2+2q^3+2q^4+q^5+q^6. \end{aligned}$$

The corresponding partitions whose length and largest part are ≤ 2 and ≤ 3 respectively are

$$\begin{aligned} 0 &= \ , \\ 1 &= 1, \\ 2 &= 1+1 = 2, \\ 3 &= 2+1 = 3, \\ 4 &= 2+2 = 3+1, \\ 5 &= 3+2, \\ 6 &= 3+3. \end{aligned}$$